Lecture 15 The Feynman Path Integral

The Path Integral provides us with an alternative way of formulating QFT. It is ultimately equivalent to canonical quantization but for many purposes it is in fact more convenient. To start with we will develop the formalism for the scalar field theory. Of course this is defined by the field $\phi(t, \underline{x})$, its conjugate momentum $\pi(t, \underline{x})$, and its Hamiltonian, H.

Canonical quantization takes ϕ and π an promotes them to operators with commutation rules.
To obtain a path integral formulation we first introduce the basic tools that we need.

The state $| \underline{\Phi}(\underline{x}), t\rangle$ is an eigenstate of the operator $\hat{\phi}(\underline{x},t)$

$$\hat{\phi} \; | \underline{\Phi}(\underline{x}), \; t\rangle = \underline{\Phi}(\underline{x}) \, | \underline{\Phi}(\underline{x}), t\rangle$$

That is to say, in this state the field configuration is definitely $\underline{\Phi}(\underline{x})$ at time t.
Similarly the state $| \Pi(\underline{x}), t\rangle$ is an eigenstate of the conjugate momentum $\hat{\pi}(\underline{x},t)$

$$\hat{\pi} \; | \Pi(\underline{x}), t\rangle = \Pi(\underline{x}) \, | \Pi(\underline{x}), t\rangle$$

These states have properties analogous to particle states; for example

$$\langle \underline{\Phi}'(\underline{x}), t \,|\, \underline{\Phi}(\underline{x}), t\rangle = \delta^{\infty}\left(\underline{\Phi}'(\underline{x}) - \underline{\Phi}(\underline{x})\right)$$

This means that the matrix element is non-zero only if Φ and Φ' take the same value at every point in space. Clearly this is an infinite number of constraints - one for each point in space - so we will write δ^∞; this is called a <u>functional delta function</u>.

To make rigorous sense of it we split space up into a lattice with sites labelled i in which the field is specified at each site

$$\Phi(\underline{x}) \rightarrow \{\Phi_i\ ,\ i=1,2\ldots\}$$

Then

$$\delta^\infty(\Phi'(\underline{x})-\Phi(\underline{x})) = \prod_{i=1\ldots} \delta(\Phi_i' - \Phi_i)$$

Similarly

$$\langle \Pi'(\underline{x}),t | \Pi(\underline{x}),t\rangle = \delta^\infty\left(\frac{\Pi'(\underline{x})-\Pi(\underline{x})}{2\pi}\right)$$

In the lattice version this is

$$\prod_{i=1\ldots} 2\pi\ \delta(\Pi_i' - \Pi_i)$$

Finally we will need the matrix element

$$\langle \Pi'(\underline{x}), t \,|\, \Phi(\underline{x}), t \rangle$$

In the discrete picture we would get

$p_j = a^3 \Pi_j$
see Notes 3.8

$$\langle \{\Pi_j'\}, t \,|\, \{\Phi_j\}, t \rangle = \prod_j e^{-i \Pi_j' \Phi_j a^3}$$

$$= e^{-i \sum_j \Pi_j' \Phi_j a^3}$$

which in the continuum becomes

$$e^{-i \int d^3\underline{x}\, \Pi'(\underline{x})\, \Phi(\underline{x})}$$

We can construct identity operators by noting that

$$|\{\Phi_i'\}, t\rangle \langle \{\Phi_i'\}, t \,|\, \{\Phi_i \ldots\}, t\rangle$$

$$= |\{\Phi_i'\}, t\rangle \prod_j \delta(\Phi_j' - \Phi_j)$$

$\longrightarrow$

If we integrate over $\underline{\Phi}'$ <u>at each site</u> we get

$$\left(\prod_i \int d\underline{\Phi}_i\right) \; |\{\underline{\Phi}_i'\}, t\rangle \; \left(\prod_i \delta(\underline{\Phi}_i' - \underline{\Phi}_i)\right)$$

$$= \left(\prod_i \int d\underline{\Phi}_i' \; \delta\,(\underline{\Phi}_i' - \underline{\Phi}_i)\right) \; | \{\underline{\Phi}_i'\}, t\rangle$$

$$= \; |\{\underline{\Phi}_i\}, t\rangle$$

That is to say

$$\int d[\underline{\Phi}'(\underline{x})] \quad |\underline{\Phi}'(\underline{x}), t\rangle \langle \underline{\Phi}'(\underline{x}), t|$$

is actually the identity operator.

