Lecture notes on

Advanced Supersymmetry

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Contents

Introduction

In this set of lectures, we will build on the formalism developed in the *Supersymmetry and Supergravity* lectures. There, we learned how to write down classical field-theory Lagrangians with supersymmetry, and we discussed important aspects of the quantum theory in the simple case of theories with chiral multiplets only.

In the present lectures, we will explore aspects of the quantum dynamics of *supersymmetric gauge theories in 4d.* These are close cousins to the gauge theories that we use in Particle Physics. Supersymmetry leads to more technical countrol, leading to many exact results that we could only dream off in 'real-world' QFT without supersymmetry. It is thus of great theoretical interest to study these models in details, for instance as 'toy models' for the much harder problem of understanding, say, QCD at strong coupling from first principles.

These short lectures will have two parts:

In part I, we will study the dynamics supersymmetric gauge theories with $\mathcal{N} = 1$ supersymmetry, The main question is simple to state: given an asymptotically-free gauge theory, we have a well-defined, perturbative, weakly-coupled QFT in the ultraviolet (UV), while the gauge coupling becomes large towards the infrared (IR). Can we say anything useful about the IR physics? We will focus our attention on a supersymmetric version of QCD, called $\mathcal{N} = 1$ SQCD.

In part II, we will consider gauge theories with $\mathcal{N} = 2$ supersymmetry. The extra supercharges give us additional analytic control, and we will see that, in some sense, one can entirely 'solve' for the IR physics in terms of the UV data. This is the celebrated Seiberg-Witten solution of $\mathcal{N} = 2$ gauge theories. We will only be able to cover some basics, but it will hopefully be enough to display the great beauty of the subject, and to give you the motivation to go and learn more about it on your own.

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References and further reading:

- Again, the lecture notes by Argyres: `http://homepages.uc.edu/~argyrepc/cu661-gr-SUSY/index.html`.
- The classic lectures by Intriligator and Seiberg [1] on 4d $\mathcal{N} = 1$ dynamics.
- The lectures by Tachikawa on $\mathcal{N} = 2$ dynamics [2], which also covers some more modern aspects.
- The original papers by Seiberg [3, 4] ($\mathcal{N} = 1$) and Seiberg and Witten [5] ($\mathcal{N} = 2$). (There hasn't been quite anything like the year 1994 for the study of supersymmetric QFT...)

Part I
Dynamics of 4d $\mathcal{N} = 1$ gauge theories

1 Quantum aspects of 4d gauge theories

In this section, we discuss some general aspects of (gauge) anomalies in QFT, as well as the role of the θ angle in gauge theories. (This is by no means self-contained, but should amply suffice for our purpose in these lectures.)

1.1 Anomalies for gauge and global symmetries

Let us first make some general comments about *quantum anomalies* in gauge and global symmetries.

1.1.1 Global symmetry, background gauge fields and *gauging*

Consider some fermions ψ and bosons ϕ that transform into some representation $\mathfrak{R}_\psi$ and $\mathfrak{R}_\phi$, respectively, of some *global symmetry group* $\widetilde{G}$. Then, by Noether theorem, we have some conserved currents j_a^μ, with the index $a = 1, \cdots, \dim\widetilde{G}$ running over the generator of the Lie algebra $\widetilde{\mathfrak{g}}$.

Whenever we have such conserved currents, we can introduce some *sources* for them:

$$\mathscr{L}[A] = \mathscr{L}_0 + A_\mu^a j_a^\mu + \cdots \ . \tag{1.1}$$

Here, $\mathscr{L}_0$ is the Lagrangian of the theory with a global symmetry, A_μ^a is the source for the current operator $j_a^\mu(x)$, and the ellipsis denotes higher-order terms in the sources. The source is nothing but a *background gauge field*—that is, a non-dynamical vector field A_μ, which is introduced to keep track of the conserved current. The path integral in terms of the sources takes the form:

$$Z[A] = \int [D\phi][D\psi] \exp\left(i \int d^4x \mathscr{L}[A]\right) \ , \tag{1.2}$$

for any fixed background A_μ.

Gauging a global symmetry. Given a global symmetry $\widetilde{G}$, a natural operation in QFT is to *gauge it.* In path-integral language, gauging a global symmetry means that we first introduce background gauge fields as in (1.2), and then integrate over all possible gauge fields:

$$\begin{aligned} Z &= \int [DA]\, e^{iS_{\rm YM}[A]+iS_{\rm top}[A]}\, Z[A] \\ &= \int [DA][D\phi][D\psi]\ \exp\left(i\int d^4x\, (\mathscr{L}[A] + \mathscr{L}_{\rm YM}[A] + \mathscr{L}_{\rm top}[A])\right) \ . \end{aligned} \tag{1.3}$$

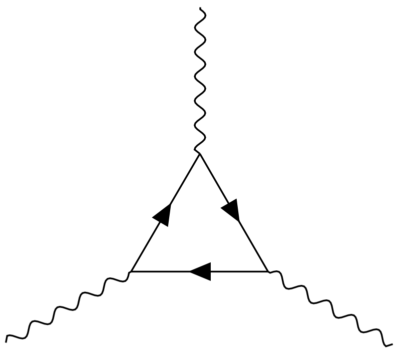

Figure 1: The triangle one-loop diagram that determines the gauge anomaly in four dimensions, with fermions running in the loop. The wiggly lines correspond to the external gauge fields; equivalently, one inserts a current j_a^μ at each vertex.

Here, we weighted each gauge-field configuration by the Yang-Mills action (the standard kinetic) term, as well as by the topological term, which we will discuss further in the next subsection:

$$\mathscr{L}_{\rm YM}[A] = -\frac{1}{4g}\,{\rm tr}\; F_{\mu\nu}F^{\mu\nu} \;, \qquad\qquad \mathscr{L}_{\rm top}[A] = -\frac{\theta}{64\pi^2}\,{\rm tr}\,(\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}) \;. \tag{1.4}$$

1.1.2 Anomaly as an obstruction to gauging a global symmetry

The gauging of a global symmetry can only be done *consistently* if the functional (1.2) is itself gauge invariant—infinitesimally:

$$\delta_\alpha \log Z[A] = 0 \;, \tag{1.5}$$

for $\delta_\alpha A_\mu = D_\mu \alpha$. The symmetry $\widetilde{G}$ has an *anomaly* if and only if:

$$\delta_\alpha \log Z[A] \neq 0 \;. \tag{1.6}$$

This can be taken as our *definition* of a gauge anomaly. In the presence of an anomaly for $\widetilde{G}$ in the original theory $\mathscr{L}_0$, the global symmetry cannot be gauged, and the corresponding gauge theory (1.3) does not exist as a consistent QFT.

While the classical Lagrangian $\mathscr{L}[A]$ itself is gauge invariant by construction, the *path-integral measure* in (1.2) might not always be—this is precisely the origin of those "quantum anomalies." In renormalisable gauge theories with scalar and fermion matter fields only, anomalies arise exclusively from *chiral fermions* in *complex representations* of the gauge group (*i.e.* $\mathfrak{R} \neq \bar{\mathfrak{R}}$).

In some appropriate regularisation scheme, the anomaly takes the explicit form:

$$\boxed{\delta_\alpha \log Z[A] = -\frac{i\mathcal{A}_{abc}}{64\pi^2}\int d^4x\, \alpha^a F^b_{\mu\nu} F^c_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \;,} \tag{1.7}$$

schematically, with $\alpha(x) = \alpha^a(x) T_a^{(\mathfrak{R}_\psi)}$ the gauge-transformation functions acting on the fermions; the sum over the $\widetilde{\mathfrak{g}}$-indices a, b, c is understood. The anomaly coefficients $\mathcal{A}_{abc}$ are given by:

$$\mathcal{A}_{abc} = \mathrm{tr}\left(T_a^{(\mathfrak{R}_\psi)}\{T_b^{(\mathfrak{R}_\psi)}, T_c^{(\mathfrak{R}_\psi)}\}\right) . \tag{1.8}$$

These are also called *cubic anomalies*, since they are cubic in the "charges" under $\widetilde{G}$. Physically, they can be extracted from the three-point function of the currents j_a^μ in the original theory with Lagrangian $\mathscr{L}_0$:

$$\left\langle j_a^\mu(p) j_b^\mu(q) j_c^\mu(-p-q)\right\rangle \sim \mathcal{A}_{abc} . \tag{1.9}$$

The anomaly coefficient (1.8) is entirely determined by the one-loop contribution to these observables, which are triangle diagrams of the type depicted in Figure 1.

We should also mention the existence of some more subtle quantum anomalies in 4d, the *linear anomalies*:

$$\mathcal{A}_a = \mathrm{tr}(T_a^{(\mathfrak{R}_\psi)}) . \tag{1.10}$$

They are "mixed anomalies" between gauge invariance and diffeomorphism invariance. They are often called "gravitational anomalies," because they contribute to the right-hand-side of (1.7) in the presence of a non-trivial metric $g_{\mu\nu}$ (with non-zero curvature)—they arise at one-loop from the same triangle diagram as in Fig 1, but with one gauge field and two gravitons for the external legs. We see from (1.10) that linear anomalies can only be non-zero for abelian symmetries, the $U(1)$ factors inside $\widetilde{G}$.

