INTRODUCTION TO QFT 2019: PROBLEM SET 5

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Here are some exercises using the Path Integral. Worked answers will be placed on the web-site; but it is recommended that you have a go at the questions before looking at the answers! The starred parts are a bit harder.

1. *Diagrams and Feynman Rules* Use the Path Integral vacuum generating functional $Z[j]$ to to generate expressions for the diagrams in Q5a of Problem Set 3. Make sure that the combinatorial factors come out as expected. You might decide to do this by hand – fine, but a better idea would be to write a simple Mathematica program.

2. *Generating Functionals I* Assume that we are working with the ϕ^4 scalar field theory.

   a) The generating functional $W[j]$ is defined by $W[j] = -i \log Z[j]$. Show that, up to an overall factor,
$$\left(\prod_{k=1}^{4}\left(-i\frac{\delta}{\delta j(x_k)}\right)\right) W[j]\Big|_{j=0} \tag{1}$$
   generates the connected four point function.

   b) * Show that
$$\left(\prod_{k=1}^{n}\left(-i\frac{\delta}{\delta j(x_k)}\right)\right) W[j]\Big|_{j=0} \tag{2}$$
   generates the connected n-point function. Use the method of induction ie assuming the result is true for n show that it is true for $n+2$ (as n must be even in this case).

3. *Generating Functionals II* For a scalar field theory:

   a) Let
$$\phi(x) = \frac{\delta W}{\delta j(x_k)} \tag{3}$$
   without setting j to zero. In the free field case solve this equation for $j(x)$ in terms of $\phi(x)$. The *effective action* $\Gamma[\phi]$ is given by
$$\Gamma[\phi] = W[j] - \int d^4x j(x)\phi(x) \tag{4}$$
   Find $\Gamma[\phi]$.

   b) * For an interacting field $\Gamma[\phi]$ is the generating function for 1PI n-point functions ie
$$\left(\prod_{k=1}^{n}\left(-i\frac{\delta}{\delta \phi(x_k)}\right)\right) \Gamma[\phi]\Big|_{\phi=0} \tag{5}$$
   is the 1PI n-point function. Consider a scalar field theory with $\mathcal{L}_{int} = \eta\phi^3 + \lambda\phi^4$ and show that 1PI functions are generated for $n = 3, 4$. You have to do this by

using the relationships (3), (4) and results for differentiating implicit functions – clearly you cannot actually compute $\Gamma[\phi]$ exactly.