INTRODUCTION TO QFT 2019: PROBLEM SET 5 SOLUTIONS TO NON-STARRED QUESTIONS

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1. *Diagrams and Feynman Rules* As the question hints this is best done by Mathematica. On the web-page is a simple Mathematica Notebook that does the job. The basic idea is that for a finite graph we don't really need functional differentiation. If we're just differentiating w.r.t a fixed number, N, of j's we label them $\{j_n, n = 1, \dots N\}$ and represent

$$\int d^4x_1 \int d^4x_2\, j(x_1) D_F(x_1 - x_2) j(x_2) \tag{1}$$

formally by

$$\sum_{k,n} j_k D_{F\,k,n}\, j_n \tag{2}$$

so the propagator is just a matrix *and* we can do ordinary partial differentiations which is built into Mathematica. This is good enough to generate a representation of the graphs in terms of propagators; at the end we simply replace $D_{F\,k,n}$ by the real expression $D_F(x_k - x_n)$ and integrate d^4x_n over internal vertices.

2. *Generating Functionals I*

a) Adopt the short-hand

$$\frac{\delta}{\delta j_k} = -i \frac{\delta}{\delta j(x_k)} \tag{3}$$

then

$$\begin{aligned}
&i \left(\prod_{k=1}^{4} \left(-i \frac{\delta}{\delta j(x_k)} \right) \right) W[j] \qquad (4)\\
&= \left(\prod_{k=1}^{3} \left(\frac{\delta}{\delta j_k} \right) \right) \frac{1}{Z[j]} \frac{\delta Z[j]}{\delta j_4}\\
&= \left(\prod_{k=1}^{2} \left(\frac{\delta}{\delta j_k} \right) \right) \left(\frac{1}{Z[j]} \frac{\delta^2 Z[j]}{\delta j_3 \delta j_4} - \frac{1}{Z[j]^2} \frac{\delta Z[j]}{\delta j_3} \frac{\delta Z[j]}{\delta j_4} \right)\\
&= \frac{\delta}{\delta j_1} \left(\frac{1}{Z[j]} \frac{\delta^3 Z[j]}{\delta j_2 \delta j_3 \delta j_4} - \frac{1}{Z[j]^2} \left(\frac{\delta^2 Z[j]}{\delta j_2 \delta j_3} \frac{\delta Z[j]}{\delta j_4} + \text{distinct perms} \right) \right.\\
&\qquad \left. + \frac{2}{Z[j]^3} \frac{\delta Z[j]}{\delta j_2} \frac{\delta Z[j]}{\delta j_3} \frac{\delta Z[j]}{\delta j_4} \right)\\
&= \frac{1}{Z[j]} \frac{\delta^4 Z[j]}{\delta j_1 \delta j_2 \delta j_3 \delta j_4} - \frac{1}{Z[j]^2} \left(\frac{\delta^2 Z[j]}{\delta j_2 \delta j_3} \frac{\delta^2 Z[j]}{\delta j_4 \delta j_1} + \frac{\delta^3 Z[j]}{\delta j_1 \delta j_2 \delta j_3} \frac{\delta Z[j]}{\delta j_4} + \text{distinct perms} \right)\\
&\qquad + \frac{2}{Z[j]^3} \left(\frac{\delta^2 Z[j]}{\delta j_1 \delta j_2} \frac{\delta Z[j]}{\delta j_3} \frac{\delta Z[j]}{\delta j_4} + \text{distinct perms} \right) - \frac{6}{Z[j]^4} \frac{\delta Z[j]}{\delta j_1} \frac{\delta Z[j]}{\delta j_2} \frac{\delta Z[j]}{\delta j_3} \frac{\delta Z[j]}{\delta j_4}
\end{aligned}$$

Now set $j = 0$; all odd derivatives of $Z[j]$ vanish because it is an even function of j. We are left with

$$\frac{1}{Z[j]}\frac{\delta^4 Z[j]}{\delta j_1 \delta j_2 \delta j_3 \delta j_4}\Big|_{j=0} - \left(\frac{1}{Z[j]}\frac{\delta^2 Z[j]}{\delta j_2 \delta j_3}\Big|_{j=0}\frac{1}{Z[j]}\frac{\delta^2 Z[j]}{\delta j_4 \delta j_1}\Big|_{j=0} + \text{distinct perms}\right) \tag{5}$$

which is precisely the full 4pt-Green's function with the components that are products of two 2pt-functions subtracted off.

3. *Generating Functionals II*

a) We have from the lectures

$$Z[j] = \exp\left(-\frac{1}{2}\int d^4x_1 \int d^4x_2\, j(x_1) D_F(x_1 - x_2) j(x_2)\right) \tag{6}$$

so

$$W(j) = \frac{i}{2}\int d^4x_1 \int d^4x_2\, j(x_1) D_F(x_1 - x_2) j(x_2) \tag{7}$$

and

$$\phi(x) = \frac{\delta W}{\delta j(x_k)} = i\int d^4x_1 D_F(x - x_1) j(x_1) \tag{8}$$

Now act on both sides with $Q = (-\partial_\mu \partial^\mu - m^2 + i\epsilon)$ to get (remember D_F is the Green's function of Q)

$$Q_x \phi(x) = i\int d^4x_1\, i\delta^4(x - x_1) j(x_1) = -j(x) \tag{9}$$

So

$$\begin{aligned}\Gamma[\phi] &= \frac{i}{2}\int d^4x_1 \int d^4x_2\, (Q_{x_1}\phi(x_1)) D_F(x_1 - x_2)(Q_{x_2}\phi(x_2)) \\ &\quad + \int d^4x\, \phi(x) Q_x \phi(x)\end{aligned} \tag{10}$$

Integrating the derivatives in Q_{x_1} by parts twice gives

$$\begin{aligned}\Gamma[\phi] &= \frac{i}{2}\int d^4x_1 \int d^4x_2\, \phi(x_1)(Q_{x_1} D_F(x_1 - x_2))(Q_{x_2}\phi(x_2)) \\ &\quad + \int d^4x\, \phi(x) Q_x \phi(x) \\ &= \frac{1}{2}\int d^4x\, \phi(x)(-\partial_\mu \partial^\mu - m^2 + i\epsilon)\phi(x)\end{aligned} \tag{11}$$

which of course is just the action for a free field.