1.1.3 Three types of anomalies

In general, the symmetry group $\widetilde{G}$ of the theory $\mathscr{L}_0$ might be a product of many simple groups and $U(1)$ factors. Let us consider:

$$\widetilde{G} = G_F \times G . \tag{1.11}$$

Here, we would like to *gauge* the factor $G \subset \widetilde{G}$, while G_F will remain as a global symmetry (often called a "flavor symmetry") of the gauge theory with gauge group G. The cubic anomalies (1.8) can then take the schematic form:

$$\mathrm{Tr}(GGG) , \qquad \mathrm{Tr}(GGG_F) , \qquad \mathrm{Tr}(GG_FG_F) , \qquad \mathrm{Tr}(G_FG_FG_F) , \tag{1.12}$$

while we have $\mathrm{Tr}(G)$ and $\mathrm{Tr}(G_F)$ for the linear anomalies. There are thus, in fact, three types of anomalies that concern us when G is gauged, from the lethal to the innocuous:

(i) Gauge anomalies. A gauge anomaly is an anomaly of the form:

$$\mathrm{Tr}(GGG) \ , \tag{1.13}$$

for three currents of the gauge group G. If it were non-zero, the theory would be inconsistent quantum-mechanically. Therefore, we have the **anomaly-free condition:**

$$\mathcal{A}_{abc} = 0 \ , \tag{1.14}$$

where a, b, c runs over the generators of G only. For a simple Lie algebra $\mathfrak{g}$, we have:

$$\mathcal{A}_{abc} = \frac{1}{2} A\big(\mathfrak{R}^{(\mathfrak{R}_\psi}\big)\, d_{abc} = 0 \ , \tag{1.15}$$

which is a non-trivial condition only for $SU(N)$ with $N > 2$. (See below equation (7.28) in the Susy&Sugra lectures.) Note also that, if the gauge group is a product:

$$G = \prod_i G_i \times \prod_j U(1)_k \ ,$$

with G_i some simple factors (as in the Standard Model, for instance), we also have non-trivial constraints:

$$\mathrm{tr}(G_i G_i U(1)_k) = 0 \ , \qquad \mathrm{tr}(U(1)_k U(1)_l U(1)_m) = 0 \ , \tag{1.16}$$

which translate to:

$$\sum_\psi q_k[\psi] T(\mathfrak{R}_i^\psi) = 0 \ , \qquad \sum_\psi q_k[\psi] q_l[\psi] q_m[\psi] = 0 \ , \tag{1.17}$$

where $T(\mathfrak{R}_i^\psi)$ is the quadratic index of the representation $\mathfrak{R}_i^\psi$ of G_i under which ψ transforms, and $q_k[\psi]$ denotes the $U(1)_k$ charges of the fermions.

(ii) Anomalous global symmetries. We could also have so-called "mixed anomalies:"

$$\mathrm{Tr}(GGG_F) \ , \qquad \mathrm{Tr}(GG_F G_F) \ , \tag{1.18}$$

between the gauge-symmetry and the global-symmetry currents. If G is semi-simple, only $\mathrm{Tr}(GGG_F)$ can be non-trivial. Let $a, b, \cdots$ run over the generators T_a of the gauge group G, and let $\alpha, \beta, \cdots$ run over the generators T_α of G_F. Then, any anomaly:

$$\mathcal{A}_{ab\alpha} \neq 0 \ , \qquad \text{or} \qquad \mathcal{A}_{a\alpha\beta} \neq 0 \ , \tag{1.19}$$

signals that the currents j^μ_α, or j^μ_α and j^μ_β, part of the naive symmetry G_F, are actually not conserved in the gauge theory. This is an *anomalous global symmetry,* which is then *not* a symmetry of the quantum system.

We will mostly focus on $G = SU(N)$ and $G_F \supset U(1)_A$, with a particular flavor symmetry $U(1)_A$ that can be anomalous. This non-conservation of $U(1)_A$ is generally called a *chiral anomaly.*

(iii) ’t Hooft anomalies. Finally, consider an anomaly-free theory with gauge group G and a *non-anomalous* global symmetry group G_F (with generators T_α). In general, the anomalies involving only the global symmetry currents can be (and generally are) non-zero:

$$\mathcal{A}_{\alpha\beta\gamma} \neq 0 \; . \tag{1.20}$$

These are called *’t Hooft anomalies.* They are an *obstruction to gauging* the symmetry G_F—per our general definition of anomalies—, but they are otherwise innocuous. In the absence of G_F background gauge fields, the currents j^μ_α are still conserved. There is a mild modification of the Ward identities for G_F that follows from the anomalous variations (1.7), but does not change their essential meaning—the symmetry G_F still implies all the usual selection rules, in particular.

1.1.4 The ’t Hooft anomaly matching condition

While ’t Hooft anomalies are innocuous, they carry some interesting information about the QFT. The reason is that they are *invariant under RG flow,* in the following sense. Let us consider some UV theory with some global symmetry G_F and a set of ’t Hooft anomalies $\mathcal{A}_{G_F}$ for that global symmetry. Then, consider an RG flow starting from the UV theory, preserving the symmetry G_F, that flows to some effective field theory in the infrared. That IR effective theory could look very different from the UV description, since the RG flow does not need to be perturbative. Nonetheless, we claim that the ’t Hooft anomalies in the UV and the IR must match [6]:

$$\boxed{\mathcal{A}_{G_F}(\mathcal{T}_{UV}) = \mathcal{A}_{G_F}(\mathcal{T}_{IR}) \; .} \tag{1.21}$$

This is because we can always add some free fermions in the UV theory, $\mathcal{T}_{UV}$ to saturate the anomalies—that is, we can add some free fermions that transform in some representation of G_F exactly so that the anomaly vanishes in the enlarged theory $\mathcal{T}_{UV} \otimes \mathcal{T}_{\rm free}$. We can then consistently *gauge* the enlarged theory $\mathcal{T}_{UV} \otimes \mathcal{T}_{\rm free}$ with some arbitrarily weakly coupled gauge field. Upon RG flow the free-free fermion sectors then behaves as a spectator, and therefore we obtain another consistent theory $\mathcal{T}_{IR} \otimes \mathcal{T}_{\rm free}$ very weakly coupled to G_F-gauge fields in the IR. It must now be true that the G_F anomaly associated to $\mathcal{T}_{IR}$ exactly cancels the one associated to $\mathcal{T}_{\rm free}$. Therefore, (1.21) must hold.

This *’t Hooft anomaly matching condition* provides a strong constraint on RG flows, even at strong coupling, as we will see in some supersymmetric examples.

1.2 Instantons, θ angle and chiral anomalies

The action of a Yang-Mills theory contains a so-called topological term:

$$S_{\rm top} = -\frac{\theta}{64\pi^2} \int d^4x \, {\rm tr} \left(\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right) \; , \tag{1.22}$$

It is also known as the instanton density, or Pontryagin density.

1.2.1 Instantons and θ angle: executive summary

The meaning of this topological term is best described in Euclidean signature. Let us also replace space-time by a general Riemannian four-manifold $\mathcal{M}_4$:

$$S_{\rm top} = -\frac{\theta}{64\pi^2} i \int_{\mathcal{M}_4} d^4x\sqrt{g}\, {\rm tr}\,(\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}) = -\frac{\theta}{16\pi^2} i \int_{\mathcal{M}_4} {\rm tr}\ F\wedge F\ , \tag{1.23}$$

where we used the form notation, with $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ in local coordinates. It is clear from the right-hand-side of (1.23) that $S_{\rm top}$ is independent of the metric on $\mathcal{M}_4$. In fact, the integrand is also a total derivative:

$$\int_{\mathcal{M}_4} {\rm tr}\ F\wedge F = \int_{\mathcal{M}_4} d\ {\rm tr}\left(A\wedge dA - \frac{2i}{3} A\wedge A\wedge A\right)\ , \tag{1.24}$$

and thus we naively expect the action to vanish.

However, the topological action can be non-trivial in the presence of a non-trivial gauge-field configuration, which are called *instantons.* Indeed, consider $\mathcal{M}_4 = \mathbb{R}^4$. A physically sensible gauge field does not need to vanish at infinity; instead, it should only be *pure gauge*:

$$\lim_{|x|\to\infty} A_\mu = i g(x)\partial_\mu g(x)^{-1}\ , \tag{1.25}$$

with $g(x)$ an gauge transformation on the three-sphere, S^3_∞, at infinity:

$$g(x)\Big|_{S^3_\infty}\ : S^3 \to G\ . \tag{1.26}$$

This map can have non-trivial *winding number* $k\in\mathbb{Z}$, corresponding to an element of the homotopy group $\pi_3(G)$. One can show that the topological action precisely computes the winding number:

$$\int_{\mathcal{M}_4} {\rm tr}\ F\wedge F = \int_{S^3_\infty} {\rm tr}\left(A\wedge A - \frac{2i}{3} A\wedge A\wedge A\right) = 16\pi^2 k\ . \tag{1.27}$$

Mathematically, the quantity $\int_{\mathcal{M}_4} F\wedge F$ is a so-called characteristic class, known as the *Pontraygin class*, which captures some of the non-trivial *topology* of a non-trivialisable G-bundle over a four-manifold $\mathcal{M}_4$—a non-trivial gauge-field configuration on $\mathcal{M}_4$. It is a non-trivial mathematical fact that this topological invariant is *always* an *integer*:

$$\frac{1}{16\pi^2}\int_{\mathcal{M}_4} {\rm tr}\ F\wedge F = k\in\mathbb{Z}\ . \tag{1.28}$$

Let us insist on this amazing fact: while the action $S[\varphi]$ in general depends on the detailed value of the fields $\varphi(x)$, the topological term (1.28) gives an integer for *any* gauge field $A_\mu(x)$. Thus, the quantity $e^{iS_{\rm top}} = e^{i\theta k}$ factors out of the path integral for each "topological sector" at fixed k.

