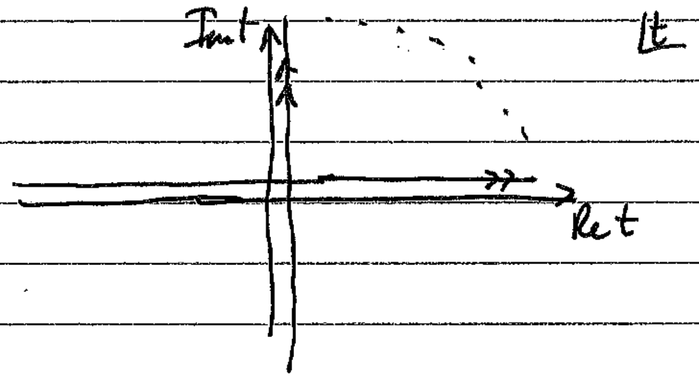
Lecture 16 Using the Path Integral

As it stands the Path Integral has a very rapidly oscillating integrand and it is not clear that the integral really exists. There are (at least) two ways of ameliorating this.

1) Replace $\mathcal{L} \to \mathcal{L} + i\epsilon\Phi^2$ ϵ real, small, +ve

This gives a real damping factor $e^{-\epsilon\int d^4x\,\Phi^2}$ which is sufficient to make the integral (at least for the free field) sensible.

2) Make a Wick rotation to imaginary time



so that $t = -i\tau$. This is essentially the conjugate of the $k_0 \to ik_{0E}$ that we made in the loop integral. Note that $k_0 t \to k_{0E}\tau$.

Now we have

$$\langle \Phi_B, \tau_f \mid \Phi_A, \tau_0 \rangle = \int D\Phi \, \exp \int d^3x \, d\tau \left(-(\partial_\tau \phi)^2 - (\nabla \phi)^2 - V(\phi) \right)$$

$$= \int D\Phi \, \exp - \int d^3x \, d\tau \left(+(\partial_\tau \phi)^2 + (\nabla \phi)^2 + V(\Phi) \right)$$

and the exponential is damped.

Both of these methods of making sense of the path integral have their uses; we will make free of both!

Let us go back to

$$\langle \Phi_B, t_f \mid \Phi_A, t_0 \rangle$$

we know that the evolution is governed by the Hamiltonian operator so this is

$$= \langle \Phi_B | \, e^{-i(t_f - t_0) H} \, | \Phi_A \rangle$$

Write $|\Phi_A\rangle$ in terms of <u>energy eigenstates</u>

$$|\Phi_A\rangle = \sum_n C_n^A \, |E_n\rangle$$

and similarly for $|\Phi_B\rangle$

then

$$\langle \Phi_B, t_f | \Phi_A, t_0 \rangle = \sum_{nm} C_m^{B*} C_n^A \langle E_m | e^{-i(t_f - t_0)H} | E_n \rangle$$
$$= \sum_m C_m^{B*} C_m^A \, e^{-i(t_f - t_0)E_n}$$

Now in the Euclidean picture $\tau = it$ we get

$$\langle \Phi_B, \tau_f | \Phi_A, \tau_0 \rangle = \sum_m C_m^{B*} C_m^A \, e^{-(\tau_f - \tau_0)E_n}$$

This is a remarkable expression because as $\tau_f - \tau_0 \to \infty$ only the state of lowest energy, the vacuum state, contributes. We can always arrange that $E_0 = 0$ by defining H appropriately so

$$\lim_{\substack{\tau_f \to \infty \\ \tau_0 \to -\infty}} \langle \Phi_B, \tau_f | \Phi_A, \tau_0 \rangle = C_0^{B*} C_0^A$$

Now consider a time-ordered product

$$\langle \Phi_B, \tau_f | T(\hat{\phi}(\tau_n) \ldots \hat{\phi}(\tau_1)) | \Phi_A, \tau_0 \rangle, \quad \tau_f \gg \tau_n > \ldots > \tau_1 \gg \tau_0$$
$$= \sum_{jk} \langle \Phi_B, \tau_n | e^{-(\tau_f - \tau_n)H} | E_j, \tau_n \rangle \langle E_j, \tau_n | T(\ldots) | E_k, \tau_1 \rangle$$
$$\times \langle E_k, \tau_1 | e^{-(\tau_1 - \tau_0)H} | \Phi_A, \tau_0 \rangle$$

Again, as $\tau_f - \tau_n \to \infty$ $\tau_1 - \tau_0 \to \infty$ only the vacuum state contributes.

$$\lim_{\substack{\tau_f \to \infty \\ \tau_0 \to -\infty}} \langle \Phi_B, \tau_f | T(\ldots) | \Phi_A, \tau_0 \rangle = C_0^{B*} C_0^A \langle \Omega | T(\ldots) | \Omega \rangle$$

That is to say

$$\langle \Omega | T(\ldots) | \Omega \rangle = \lim_{\substack{\tau_f \to \infty \\ \tau_0 \to -\infty}} \frac{\langle \Phi_B, \tau_f | T(\ldots) | \Phi_A, \tau_0 \rangle}{\langle \Phi_B, \tau_f | \Phi_A, \tau_0 \rangle}$$

and that this statement is independent of what we choose for $\Phi_{A,B}$ (provided we choose the same for numerator and denominator of course).