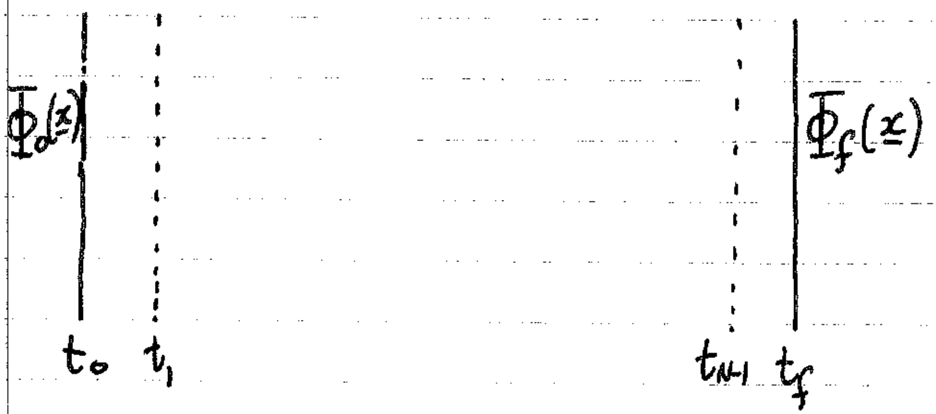
$$\int d[\underline{\Phi}'(\underline{x})] \;\leftrightarrow\; \prod_i \int d\underline{\Phi}_i'$$

– it means integrate <u>separately</u> over the value of $\underline{\Phi}'$ at each point in space. If you think about it for a minute this is the same as summing <u>over all possible configurations</u> $\underline{\Phi}'(\underline{x})$.

Similarly we can make an identity operator out of

$$\int d\left[\frac{\underline{\Pi}'(\underline{x})}{2\pi}\right] \quad |\underline{\Pi}'(\underline{x}), t\rangle \langle \underline{\Pi}'(\underline{x}), t|$$

Now let's consider a situation in which at time t_0 the field configuration is $\Phi_0(\underline{x})$ and at time t_f it is $\Phi_f(\underline{x})$



and calculate the matrix element

$$M_{f0} = \langle \Phi_f(\underline{x}), t_f \,|\, \Phi_0(\underline{x}), t_0 \rangle$$

Now introduce the identity operator at t_1

$$M_{f0} = \int d[\Phi_1(\underline{x})] \; \langle \Phi_f(\underline{x}), t_f \,|\, \Phi_1(\underline{x}), t_1 \rangle \langle \Phi_1(\underline{x}), t_1 \,|\, \Phi_0(\underline{x}), t_0 \rangle$$

Now we are summing over all possible field configurations at intermediate time t_1. We can carry on introducing the identity operator at N-1 intermediate time slices to get

$$M_{f0} = \left(\prod_{i=1}^{N-1} \int d[\Phi_i(\underline{x})] \right) \langle \Phi_f(\underline{x}), t_f \,|\, \Phi_{N-1}(\underline{x}), t_{N-1} \rangle \langle \Phi_{N-1}(\underline{x}), t_{N-1} \,|\, \Phi_{N-2}(\underline{x}), t_{N-2} \rangle$$
$$\ldots\ldots \langle \Phi_2(\underline{x}), t_2 \,|\, \Phi_1(\underline{x}), t_1 \rangle \langle \Phi_1(\underline{x}), t_1 \,|\, \Phi_0(\underline{x}), t_0 \rangle$$