Given a gauge field A_μ with field-strength $F_{\mu\nu}$, define the *dual (or "magnetic") field-strength* as:

$$\widetilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \ . \qquad (1.29)$$

Definition: An (anti)-instanton is a gauge field configuration on $\mathcal{M}_4$ which is (anti)-self dual:

$$\boxed{F_{\mu\nu} = \pm \widetilde{F}_{\mu\nu} \ ,} \qquad (1.30)$$

with the + and − signs for the instanton and anti-instanton, respectively. In QFT, we consider, in particular, $\mathcal{M}^4 = \mathbb{R}^4$, which can be compactified to a sphere S^4.

One physical significance of instantons is that they are the non-trivial *classical saddles* of the Yang-Mills action. Indeed, we have:

$$\frac{1}{2}\int |F + \widetilde{F}|^2 = \int |F|^2 \pm \int F \wedge F \geq 0 \ , \qquad (1.31)$$

schematically, which implies that the YM action in always larger or equal to the topological action, the sense that:

$$\int |F|^2 \geq \left| \int F \wedge F \right| = 16\pi^2 |k| \ , \qquad (1.32)$$

with the inequality saturated for the (anti)-instanton configuration. We have $k > 0$ for an instanton and $k < 0$ for an anti-instanton. On any (anti)-instanton background, we have:

$$\frac{1}{4g^2}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu} = \pm\frac{1}{8g^2}\int d^4x\sqrt{g}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{8\pi^2}{g^2}|k| \ . \qquad (1.33)$$

Thus, a k-instanton gauge field, $A_\mu^{(k)}$, is weighted by a numerical factor:

$$e^{-S_{\rm YM}[A^{(k)}]} = e^{-\frac{8\pi^2|k|}{g^2}} \ , \qquad (1.34)$$

in the path integral. Note the similarlity to the (one-loop) strong-coupling scale:

$$\Lambda = \mu e^{-\frac{8\pi^2}{b_0 g^2(\mu)}} \ , \qquad (1.35)$$

which we mentioned in previous lectures. Each *topological sector*, for each $k \in \mathbb{Z}$—that is, the set of all gauge fields with a non-zero instanton number (1.28)—gives a *non-perturbative* contribution to the path integral. For g^2 very small and $k \neq 0$, the instanton factor (1.34) is *extremely* small, and can be neglected for most purposes (hence the name, non-perturbative [1]). However, as g^2 grows under RG flow (in

[1] Note also that the function:

$$f(g) = e^{-\frac{1}{g^2}}$$

is non-differentiable at $g = 0$. The apparence of such a factor is always the hallmark of a non-perturbative correction.

the IR, for an asymptotically-free gauge theory), the instantons will start giving increasingly important contributions to the dynamics. At strong coupling, we would even expect them to dominate.

While the trivial saddle $A_\mu = 0$ (corresponding to $k = 0$) is the starting point of ordinary perturbation theory, one can (and should) do a similar perturbative expansion around all saddles. The computation of any given observable will *a priori* receive contributions from all saddles (plus perturbative fluctuations). For a simple, simply-connected gauge group, such as $SU(N)$ (for simplicity), we have:

$$S_{\text{top}} = -i\theta k \ , \tag{1.36}$$

in Euclidean signature, in the presence of a k-instanton. Then, the Yang-Mills path integral takes the schematic form:

$$Z = \int [DA_\mu]\, e^{iS_{\text{YM}}+iS_{\text{top}}} = \sum_{k\in\mathbb{Z}} e^{i\theta k} e^{-\frac{8\pi^2|k|}{g^2}} Z_k \ , \tag{1.37}$$

where Z_k is the perturbative contribution of each *topological sector*. Note that, since a shift $\theta \to \theta + 2\pi$ does not modify the exponentiated action, θ is indeed an angle, with period 2π. We can also write (1.37) as:

$$Z = Z_0 + \sum_{k>0} q^k Z_k + \sum_{k<0} \bar{q}^{-k} Z_k \ , \qquad q \equiv e^{2\pi i\tau} \ , \tag{1.38}$$

with the holomorphic gauge coupling τ defined in previous lectures:

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \ . \tag{1.39}$$

1.2.2 θ-term and chiral anomaly

Consider a chiral anomaly—namely, a simple gauge group G and an anomalous flavor current $U(1)_A$, the anomalous chiral symmetry. Denoting by q_A the $U(1)_A$ charge, we have the anomaly coefficients:

$$\mathcal{A}_{Aab} = 2\sum_\psi q_A[\psi]\, T_a[\mathfrak{R}^\psi] T_b[\mathfrak{R}^\psi] = \delta_{ab}\sum_\psi q_A[\psi]\, T(\mathfrak{R}^\psi) \ , \tag{1.40}$$

where the sum is over all the chiral fermions ψ with non-zero $U(1)_A$ charges $q[\psi]$. Each such ψ sits in a representation $\mathfrak{R}^\psi$ of the gauge group G, with quadratic index $T(\mathfrak{R}^\psi)$. Let us define, then, the chiral anomaly coefficient:

$$\boxed{A_{U(1)_A} \equiv \sum_\psi q_A[\psi]\, T(\mathfrak{R}^\psi) \ .} \tag{1.41}$$

Under a chiral symmetry transformation (with symmetry parameter $\alpha \in \mathbb{R}$, a constant), we have:

$$\delta_\alpha \log Z = -\frac{i\alpha\, A_{U(1)_A}}{64\pi^2} \int d^4x \, \text{tr}\left(\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right) \ , \tag{1.42}$$

as a special case of (1.7). But this is equivalent to a shift of the θ angle:

$$\theta \to \theta + \alpha\, A_{U(1)_A} \ . \tag{1.43}$$

Thus, even though the axial symmetry $U(1)_A$ does not exists in the quantum theory if $A_{U(1)_A} \neq 0$, one can keep track of it (and extract physical consequences) by assigning the $U(1)_A$ transformation (1.43) to θ, viewing θ itself as a background field.

2 General aspects of supersymmetric gauge theories

In the following, we would like to study some aspects of renormalisable gauge theories with $\mathcal{N} = 1$ supersymmetry in four dimensions.

Let us take the gauge group G to be a simple compact Lie group, and consider some matter field in chiral multiplet, Φ, in some (generally reducible) representation $\mathfrak{R}$. The full Lagrangian for the vector and chiral multiplets can be written compactly, in superspace, as:

$$\begin{aligned}\mathscr{L} &= \int d^2\theta d^2\bar\theta\, \bar\Phi e^{-2V}\Phi - \frac{\tau}{16\pi i}\int d^2\theta\, \mathrm{tr}(\mathcal{W}\mathcal{W}) + \frac{\bar\tau}{16\pi i}\int d^2\bar\theta\, \mathrm{tr}(\bar{\mathcal{W}}\bar{\mathcal{W}}) \\ &\quad + \int d^2\theta W(\Phi) + \int d^2\bar\theta \bar W(\bar\Phi) \ .\end{aligned} \tag{2.1}$$

Here, $W(\Phi)$ is some *gauge invariant* holomorphic polynomial in Φ (which we take to be at most cubic in the renormalisable theory).

Note that the Lagrangian of any renormalisable 4d $\mathcal{N} = 1$ supersymmetric gauge theory is fully determined by the data of:

- The gauge group G with gauge coupling(s) $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.
- The representation $\mathfrak{R}$ for the chiral multiplets.
- The superpotential $W(\Phi)$.

All the various interactions terms are then determined by the combination of gauge invariance and supersymmetry, as well as by W.

2.1 Vacuum and beta functions

2.1.1 Classical scalar potential and vacuum manifold

By looking at the classical Lagrangian in components, it is easy to study the classical scalar potential of the gauge theory. The adjoint-valued auxiliary field D enters as:

$$\mathscr{L} \supset \frac{1}{2g^2}D^2 - \bar\phi D\phi \ , \tag{2.2}$$

in the WZ gauge. The equations of motions for the auxiliary fields $D = D^a T_a$ give:

$$D_a = g^2\, \bar\phi T_a^{(\mathfrak{R})} \phi \ , \qquad a = 1, \cdots, \dim(G) \ . \tag{2.3}$$

Integrating out D, we then find the scalar potential:

$$\boxed{V_0 = \sum_\phi \left| \frac{\partial W}{\partial \phi} \right|^2 + \frac{g^2}{2} \sum_{a=1}^{\dim(G)} \left(\bar\phi T_a^{(\mathfrak{R})} \phi \right)^2 \ .} \tag{2.4}$$

The first term is the contribution from the superpotential, which we discussed in previous sections, while the second term can be viewed as a contribution from the gauge interactions themselves. The real operators:

$$\mu_a(\phi, \bar\phi) \equiv \bar\phi T_a^{(\mathfrak{R})} \phi \tag{2.5}$$

are often called the "moment map operators."