The r.h.s. can be written in terms of Path Integrals

$$\langle \Omega | T(\ldots) | \Omega \rangle = \frac{\int D\Phi \, \Phi(\tau_n) \ldots \Phi(\tau_1) \, e^{-S[\Phi]}}{\int D\Phi \, e^{-S[\Phi]}}$$

where the integrals extend over all three-space and from $\tau = -\infty$ to $\tau = +\infty$; any fixed field configuration at $\tau = \pm\infty$ will do.

The equivalent result in Minkowski space is

$$\langle \Omega | T \ldots | \Omega \rangle = \frac{\int D\Phi \, \Phi(t_n) \ldots \Phi(t_1) \, e^{i \int d^4x \, (\mathcal{L}(\Phi) + i\epsilon \Phi^2)}}{\int D\Phi \, e^{i \int d^4x \, (\mathcal{L}(\Phi) + i\epsilon \Phi^2)}}$$

The Path Integral in this form has many applications including approaches to calculating in QFT which do not use perturbation theory (they are called "non-perturbative" therefore). Here we will focus on perturbation theory. It's convenient to define a <u>functional derivative</u> which has the property (from now on we'll drop Φ and revert to ϕ which is a function...)

$$\frac{\delta}{\delta j(x)} F(j(y)) = \delta^4(x-y) \, F'(j(y))$$

↑ ordinary derivative of F

Then

$$\frac{\delta}{\delta j(x)} \exp\left(i\int d^4y \; j(y)\,\phi(y)\right)$$

$$= \frac{\delta}{\delta j(x)} \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\int d^4y \; j(y)\,\phi(y)\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{n}{n!}\left(i\int d^4y \; j(y)\,\phi(y)\right)^{n-1} i\,\frac{\delta}{\delta j(x)}\int d^4y' \, j(y')\,\phi(y')$$

$$= \exp\left(i\int d^4y \, j(y)\,\phi(y)\right) \, i\int d^4y' \, \delta^4(x-y')\,\phi(y')$$

$$= i\,\phi(x)\,\exp\left(i\int d^4y \, j(y)\,\phi(y)\right)$$

From which it follows that

$$\int D\phi \; \phi(x_n) .. \phi(x_1) \; e^{i\int d^4x \left(\mathcal{L}(\phi) + i\epsilon\phi^2 + j(x)\,\phi(x)\right)}$$

here we have emphasized that these are function of x, implied in the other terms

$$= \prod_{k=1}^{n}\left(-i\frac{\delta}{\delta j(x_k)}\right) \int D\phi \; e^{i\int d^4x \left(\mathcal{L}(\phi) + i\epsilon\phi^2 + j\phi\right)}$$

$$= \prod_{k=1}^{n}\left(-i\frac{\delta}{\delta j(x_k)}\right) \; Z[j]$$

so

$$\langle\Omega|T\ldots|\Omega\rangle = \left.\frac{\prod_{k=1}^{n}\left(-i\frac{\delta}{\delta j(x_k)}\right) \cancel{\log}\; Z(j)}{Z[0]}\right|_{j=0} \qquad (*)$$

where $Z[j] = \int D\phi \; e^{i\int d^4x\,(\mathcal{L}(\phi) + i\epsilon\phi^2 + j\phi)}$

is called the vacuum generating functional.

From this we can get the Feynman propagator

for a free field by setting $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

so

$$Z[j] = \int D\phi \; e^{i\int d^4x\,\frac{1}{2}\phi(-\Box - m^2 + i\epsilon)\phi + j\phi}$$

where we have integrated by parts and assumed suitable behaviour at ∞ as usual. Define $Q = -\Box - m^2 + i\epsilon$

$$Z[j] = \int D\phi \; e^{i\int d^4x\left(\frac{1}{2}(\phi + jQ^{-1})Q(\phi + Q^{-1}j) - \frac{1}{2}j(Q^{-1}j)\right)}$$

$$= \exp\left(-\frac{i}{2}\int d^4x\; j\,Q^{-1}j\right) \; Z[0]$$

where $(Q^{-1}j)(x) = -i\int d^4y\; D_F(x-y)\,j(y)$

because we defined $(-\partial^2 - m^2 + i\epsilon)\,D_F(x-y) = i\,\delta^4(x-y)$

So $Z[j] = \exp\left(-\frac{1}{2}\int d^4x \int d^4y \; j(x)\, D_F(x-y)\, j(y)\right) Z[0]$

inserting this back into (*) on 16.2 gives

$\langle \Omega | T\, \phi(x)\, \phi(y) | 0 \rangle = D_F(x-y)$

so we see that the $i\epsilon$ prescription in the path integral singles out the Feynman propagator.