We matrix elements between states at different times. To deal with these we insert a complete set of momentum eigenstates to give eg

$$M_{21} = \langle \Phi_2(\underline{x}), t_2 | \Phi_1(\underline{x}), t_1 \rangle$$

$$= \int d\left[\frac{\Pi_2'(\underline{x})}{2\pi}\right] \langle \Phi_2(\underline{x}), t_2 | \Pi_2'(\underline{x}), t_2 \rangle \langle \Pi_2'(\underline{x}), t_2 | \Phi_1(\underline{x}), t_1 \rangle$$

$$= \int d\left[\frac{\Pi_2'(\underline{x})}{2\pi}\right] e^{i\int d^3\underline{x}\, \Pi_2'(\underline{x})\, \Phi_2(\underline{x})} \langle \Pi_2'(\underline{x}), t_2 | \Phi_1(\underline{x}), t_1 \rangle$$

Now[‡]

$$|\Pi_2'(\underline{x}), t_2\rangle = e^{iH(t_2 - t_1)} |\Pi_2'(\underline{x}), t_1\rangle$$

and we assume $t_2 - t_1 = \delta t$ is small so we can expand the exponential. We get

$$M_{21} = \int d\left[\frac{\Pi_2'(\underline{x})}{2\pi}\right] e^{i\int d^3\underline{x}\, \Pi_2'(\underline{x})\, \Phi_2(\underline{x})} \times$$

$$\langle \Pi_2'(\underline{x}), t_1 | (1 - i\,\delta t\, H) | \Phi_1(\underline{x}), t_1 \rangle \qquad (*)$$

Now order H so that all π's are to the left of all ϕ's so each operator is acting directly on one of its eigenstates in (*)

We get

$$M_{21} = \int d\left[\frac{\Pi_2'(\underline{x})}{2\pi}\right] e^{i\int d^3\underline{x}\, \Pi_2'(\underline{x})\,(\Phi_2(\underline{x}) - \Phi_1(\underline{x}))} \times \left(1 - i\,\delta t\, H(\Pi_2'(\underline{x}), \Phi_1(\underline{x}))\right)$$

‡ Footnote on next page

<u>Footnote</u> Remember that $\phi(t) = e^{iHt}\phi(0)e^{-iHt}$.

Hence, for $\phi(t)|\Phi,t\rangle = \Phi|\Phi,t\rangle$ and $\phi(t')|\Phi,t'\rangle = \Phi|\Phi,t'\rangle$

we have
$$e^{iH(t'-t)}\phi(t)\,e^{-iH(t'-t)}|\Phi,t'\rangle = \Phi|\Phi,t'\rangle$$

implying $\quad \phi(t)\,e^{-iH(t'-t)}|\Phi,t'\rangle = \Phi\,e^{-iH(t'-t)}|\Phi,t'\rangle$

and hence $\quad e^{-iH(t'-t)}|\Phi,t'\rangle = |\Phi,t\rangle$

and $\quad |\Phi,t'\rangle = e^{iH(t'-t)}|\Phi,t\rangle$

We can put this back into M_{fo} to get (re-exponentiating the Hamiltonian factor and assuming $(\delta t)^2$ terms can be ignored)

$$M_{fo} = \prod_{i=1}^{N-1} \int d\left[\Phi_i(\underline{x})\right] d\left[\frac{\Pi_i(\underline{x})}{2\pi}\right]$$

$$\times \exp i \int d^3\underline{x} \sum_{i=0}^{N-1} \left(\Pi_{i+1}(\underline{x})\left(\Phi_{i+1}(\underline{x}) - \Phi_i(\underline{x})\right)\right)$$

$$\times \exp -i\,\delta t \sum_{i=0}^{N-1} H\left(\Pi_{i+1}(\underline{x}), \Phi_i(\underline{x})\right) \quad (**)$$

with the identifications $\Phi_f \equiv \Phi_N$

$$\Pi_f \equiv \Pi_N$$

We can now do two things

1) Formally take the limit $N \to \infty$, $\delta t \to 0$
such that $N\,\delta t = t_f - t_o$