Since the scalar potential is again a sum of perfect squares, the classical vacuum equations of a supersymmetric gauge theory are:

$$\boxed{\partial_\phi W = 0 \ , \quad \forall \phi \ , \qquad\qquad \mu_a(\phi, \bar\phi) = 0 \ , \quad \forall a \ .} \tag{2.6}$$

Any two solutions to (2.6) related by a (constant) gauge transformations are physically equivalent. So, we introduce the equivalence relation on the space of constant field values:

$$\phi' \sim \phi \qquad \text{if} \qquad \exists \ (\alpha^a) \in \mathbb{R}^{\dim(G)} \ \text{ such that } \ \phi' = e^{i\alpha^a T_a^{(\mathfrak{R})}} \phi \ . \tag{2.7}$$

The constant values of the scalar field $\phi \in \Phi$ span the vector space:

$$V_\mathfrak{R} \cong \mathbb{C}^n \ , \qquad n \equiv \dim(\mathfrak{R}) \ , \tag{2.8}$$

on which the representation $\mathfrak{R}$ acts. Then, the *vacuum manifold* of the gauge theory takes the general form:

$$\boxed{\mathcal{M} = \{\phi \in V_\mathfrak{R} \mid \partial_\phi W = 0 \ , \ \mu_a = 0\}/G \ ,} \tag{2.9}$$

where the quotient by the gauge group corresponds to the equivalence relation (2.7). In our discussion of theories with only chiral multiplets, we saw that the vacuum moduli space was a purely algebraic object—in particular, everything was holomorphic in ϕ. This is apparently not the case in a gauge theory, since the formula (2.9) is non-holomorphic in two ways: the moment maps μ_a are real, and the gauge equivalence (2.7) is in terms of real gauge parameters α^a.

Nonetheless, there is a simple-looking (although by no mean obvious) way to rewrite (2.9) more algebraically. It turns out that imposing the vanishing of the moment maps, $\mu_a = 0$, and then dividing by G, is *equivalent* to dividing by the *complexified gauge group:*

$$\boxed{\mathcal{M} = \{\phi \in V_{\mathfrak{R}} \mid \partial_\phi W = 0\}/G_{\mathbb{C}}\ .} \tag{2.10}$$

In this approach, we are considering the space of complexified gauge orbits (or, more precisely, their closure), under the $G_{\mathbb{C}}$ action:

$$\phi' \sim \phi \qquad \text{if} \qquad \exists\ (\omega^a) \in \mathbb{C}^{\dim(G)}\ \text{ such that }\ \phi' = e^{i\omega^a T_a^{(\mathfrak{R})}}\phi\ . \tag{2.11}$$

The fact that the two approaches (2.9) and (2.10) reproduce the same moduli space was shown explicitly in [7]. [2]

Conceptually, this was to be expected: the fact that we only divide by real gauge transformations in (2.9) is an artefact of the WZ gauge. The supersymmetric gauge transformations on chiral superfields,

$$\delta_\Omega \Phi = e^{i\Omega}\Phi\ , \tag{2.12}$$

are really $G_{\mathbb{C}}$-valued gauge transformations. More generally, the F-term contributions to the Lagrangian of any supersymmetric gauge theory are invariant under the complexified gauge group $G_{\mathbb{C}}$, while the total Lagrangian (in particular, the D-term kinetic term for matter fields) is only G-invariant.

Finally, it is non-obvious but nonetheless true that the vacuum moduli space $\mathcal{M}$ of a gauge-theory is also a Kähler manifold, just like in the case without gauge fields.

2.1.2 β-function and chiral anomalies

The one-loop β function of the YM coupling, in our supersymmetric gauge theory, is given by:

$$\beta\left(\frac{1}{g^2}\right) = \frac{b_0}{8\pi^2}\ , \qquad b_0 = \frac{3}{2}T(\text{adj}) - \frac{1}{2}T(\mathfrak{R})\ . \tag{2.13}$$

Here, we simply specialised the general result (see eq.(7.21) in the Susy&Sugra lectures) to the supersymmetric matter content, taking into account that the gaugino transforms in the adjoint representation.

Anomalous symmetries (with $W = 0$). Consider the axial symmetry $U(1)_A$ that gives a charge 1 to all chiral multiplets. This is a symmetry of the theory without superpotential ($W = 0$), which is however anomalous, since:

$$A_{U(1)_A} = T(\mathfrak{R})\ . \tag{2.14}$$

[2]Mathematically, it is a non-trivial equivalence between Kähler quotients (corresponding to (2.9)) and Geometric Invariant Theory (GIT) quotients (corresponding to (2.10)).

More generally, we should decompose $\mathfrak{R}$ into irreducible representations:

$$\mathfrak{R} = \oplus_i \mathfrak{R}_i \ , \tag{2.15}$$

and define the symmetry $U(1)_i$, which charge q_i, that acts only on Φ_i (in the representation $\mathfrak{R}_i$) with charge 1:

$$q_i[\Phi_j] = \delta_{ij} \ . \tag{2.16}$$

The symmetry is anomalous, with chiral anomaly $A_i = T(\mathfrak{R}_i)$, which acts as:

$$\Phi_j \to e^{i\delta_{ij}\alpha}\Phi_j \ , \qquad \theta \to \theta + T(\mathfrak{R}_i)\,\alpha \ , \tag{2.17}$$

including the anomalous shift of the θ angle. Another symmetry which is anomalous is the "reference" R-charge R_0 with charges:

$$R_0[V] = 0 \ , \qquad R_0[\Phi_i] = 0 \qquad \Rightarrow \qquad R_0[\lambda] = 1 \ , \quad R_0[\psi_i] = -1 \ . \tag{2.18}$$

Any mixing of an R-symmetry with a non-R (flavor) symmetry gives another R-symmetry. Let us introduce a generic R-symmetry:

$$\boxed{R = R_0 + \sum_i r_i q_i \ ,} \qquad r_i \in \mathbb{R} \ , \tag{2.19}$$

where the parameters r_i are the R-charges of the chiral multiplets, $R[\Phi_i] = r_i$ (so the corresponding chiral fermions have $R[\psi_i] = r_i - 1$). One can then often find a *non-anomalous R-symmetry*, by choosing the R-charges r_i such that:

$$A_{U(1)_R} = T(\text{adj}) + \sum_i (r_i - 1) T(\mathfrak{R}_i) = 0 \ . \tag{2.20}$$

A digression: The reader might wonder about the fact that the same quadratic indices of the gauge representations appear in the expression for the YM β function (2.13) and for the chiral anomalies. In fact, one can easily define an R-symmetry:

$$R_c \equiv R_0 + \frac{2}{3}A = R_0 + \frac{2}{3}\sum_i q_i \ , \tag{2.21}$$

which is such that the chiral anomaly is exactly proportional to the β function:

$$A_{U(1)_{R_c}} = \frac{2}{3} b_0 \ . \tag{2.22}$$

The R-charge R_c assigns $r_c = \frac{2}{3}$ to all chiral multiplets, which is compatible with a classically-marginal superpotential, $W = \Phi^3$ (schematically). In fact, in the far UV, such a super-Yang-Mills (SYM) theory is classically conformal (in particular, scale invariant), and the R-charge combines with the supersymmetry current and the energy-momentum tensor into a larger algebraic structure (known as $\mathcal{N} = 1$ superconformal multiplet). Quantum corrections break both conformal invariance (giving the running of g^2, which is a "quantum anomaly" of scale invariance) and the R-symmetry (through the chiral anomaly), but supersymmetry relates these two "quantum anomalies" exactly as in (2.22).

2.2 Renormalisation of the holomorphic gauge coupling

In a supersymmetric gauge theory, we now have a "generalised superpotential:"

$$\mathbf{W}_{\mu_0}(\mathcal{W}, \Phi, \tau, \lambda) = -\frac{\tau(\mu_0)}{16\pi i} \operatorname{tr} \mathcal{W}^\alpha \mathcal{W}_\alpha + W(\Phi, \lambda; \mu_0) \ . \tag{2.23}$$

Here, λ denote the ordinary superpotential coupling. This F-term must depend holomorphically on both τ and λ, which restricts the possible quantum corrections to $\mathbf{W}$, at least in some appropriate "holomorphic scheme." We also introduced the UV scale μ_0 explicitly in (2.23).

Perturbatively in g^2, the holomorphy in τ gives some very severe restriction when combined with the fact that θ is an *angle* with period 2π, which means that the theory is invariant under:

$$\tau \sim \tau + 1 \ . \tag{2.24}$$

This implies that τ can only appear *linearly* in $\mathbf{W}_\mu$, at any scale μ, and only in the precise form:

$$\mathbf{W}_\mu \supset -\frac{\tau(\mu)}{16\pi i} \operatorname{tr} \mathcal{W}^\alpha \mathcal{W}_\alpha \ . \tag{2.25}$$

Indeed, in that case θ multiplies the topological term as in (1.23). Since it is a topological invariant, it cannot depend on μ at all. The one-loop running of the *holomorphic coupling* τ is given by:

$$\tau(\mu) = \tau(\mu_0) - \frac{b_0}{2\pi i} \log \frac{\mu}{\mu_0} \ . \tag{2.26}$$

It is *exact* in perturbation theory. This is because any higher-order terms in $\beta(\tau)$ would be given in terms of $\mathrm{Im}(\tau) = \frac{4\pi}{g^2}$, which is incompatible with holomorphy. [3] One can also rule out other perturbative corrections to the superpotential, like in the case without gauge interactions. This leaves the possibility of non-perturbative corrections.