Moving on to an interacting field we have, for a ϕ^4 interaction,

$$Z_\lambda[j] = \int D\phi \; e^{i\int d^4x \left(\frac{1}{2}\phi(-\Box - m^2 + i\epsilon)\phi - \frac{\lambda\phi^4}{4!} + j\phi\right)}$$

$$= \int D\phi \; e^{-i\int d^4x\, \lambda\phi^4/4!} \; e^{i\int d^4y \left(\frac{1}{2}\phi(-\Box - m^2 + i\epsilon)\phi + j\phi\right)}$$

$$= \int D\phi \; e^{-i\int d^4x \frac{\lambda}{4!}\left(\frac{i\delta}{\delta j(x)}\right)^4} \; e^{i\int d^4y \left(\frac{1}{2}\phi(-\Box - m^2 + i\epsilon)\phi + j\phi\right)}$$

$$= e^{-i\int d^4x \frac{\lambda}{4!}\left(\frac{i\delta}{\delta j(x)}\right)^4} \; e^{-\frac{1}{2}\int d^4y \int d^4z \; j(y)\, D_F(y-z)\, j(z)} \; Z[0]$$

and from this expression we can immediately read off the original position space Feynman Rules for Green's Functions. Each pair of derivatives

$\left(-i\frac{\delta}{\delta j(x)}\right)\left(-i\frac{\delta}{\delta j(y)}\right)$ pulls down a $D_F(x-y)$

→ If you are uncomfortable with this consider expanding the exponential of the interaction term. We get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \int d^4x \, \frac{\lambda}{4!} \left(-i \frac{\delta}{\delta j(x)} \right)^4 \right)^n \ldots e^{i \int d^4y \, j(y) \phi(y)}$$

now use the result on page 16.1. Each $-i \frac{\delta}{\delta j(x)}$ pulls down a $\phi(x)$ so we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \int d^4x \, \frac{\lambda}{4!} \, \phi(x)^4 \right)^n \ldots e^{i \int d^4y \, j(y) \phi(y)}$$

and resumming gives the claimed result.

and each vertex x gets a factor $-i\lambda\int d^4x$ and has four legs.
The combinatoric factor comes from the number of ways you can do the derivatives multiplying $\frac{1}{4!}$ for each vertex, $\frac{1}{n!}$ for n vertices and $\frac{1}{l!}$ for l lines.

$Z_\lambda[0]$ just generates the vacuum bubble graphs and you can show these cancel out in $\langle\Omega|T\ldots|\Omega\rangle$ by exactly the same analysis as used in one of the questions in Problem Set 3.

Of course these Feynman Rules also lead to the momentum space rules by replacing

$$D_F(x-y) \rightarrow \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

and integrating $\int d^4z$ at every vertex.

<u>What's Quantum + What's Classical?</u>

A QFT is an extension of a classical QFT to the case when $\hbar \neq 0$. It ought to contain the classical QFT somehow. For a real world example think about Quantum Electrodynamics. This should contain classical electromagnetism. How does this work? The problem is that by setting $\hbar = 1$ many weeks ago we have obscured the issue. The Path Integral makes reinstating $\hbar$ easy because the action has dimensions of $\hbar$ and really the PI is

$$Z[j] = \int D\phi \; e^{\frac{i}{\hbar}\int d^4x \; \mathcal{L}(\phi) + j\phi}$$

We see that every λ gets a $1/\hbar$ factor and every propagator is the inverse of $\frac{-\Box - m^2 + i\epsilon}{\hbar}$ so gets an $\hbar$ factor. Examining amputated matrix elements we see that a Feynman diagram with I internal lines and V vertices acquires a factor $\hbar^{I-V} = \hbar^{L-1}$ hence in general

$$\mathcal{M} = \frac{1}{\hbar}\Big(\underbrace{M_0}_{\text{no loops}} + \hbar \underbrace{M_1}_{\text{1 loop}} + \hbar^2 \underbrace{M_2}_{\text{2 loops}} + \dots \Big)$$

Now the cross section in $\hbar = c = 1$ units for $2 \to 2$ scalar scattering is

$$\sigma = \text{constants} \; \frac{1}{s} \; |M|^2 \qquad \left(\begin{array}{c} \text{where } |M| = \lambda \\ \text{at tree graph} \end{array} \right)$$

To convert to $c = 1$ units we must multiply by $\hbar^2$

because $\left[\frac{\hbar^2}{s}\right] = \left[\frac{\hbar^2}{E^2}\right] = \left[\frac{\hbar^2}{p^2}\right] = [x^2]$

which is correct. So we have

$$\sigma = \text{constants} \; \frac{\hbar^2}{s} \cdot \frac{1}{\hbar^2} \; | M_0 + \hbar M_1 + .. |^2$$

$$= \text{constants} \; \frac{1}{s} \left(|M_0|^2 + \hbar \, 2 \, \mathrm{Re} \, M_0^* M_1 + \dots \right).$$

We see that the leading, tree graph, contribution does not depend on $\hbar$ — it is the classical cross-section. The first quantum correction comes at 1 loop, the second at 2 loops and so on. This explains why ~~that~~ we organise the renormalization program by number of loops - we are doing successively higher order calculations in $\hbar$.