Then $\Phi_{i+1}(\underline{x}) - \Phi_i(\underline{x}) = \delta t\ \dot{\Phi}_i(\underline{x})$

and $\sum_{i=0}^{N-1} \delta t \ldots \to \int_{t_o}^{t_f} dt \ldots$

so

$$M_{fo} \simeq \int D\Phi\ D\left(\frac{\Pi}{2\pi}\right) \exp i \int d^3\underline{x} \int_{t_o}^{t_f} dt \left\{ \Pi(\underline{x},t)\ \dot{\Phi}(\underline{x},t) - \mathcal{H}\left(\Pi(\underline{x},t), \Phi(\underline{x},t)\right) \right\}$$

where the <u>path integral measure</u> $D\Phi\, D\Pi$ is defined through (**) as integrating over all values of Φ and Π at all space points and all time values $t_o < t < t_f$.

There is a boundary condition which is that at

$t = t_0 \quad \Phi(\underline{x}, t_0) = \Phi_A(\underline{x})$

$t = t_f \quad \Phi(\underline{x}, t_f) = \Phi_B(\underline{x})$

2) Consider for example the Hamiltonian

$$H = \int d^3\underline{x} \; \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

ie a simple scalar field theory. Then the Π integral in (**) is

$$\prod_{i=1}^{N-1} \int d\left[\frac{\Pi_i(\underline{x})}{2\pi}\right] e^{i\sum_{i=0}^{N-1}\int d^3\underline{x} \; \delta t \left(\Pi_{i+1}(\underline{x})\dot{\Phi}_i(\underline{x}) - \frac{1}{2}\Pi_{i+1}^2(\underline{x})\right)}$$

This is just a huge number of copies of the integral

$$\int d\Pi \; e^{i\,\delta t\,(\Pi\dot{\Phi} - \frac{1}{2}\Pi^2)}$$

– one for each space point on **each** time slice.

$$= \int d\Pi \; e^{i\,\delta t\,(-\frac{1}{2}(\Pi - \dot{\Phi})^2 + \frac{1}{2}\dot{\Phi}^2)} = K \; e^{i\,\delta t\,\frac{1}{2}\dot{\Phi}^2} \quad (*)$$

so we end up with (letting $N \to \infty$ as before)

$$M_{fo} = \int \mathcal{D}\Phi \; \exp i\int d^3\underline{x} \int_{t_0}^{t_f} dt \left\{ \frac{1}{2}\dot{\Phi}^2(t,\underline{x}) - \frac{1}{2}(\nabla\Phi(t,\underline{x}))^2 - V(\Phi(t,\underline{x})) \right\}$$

$$= \int \mathcal{D}\Phi \; e^{i\,S[\Phi]} = \int \mathcal{D}\Phi \; e^{i\int d^4x \, \mathcal{L}(\partial_\mu\Phi, \Phi)}$$

where S is the action. Note that there is an undetermined constant from (*) absorbed into $\mathcal{D}\Phi$

This integral ~~is~~ has the limits that $\Phi = \Phi_A$ at t_0 and $\Phi = \Phi_B$ at t_f.

We could also make field measurements at intermediate times in which case the appropriate matrix element is

$\langle \Phi_B, t_f | \, T\left(\prod_{k=1}^{K} \phi(t_k)\right) | \Phi_A, t_0 \rangle$

Now when we insert the states we will have an extra operator $\phi(t_k)$ acting on $|\Phi_k, t_k\rangle$ which will just create a factor, so $\Phi_k |\Phi_k, t_k\rangle$. Thus the path integral becomes

$$\langle \Phi_B, t_f | \, T\left(\prod_{k=1}^{K} \phi(t_k)\right) | \Phi_A, t_0 \rangle$$

$$= \int D\Phi \left(\prod_{k=1}^{K} \Phi(t_k, \underline{x}_k)\right) e^{iS(\Phi)}$$

with the same boundary conditions as before, and we have restored the $\underline{x}$ argument for completeness.