It is useful to introduce a complexified dynamically-generated scale, generalizing (1.35), defined as:

$$\Lambda = e^{i\frac{\theta}{b_0}} |\Lambda| = \mu_0 e^{\frac{2\pi i \tau(\mu_0)}{b_0}} \ . \tag{2.27}$$

Then, we have:

$$\Lambda^{b_0} = \mu^{b_0} e^{2\pi i \tau(\mu)} \ , \tag{2.28}$$

which is a well-defined quantity, invariant under the shift $\theta \sim \theta + 2\pi$. Any non-perturbative effects would appear as:

$$\mathbf{W}_\mu = -\frac{\tau(\mu)}{16\pi i} \operatorname{tr} \mathcal{W}^\alpha \mathcal{W}_\alpha + W(\Phi, \lambda; \mu) + \sum_{k=1}^{\infty} \Lambda^{b_0 k} g_n(\Phi, \lambda, \cdots) \ , \tag{2.29}$$

[3] One could also expect perturbative corrections in the superpotential couplings λ in $\beta(\tau)$, but that can be ruled out by considering the weak-coupling limit $g^2 \to 0$.

since the theory should be regular in the limit $\Lambda \to 0$. Note the expression:

$$\boxed{\tau(\mu) = \frac{b_0}{2\pi i} \log \frac{\Lambda}{\mu}\ .} \tag{2.30}$$

One could still constraint more carefully the form of the possible quantum corrections to $\mathbf{W}$. This sort of analysis, however, is only reliable at weak coupling. As we RG flow from an asymptotically-free theory in the UV toward $\mu \sim |\Lambda|$, the theory become strongly coupled and we need new methods to explore the infrared physics. As we will see in a particular example (SQCD), supersymmetry and general symmetry arguments sometimes are exceptionally powerful in order to "guess" what the infrared physics is.

2.3 The "exact" β-function

We saw that, at the level of the F-terms, the *holomorphic* gauge coupling τ is *exact* at one-loop (up to, possibly, non-perturbative corrections), and is given by (2.30). This is true in the holomorphic scheme—that is, when we choose to preserve the holomorphy of the F-terms. On the other hand, the D-terms in (2.1) are renormalised non-trivially, with:

$$S_{\text{D-term},\mu} = \int d^2\theta d^2\bar{\theta} \left(Z_\Phi \bar{\Phi} e^{-2V} \Phi + \cdots \right)\ , \tag{2.31}$$

where the ellipsis denotes contributions from higher-dimensional operators, and the wave function renormalisation factor depends on all the coupling constants:

$$Z_\Phi = Z_\Phi(g, |\lambda|; \mu)\ , \tag{2.32}$$

and can be computed, in principle, at any order in perturbation theory. Moreover, recall that the *physical gauge coupling,* is the one obtained by rescaling:

$$A_\mu \to g_c A_\mu\ , \tag{2.33}$$

so that positive powers g appears in interactions vertices (including through the covariant derivative $D_\mu = \partial_\mu - i g_c A_\mu^{(c)}$). Here, for the moment, we write g_c for this 'physical coupling' to distinguish it from g that appears holomorphically in τ. In the holomorphic scheme, we have:

$$\mathscr{L}_\mu = \int d^2\theta d^2\bar{\theta}\, Z_\Phi(\mu) \bar{\Phi} e^{-2V} \Phi + \int d^2\theta \left(\frac{i\theta}{32\pi^2} - \frac{1}{4g^2(\mu)} \right) \mathcal{W}^2 + \text{h.c.}\ , \tag{2.34}$$

schematically. Here, we set $W = 0$ for simplicity of notation. Note also that there is really one distinct wavefunction renormalisation factor Z_{Φ^i} for each irreducible gauge representation $\mathfrak{R}_i$. We would like to define the "physical" fields:

$$\Phi_R^i = \sqrt{Z_{\Phi^i}} \Phi^i\ , \qquad V_c = \frac{1}{g_c} V\ . \tag{2.35}$$

As we did for theories of chiral multiplets. It turns out, however, that these field redefinitions are *anomalous* in the presence of the gauge interactions—that is, the change of variable in the path integral gives a non-trivial Jacobian [8]. Heuristically, this can be understood as follows. The symmetry group of the F-terms is *complexified*—we mentioned this fact before for the gauge group G, but that is true of any global symmetries as well. The rescaling (2.35), in particular, can be understood as a complexified "chiral rotation." Since that chiral symmetry is anomalous, the rotation shifts the θ angle by an imaginary amount. This, effectively, shift $\frac{1}{g^2}$ by a real quantity.

2.3.1 Rescaling of the chiral superfields

It is useful to break down the field redefinition (2.35) in two steps. First, consider:

$$\Phi^i_R = \sqrt{Z_{\Phi^i}}\Phi^i \ , \tag{2.36}$$

without touching V. This change of variable can be achieved by some complexified chiral rotations,

$$\Phi^i_R = e^{i\alpha_i}\Phi^i \ , \qquad\qquad \alpha_i = -\frac{i}{2}\log Z_{\Phi^i} \ , \tag{2.37}$$

acting on each Φ^i independently. Then, according to (2.17), this gives a shift to the θ angle by:

$$\theta \to \theta' = \theta + \sum_i \alpha_i T(\mathfrak{R}_i) = \theta - \frac{i}{2}\sum_i T(\mathfrak{R}_i)\log Z_{\Phi^i} \ . \tag{2.38}$$

We then find the F-term Lagrangian:

$$\int d^2\theta \left(\frac{i\theta}{32\pi^2} - \frac{1}{4g_R^2}\right)\mathcal{W}^\alpha \mathcal{W}_\alpha \ , \tag{2.39}$$

with:

$$\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{1}{16\pi^2}\sum_i T(\mathfrak{R}_i)\log Z_{\Phi^i} \ . \tag{2.40}$$

This effective coupling g_R^2 has a β-function:

$$\begin{aligned}\beta\left(\frac{1}{g_R^2}\right) &= \frac{b_0 + \frac{1}{2}\sum_i T(\mathfrak{R}_i)\gamma_{\phi^i}}{8\pi^2} \\ &= \frac{1}{16\pi^2}\Big(3T(\text{adj}) - \sum_i T(\mathfrak{R}_i)(1-\gamma_{\phi^i})\Big) \ ,\end{aligned} \tag{2.41}$$

where we made use of the definition:

$$\gamma_\phi = -\mu\frac{\partial}{\partial\mu}\log Z_\phi \ , \tag{2.42}$$

for the anomalous dimensions γ_ϕ of Φ. This receives contributions from every loop order in perturbation theory, but only through γ_ϕ, similarly to our discussion of the 'physical superpotential couplings' in previous lectures.

2.3.2 Rescaling of the vector superfield

Now, start from (2.39) and introduce the canonically-normalised vector multiplet:

$$V = g_c V_c \ , \tag{2.43}$$

to obtain the canonically normalised gauge field. This change of variables also has a non-trivial Jacobian, which gives [8]:

$$\int d^2\theta \left(\frac{i\theta}{32\pi^2} - \frac{1}{4g_R^2} + \frac{T(\text{adj})}{32\pi^2} \log(g_c) \right) \mathcal{W}^\alpha \mathcal{W}_\alpha \ . \tag{2.44}$$

Then, equating the real coefficient inside the parenthesis with the canonically-normalised coupling $-\frac{1}{4g_c^2}$, we get:

$$\frac{1}{g_c} = \frac{1}{g_R^2} + \frac{T(\text{adj})}{16\pi^2} \log\left(\frac{1}{g_c^2}\right) \ , \tag{2.45}$$

which gives g_c in terms of g_R, which is itself defined in (2.40) in terms of the “holomorphic coupling” g. In particular, we find the β-function:

$$\boxed{\beta\left(\frac{1}{g_c^2}\right) = \frac{b_0 + \frac{1}{2}\sum_i T(\mathfrak{R}_i)\gamma_{\phi^i}}{8\pi^2 - \frac{1}{2}T(\text{adj})g_c^2} \ .} \tag{2.46}$$

Note that the denominator is the same as in (2.41). This is the famous NSVZ β-function, which was first derived by completely different methods [9]. It is sometimes called “the exact β-function,” in the sense that it depends only on the anomalous dimensions of the fields. Note that the anomalous dimensions themselves depend on g_c^2 (and on any other superpotential couplings), so it just tells us that the β function is known is we know the exact anomalous dimensions. The latter depend on the details of the Kähler potential, and thus we cannot have exact formulas for them. This is thus similar to the “exact” β-function for the superpotential coupling that we discussed previously, [4] namely:

$$\beta(\widetilde{\lambda}_R) = \big(-3 + \sum_i \big(1 + \frac{1}{2}\gamma_{\phi^i}\big) d_i\big)\widetilde{\lambda}_R \ . \tag{2.47}$$

2.3.3 Looking for non-trivial fixed points

The physical significance of the denominator in (2.46) is not entirely clear, because perturbation theory becomes unreliable before the denominator can have any important effect on the RG running. At the level of the present discussion, we should simply view the NSVZ β function as a beautiful example of supersymmetry leading to huge simplifications in the analysis of perturbative RG flows.

[4] See the “Supersymmetry and Supergravity” HT2020 lectures, section 6.3.

One classic use that has been made of these "exact" results is to look for perturbatively-exact *fixed points* of the RG flow. Combining the results for the gauge-coupling and superpotential coupling constants, (2.46) and (2.47), we can ask whether it is possible to find an solution to the equations:

$$\beta\left(\frac{1}{g_R^2}\right) = 0 \; , \qquad \beta(\widetilde\lambda_R) = 0 \; . \tag{2.48}$$

Here we used g_R, since its β-function has the same zeros as g_c. The existence of such fixed points should be completely scheme-independent. The equations (2.48) give strong constraints on the anomalous dimensions γ_ϕ that can arise at any candidate fixed point, even at strong coupling.

3 SQCD: Lagrangian, symmetries and classical moduli space

Let us now focus on some particularly nice 4d $\mathcal{N}=1$ gauge theory, supersymmetric QCD (a.k.a. SQCD). This is an $SU(N_c)$ $\mathcal{N}=1$ supersymmetric gauge theory with N_f flavors. This means that we have N_f chiral multiplets in the fundamental representation of $SU(N_c)$, and N_f chiral multiplets in the anti-fundamental representation. We denote them by:

$$Q_i \; , \quad i = 1, \cdots, N_f \; , \qquad\qquad \widetilde{Q}^j \; , \quad j = 1, \cdots, N_f \; , \tag{3.1}$$

respectively. For obvious reasons, the numbers N_c and N_f are called the number of colors and flavors, respectively, and the scalar fields $Q, \widetilde{Q}$ are called the squarks.

The supersymmetric Lagrangian takes the form:

$$\begin{aligned} \mathscr{L} \;&=\; \int d^2\theta d^2\bar\theta \left(\sum_{i=1}^{N_f} \bar{Q}^i e^{-2V} Q_i + \sum_{j=1}^{N_f} \bar{\widetilde{Q}}_j e^{-2V} \widetilde{Q}^j\right) \\ &\quad - \frac{\tau}{16\pi i}\int d^2\theta \, \mathrm{tr}(\mathcal{W}\mathcal{W}) + \frac{\bar\tau}{16\pi i}\int d^2\bar\theta \, \mathrm{tr}(\bar{\mathcal{W}}\bar{\mathcal{W}}) \; . \end{aligned} \tag{3.2}$$

More precisely, this is massless SQCD, with vanishing superpotential. We could also consider adding Dirac masses for the quarks, through a superpotential:

$$W = \mu^i{}_j \, \widetilde{Q}^j Q_i \; , \tag{3.3}$$

with $\mu^i{}_j$ the mass matrix. Note that all gauge indices are implicit. We denote by $a = 1, \cdots, N_c$ the gauge indices in the fundamental representation. Then,

$$\widetilde{Q}^j Q_i \equiv \widetilde{Q}^j_a Q_i^a \; ,$$

which is obviously gauge invariant, and similarly for the contraction of the gauge indices in (3.2).

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_A$	$U(1)_B$	$U(1)_R$
(Q_i^a)	$(\mathbf{N_c})$	$(\overline{\mathbf{N}}_\mathbf{f})$	$(\mathbf{1})$	1	1	r
$(\widetilde{Q}_a^j)$	$(\overline{\mathbf{N}}_\mathbf{c})$	$(\mathbf{1})$	$(\mathbf{N_f})$	1	-1	r

Table 1: Gauge and global symmetry representations of the chiral multiplets $\widetilde{Q}$ and Q of SQCD.

The symmetries of the classical Lagrangian of massless SQCD are:

$$\widetilde{G}_F = SU(N_f) \times SU(N_f) \times U(1)_A \times U(1)_B \ , \tag{3.4}$$

times an R-symmetry $U(1)_R$. The $U(1)_A$ and $U(1)_B$ factors are called the axial symmetry and the baryonic symmetry, respectively. The charges of the chiral superfields under the gauge and global symmetries are summarised in Table 1. The R-charge shown if for an R-charge:

$$R = R_0 + rA \ , \tag{3.5}$$

with A the generator of $U(1)_A$ and R_0 the "reference" R-charge under which the chiral multiplets are neutral.

3.1 Anomalies and anomaly-free R-symmetry

One can easily check that the gauge group $SU(N_c)$ of SQCD is anomaly free—that is the reason we needed as many fundamental as antifundamental chiral multiplets.

As a special case of the discussion in section 2.1.2, we see that $U(1)_A$ suffers from a chiral anomaly:

$$A_{U(1)_A} = \sum_i T(\mathfrak{R}_i) = 2N_f \ , \tag{3.6}$$

and so does $U(1)_R$, for a general mixing parameter $r \in \mathbb{R}$:

$$A_{U(1)_R} = 2N_c + 2N_f(r-1) \ . \tag{3.7}$$

However, there is a unique choice of r such that the $U(1)_R$-$SU(N_c)^2$ anomaly (3.7) vanishes, namely:

$$\boxed{r = 1 - \frac{N_c}{N_f} \ .} \tag{3.8}$$

The other symmetries are non-anomalous. Thus, the global symmetry of SQCD, at the quantum level, is:

$$G_F \times U(1)_R \ , \qquad G_F \equiv SU(N_f) \times SU(N_f) \times U(1)_B \ , \tag{3.9}$$

with this particular choice of R-charge.

3.2 The classical vacuum moduli space of SQCD and gauge-invariant operators

In the absence of superpotential, SQCD can have a large vacuum moduli space. Let us denote by Q^a_i and $\widetilde{Q}^j_a$ the scalars in the corresponding superfields, and also their VEVs. We then have:

$$\mathcal{M} = \left\{ (Q^a_i, \widetilde{Q}^j_a) \in \mathbb{C}^{2N_c N_f} \;\Big|\; \mu = 0 \right\} / SU(N_c) \; . \tag{3.10}$$

Here, the "D-term constraints" can be written as the vanishing of the traceless $N_c \times N_c$ matrices: [5]

$$\mu^a_b \equiv \sum_{i=1}^{N_f} \left({Q^\dagger}^i_b Q^a_i - \frac{\delta^a_b}{N_c} \operatorname{tr}({Q^\dagger}^i Q_i) \right) - \sum_{j=1}^{N_f} \left(\widetilde{Q}^{\dagger a}_j \widetilde{Q}^j_b - \frac{\delta^a_b}{N_c} \operatorname{tr}(\widetilde{Q}^\dagger_j \widetilde{Q}^j) \right) = 0 \; . \tag{3.11}$$

Note that the matrix μ^a_b is obviously $SU(N_c) \times SU(N_c) \times U(1)_B$-invariant.

For any fixed number of "colors" N_c, the structure of the $SU(N_c)$ SQCD moduli space changes as we vary the number of flavors, N_f. The basic physical reason is the *Higgs mechanism.* A non-zero VEV for a single fundamental scalar $Q = (Q^a)$ of $SU(N_c)$ breaks the gauge group as:

$$\langle Q^a \rangle \neq 0 \qquad \Rightarrow \qquad SU(N_c) \to SU(N_c - 1) \; . \tag{3.12}$$

By a gauge transformation, we can take the vector $\langle Q^a \rangle$ to be $(q_1, 0, 0, \cdots, 0)$, which is obviously preserved by the $SU(N_c - 1)$ subgroup of $SU(N_c)$. More generally, a generic VEV for the N_f squarks breaks the gauge group according to:

$$\langle Q^a_i \rangle \neq 0 \; , \qquad i = 1, \cdots, N_f \qquad \Rightarrow \qquad SU(N_c) \to SU(N_c - N_f) \; . \tag{3.13}$$

For $N_f \geq N_c$, the $SU(N_c)$ gauge group is entirely broken at a generic point on the vacuum moduli space. The VEVs also have to satisfy the D-term conditions (3.11) in order to preserve supersymmetry.

Another approach to analysing the moduli space is to use the description (2.10). In the absence of superpotential, this tells us that:

$$\mathcal{M} = \left\{ (Q^a_i, \widetilde{Q}^j_a) \in \mathbb{C}^{2N_c N_f} \right\} / SL(N_c, \mathbb{C}) \; . \tag{3.14}$$

This space can be constructed algebraically by building all the possible *gauge invariant chiral operators, X,* and then imposing relations between the fields X that follow from their definition—these are known as syzygies. This is a classic problem in invariant theory. We will present explicit examples below.

[5] Here we take:

$$(T^{\mathbf{a}})^a_b = (T^c_d)^a_b = \delta^c_b \delta^a_d - \frac{1}{N_c} \delta^a_b \delta^c_d \; ,$$

for the fundamental of $SU(N_c)$, and then $(\bar{T}^{\mathbf{a}})^a_b = -(T^{\mathbf{a}})^a_b$ for the anti-fundamental.

3.2.1 The case $N_f < N_c$

If $N_f < N_c$, we can pick the VEVs of the fundamental squarks to be:

$$(Q_i^a) = \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ & \vdots & \ddots & \\ 0 & 0 & \cdots & q_{N_f} \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} . \tag{3.15}$$

Plugging this into (3.11), we see that this solves the D-term condition if and only:

$$(\widetilde{Q}_a^i) = \begin{pmatrix} \widetilde{q}_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \widetilde{q}_2 & \cdots & 0 & 0 & \cdots & 0 \\ & \vdots & \ddots & & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \widetilde{q}_{N_f} & 0 & \cdots & 0 \end{pmatrix} , \qquad \text{with} \quad |q_i| = |\widetilde{q}_i| \; , \; \forall i \; . \tag{3.16}$$

This is not the most general solution to (3.11), but any other allowed VEV can be reached by considering the $SU(N_f) \times SU(N_f)$ orbit of (3.15)-(3.16). It turns out that:

$$\dim_{\mathbb{C}}(\mathcal{M}) = N_f^2 \; , \tag{3.17}$$

as a complex space. This can be understood easily in terms of the Higgs mechanism. At a generic point on $\mathcal{M}$, the gauge symmetry is broken as in (3.12) and so:

$$N_W = (N_c^2 - 1) - ((N_c - N_f)^2 - 1) = 2N_f N_c - N_f^2 \; , \tag{3.18}$$

gauge bosons get a mass, by each "eating" a complex scaler ϕ. Thus, out of the $2N_f N_c$ complex scalars in the UV, $2N_f N_c - N_W = N_f^2$ scalars survive on the IR moduli space, matching the counting (3.17).

We can also see this in the purely algebraic description (3.14). In this language, we should construct all the gauge invariant scalars build out of the fundamental squarks Q and $\widetilde{Q}$. There are only N_f^2 of them, which we denote by:

$$\boxed{M^j{}_i \equiv \widetilde{Q}_a^j Q_i^a \; .} \tag{3.19}$$

They are usually called the SQCD *mesons,* since they are made of two fundamental squarks, in analogy with the mesons of real-world QCD which are made of two quarks. They are the natural coordinates on the moduli space for $N_f < N_c$, with:

$$\mathcal{M} \cong \mathbb{C}^{N_f^2} \; . \tag{3.20}$$

Note also that the case $N_f = N_c - 1$ is special, since the gauge group is completely broken. (The "$SU(1)$" group is trivial.)

	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$	$U(1)_A$
M	$(\overline{\mathbf{N_f}})$	$(\mathbf{N_f})$	0	$2r$	2
B	$(\otimes_{\mathbf{A}}^{\mathbf{N_c}}\overline{\mathbf{N_f}})$	$(\mathbf{1})$	N_c	rN_c	N_c
$\widetilde{B}$	$(\mathbf{1})$	$(\otimes_{\mathbf{A}}^{\mathbf{N_c}}\mathbf{N_f})$	$-N_c$	rN_c	N_c
Λ^{b_0}	$(\mathbf{1})$	$(\mathbf{1})$	0	$2N_c + 2N_f(r-1)$	$2N_f$

Table 2: Global symmetry representations the gauge-invariant chiral operators of SQCD. Here, $U(1)_A$ is anomalous, and so is $U(1)_R$ unless we choose the R-charge as in (3.8). This induces A and R charges for the SQCD scale Λ^{b_0}, with $b_0 = 3N_c - N_f$.

3.2.2 The case $N_f \geq N_c$

For $N_f \geq N_c$, the gauge group is completely Higgsed at a generic point on the moduli space. By the same Higgs-mechanism argument as above, we should have:

$$\dim_{\mathbb{C}}(\mathcal{M}) = 2N_f N_c - N_c^2 + 1 . \tag{3.21}$$

Let us see how this comes about in the language of gauge-invariant chiral fields. Now, in addition to the mesons (3.19), we can build other gauge-invariants, by making use of the fully-antisymmetric invariant tensor of $SU(N_c)$. These are the so-called "baryons:"

$$\boxed{B_{i_1 i_2 \cdots i_{N_c}} \equiv \epsilon_{a_1 a_2 \cdots a_{N_c}} Q^{a_1}_{i_1} Q^{a_2}_{i_2} \cdots Q^{a_{N_c}}_{i_{N_c}} \ ,} \tag{3.22}$$

and the "anti-baryons:"

$$\boxed{\widetilde{B}^{j_1 j_2 \cdots j_{N_c}} \equiv \epsilon^{a_1 a_2 \cdots a_{N_c}} \widetilde{Q}^{j_1}_{a_1} \widetilde{Q}^{j_2}_{a_2} \cdots \widetilde{Q}^{j_{N_c}}_{a_{N_c}} \ .} \tag{3.23}$$

Again, the name comes from QCD, where a baryon is a gauge-invariant combination of 3 fermions in the fundamental of the gauge group $SU(3)$.

For future reference, we collect the symmetry charges of the mesons and baryons in Table 2. In total, there are:

$$N_X \equiv N_f^2 + 2\begin{pmatrix} N_f \\ N_c \end{pmatrix} , \tag{3.24}$$

gauge-invariant operators, since the baryons B transform in N_c-index antisymmetric representations of $SU(N_f)$, and similarly for the anti-baryons $\widetilde{B}$. This is larger than the expected dimension of the moduli space, (3.21).

The reason is that they are *relations* amongst the so-called *generators* of $\mathcal{M}$ seen as an algebraic variety:

$$X = (M, B, \widetilde{B}) , \tag{3.25}$$

which follow directly from their definition in terms of fundamental fields. One can easily check that:

$$P^{j_1\cdots j_{N_c}}_{i_1\cdots i_{N_c}}(X) \equiv \widetilde{B}^{j_1 j_2 \cdots j_{N_c}} B_{i_1 i_2 \cdots i_{N_c}} - M^{[j_1}_{i_1} \cdots M^{j_{N_c}]}_{i_{N_c}} = 0 \ , \tag{3.26}$$

where the square bracket denotes anti-symmetrization of the indices. Moreover, since the anti-symmetrisation of $n > N_c$ squarks must vanish, we have:

$$Q^{i_{N_c+2}\cdots i_{N_f} j}(X) \equiv \epsilon^{i_1\cdots i_{N_f}} B_{i_1\cdots i_{N_c}} M^j_{i_{N_c+1}} = 0 \ , \tag{3.27}$$

if $N_f > N_c$, and similarly:

$$\widetilde{Q}_{j_{N_c+2}\cdots j_{N_f} i}(X) \equiv \epsilon_{j_1\cdots j_{N_f}} B^{j_1\cdots j_{N_c}} M^{j_{N_c+1}}_i = 0 \ . \tag{3.28}$$

The moduli space is then given explicitly in terms of generators and relations, as:

$$\mathcal{M} = \left\{ X \in \mathbb{C}^{N_X} \ \middle| \ P(X) = 0 \ , \ Q(X) = 0 \ , \ \widetilde{Q}(X) = 0 \right\} \ . \tag{3.29}$$

Mathematically, this is known as a *affine variety* (the zero set of a some polynomials in $\mathbb{C}^n$).

3.2.3 The case $N_f = N_c$

Consider the case $N_f = N_c$ in more detail. According to the description (3.29), we have:

$$N_X = N_c^2 + 2 \tag{3.30}$$

generators, while the moduli space is of dimension $N_c^2 + 1$. Indeed, the generators are:

$$M^j_i \ , \qquad \mathcal{B} \ , \qquad \widetilde{\mathcal{B}} \ , \tag{3.31}$$

where we defined:

$$\mathcal{B} = \epsilon^{i_1\cdots i_{N_c}} B_{i_1\cdots i_{N_c}} \ , \qquad \widetilde{\mathcal{B}} = \epsilon_{j_1\cdots j_{N_c}} \widetilde{B}^{j_1\cdots j_{N_c}} \ , \tag{3.32}$$

using the $SU(N_f) \times SU(N_f)$-invariant ϵ-symbols (with $N_f = N_c$). There is a single relation, of the form (3.26), amongst the generators, which can be written as:

$$\boxed{\det(M) - \widetilde{\mathcal{B}}\mathcal{B} = 0 \ .} \tag{3.33}$$

The resulting affine variety $\mathcal{M}$ is an example of a *hypersurface* in algebraic geometry.

3.2.4 The case $N_f = N_c + 1$

In this case, we have:

$$N_X = N_f^2 + 2N_f \tag{3.34}$$

generators, the mesons and baryons:

$$M_i^j \ , \qquad \mathcal{B}^{i_0} = \epsilon^{i_0 i_1 \cdots i_{N_c}} B_{i_1 \cdots i_{N_c}} \ , \qquad \widetilde{\mathcal{B}}_{j_0} = \epsilon_{j_0 j_1 \cdots j_{N_c}} \widetilde{B}^{j_1 \cdots j_{N_c}} \ . \tag{3.35}$$

On the other hand, the expected dimension of the moduli space is:

$$\dim(\mathcal{M}) = 2N_f(N_f - 1) - (N_f - 1)^2 + 1 = N_f^2 \ . \tag{3.36}$$

The relations amongst the generators take the form:

$$\mathcal{B}^i \widetilde{\mathcal{B}}_j - \text{Minor}(M)^i_j = 0 \ , \qquad M_i^j \mathcal{B}^i = 0 \ , \qquad \widetilde{\mathcal{B}}_j M_i^j = 0 \ . \tag{3.37}$$

Note that there are $N_P = N_f^2 + 2N_f$ relations amongst the $N_X = N_f^2 + 2N_f$ generators, but:

$$N_P > N_X - \dim(\mathcal{M}) \ . \tag{3.38}$$

Thus, the relations cannot be all independent, but nonetheless there does not exist a smaller set of relations. This is a common feature of algebraic varieties. (Variety whose dimensions is given by the number of generators minus the number of relations are called *complete intersections.* The SQCD moduli space for $N_f > N_c$ is not a complete intersection.)

3.3 The IR phases of SQCD

In this section, we discuss how the classical picture of the SQCD vacuum is modified quantum mechanically. We will only be able to touch upon the subject, for lack of space and time, but hopefully it will be enough to give you some the desire to go and learn more about this beautiful chapter of mathematical physics.

3.4 Infrared phases of gauge theories

Asymptotically-free gauge theories run to strong coupling at low energy, and perturbation theory breaks down. Therefore, exploring the infrared physics of gauge theories is a very challenging problem, theoretically.

Given an RG flow from any weakly coupled UV theory (in particular, a gauge theory), there are roughly three possibilities for what may happen in the infrared:

- **Mass gap.** The theory might be gapped. That is, there are no excitations below a finite energy E_0, and therefore the far-infrared physics is trivial.

  For instance, pure Yang-Mills theory in 4d is expected to have a mass gap. (Proving that conjecture is a Millennium Prize Problem, literally and figuratively worth \$1,000,000.)

- **IR free.** The infrared theory consists of free massless particles.

  This happens, for instance, in QED below the electron mass, where we only have a free photon. This also happens in theories with spontaneous symmetry breaking, where the Goldstone bosons are the massless particles.

- **Non-trivial fixed point (CFT).** The infrared theory may be at a non-trivial fixed point of the renormalisation group flow. In that case, the IR theory is a non-trivial conformal field theory (CFT). [6]

  While there are no obvious examples of this kind of RG flow in real-world particle physics, there are plenty of examples in condensed-matter physics.

Note that the vacuum at a generic point of the classical SQCD moduli space is an IR-free theory, consisting of n free chiral multiplets, with $n = \dim(\mathcal{M})$. On the other hand, it is much more challenging to understand what happens *at the origin* of the moduli space, where the gauge group $SU(N_c)$ is unbroken and we expect the strongly-coupled gauge dynamics at scales $\mu \leq \Lambda$ to be dominant.

3.5 Aspects of the quantum vacuum of SQCD

3.5.1 $N_f = 0$: SYM theory

In the case of super-Yang-Mills theory ($N_f = 0$), we have a ordinary pure YM theory $SU(N_c)$ coupled to a fermion λ in the adjoint representation. In the UV, the theory has a $U(1)_R$ symmetry classically. Due to the chiral anomaly:

$$A_{U(1)_R} = 2N_c \ , \tag{3.39}$$

from the gaugino, the R-symmetry is broken to a discrete subgroup:

$$U(1)_R \to \mathbb{Z}_{2N_c} \ . \tag{3.40}$$

It is expected that the theory *confines* and develops a mass gap in the IR, just like pure YM theory. Moreover, the theory has N_c distinct vacua, in which the R-symmetry $\mathbb{Z}_{2N_c}$ is spontaneously broken to $\mathbb{Z}_2$, due to the appearance of a *gaugino condensate* in the supersymmetry-preserving vacuum:

$$\langle \lambda^\alpha \lambda_\alpha \rangle = \Lambda^3 e^{\frac{2\pi i n}{N_c}} \ , \qquad n = 1, \cdots, N_c \ . \tag{3.41}$$

The fact that the theory has (at least) N_c distinct vacua can also be inferred from the Witten index of the theory, which is equal to N_c [10].

[6] Free massless particles are CFTs too, but trivial ones.

3.5.2 $0 < N_f < N_c$: runaway supersymmetry breaking

In the case $0 < N_f < N_c$, the classical moduli space is spanned by the N_f^2 mesons M_i^j. This vacuum structure is preserved in perturbation theory, but might be modified by non-perturbative effects.

Any such non-perturbative correction should appear as new operators in the superpotential:

$$W_{\text{eff}} = W(M, \Lambda) \ . \tag{3.42}$$

Here, W_{eff} can only be an holomorphic function of the complex scale Λ and of M_j^i (instead of Q and $\widetilde{Q}$ individually, by gauge invariance). In order to preserve the symmetries, with charges given in the Table 2, we must have:

$$W_{\text{eff}} = \alpha (\det M)^{c_1} \Lambda^{b_0 c_2} \ , \tag{3.43}$$

with the dependence on $\det M$ only, to preserve $SU(N_f) \times SU(N_f)$, and the coefficients:

$$c_1 2N_f + c_2 2N_f = 0 \ , \qquad 2rc_1 N_f + c_2(2N_c + 2N_f(r-1)) = 2 \ , \tag{3.44}$$

for consistency with $U(1)_A \times U(1)_R$. This gives:

$$W_{\text{eff}} = \alpha \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}} \ , \tag{3.45}$$

up to some undetermined dimensionless coupling constant. In fact, it is a famous result by Affleck, Dine and Seiberg (ADS) [11] that this superpotential is generated by a *one-instanton effect* (that is, at first order in Λ^{b_0}) when $N_f = N_c - 1$, giving:

$$W_{\text{ADS}, N_f = N_c - 1} = \frac{\Lambda^{3N_c - N_f}}{\det M} \ . \tag{3.46}$$

By consistency with various decoupling limits, this fixes:

$$\boxed{W_{\text{ADS}} = (N_c - N_f) \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}} \ ,} \tag{3.47}$$

for any $0 < N_f < N_c$.

The effect of the ADS superpotential is quite dramatic. While there is a large moduli space of vacua classically, and at all orders in perturbation theory, a non-perturbative correction—a “tiny” correction of order $e^{-8\pi^2/g^2}$—destabilises the vacuum. Indeed, the vacuum equations that follow from this superpotential are:

$$\partial_Q W_{\text{ADS}} = 0 \qquad \partial_{\widetilde{Q}} W_{\text{ADS}} = 0 \ . \tag{3.48}$$

These equations have no solution except in the limit:

$$\langle M \rangle \to \infty \ . \tag{3.49}$$

Thus, supersymmetry is spontaneously broken—since the breaking is due to dynamical effect, this is an example of what is known as *dynamical supersymmetry breaking.* In fact, there is still a supersymmetry vacuum asymptotically, "at infinity" in field space.

Exercice: Consider the case of massive SQCD, with the tree-level Dirac mass term (3.3). We still have an ADS superpotential generated at one-loop, so that:

$$W_{\text{eff}} = \mu^i_j M^j_i + W_{\text{ADS}} \ . \tag{3.50}$$

Show that, in that case, we have a finite number (equal to N_c) of supersymmetric vacua. In the massless limit $\mu \to 0$, these vacuum are pushed to infinity in field space. This is known as *runaway supersymmetry breaking.*

3.5.3 Are there low-energy σ-models for $N_f \geq N_c$?

Consider now the case $N_f \geq N_c$. Far away on the moduli space, as field distances $\langle\phi\rangle \gg \Lambda$, the physics is the one of the weakly-coupled Higgs mechanism, which gave us a simple way to compute $\dim(\mathcal{M})$ in (3.21). The low energy excitations are given in terms of mesons and baryons—even though there are constraints relating those gauge-invariant fields, we can always solve those constraint locally at a generic point on $\mathcal{M}$, to keep $n = \dim(G)$ coordinates.

The hard question is to understand what happens quantum-mechanically near the origin of the moduli space, at field distances:

$$\langle\phi\rangle \leq \Lambda \ . \tag{3.51}$$

At $\phi = 0$, classically, we have the massless bosons of the $SU(N_c)$ vector multiplet, but their dynamics is strongly coupled in the infrared. Assuming that supersymmetry is not broken, one hypothesis is that the low-energy effective theory can be written entirely in terms of gauge-invariant massless states, the mesons and baryons X, sitting in chiral multiplets, and interacting through a superpotential.

The simplest form of this hypothesis is that there the low energy theory is just a *supersymmetric σ-model* of chiral multiplets:

$$\mathscr{L}_{\sigma\text{-model}} = \int d^2\theta d^2\bar\theta\, K(\bar X, X) + \int d^2\theta W(X) + \int d^2\theta W(X) \ , \tag{3.52}$$

for the "meson" and "baryon" fields X in (3.25), which we now view as the *"fundamental fields"* in the low-energy description. The effective Lagrangian at scale $\mu \ll \Lambda$ is then given in terms of some unknown (and presumably complicated) Kähler potential K, and some holomorphic superpotential $W(X)$.

We can test this hypothesis using the 't Hooft anomaly matching condition. There are many non-trivial 't Hooft anomalies for global symmetry group $G_F \times U(1)_R$ of the UV theory, massless SQCD. For instance:

$$\text{tr}(SU(N_f)^3_+) = -N_c \ , \tag{3.53}$$

for the $SU(N_f)$ factor under which Q_i transforms. We also have:

$$\mathrm{tr}(SU(N_f)^2 U(1)_B) = N_c \ , \tag{3.54}$$

and:

$$\mathrm{tr}(U(1)_R) = -N_c^2 - 1 \ , \qquad \mathrm{tr}(U(1)^3) = N_c^2 - 2\frac{N_c^4}{N_f^2} - 1 \ , \tag{3.55}$$

where we used the non-anomalous R-charge (3.8). *There are still more 't Hooft anomaly coefficients, whose computation is left as an exercice.*

On the other hand, the naive theory (3.52) would have:

$$\mathrm{tr}(SU(N_f)^3_+) = -N_f + A\left(\otimes_{\mathbf{A}}^{\mathbf{N_c}} \overline{\mathbf{N}}_{\mathbf{f}}\right) \ , \tag{3.56}$$

where the first term is the contribution from the mesons M, and the second term is the contribution from the baryons B to the cubic anomaly. In particular, we have:

$$\mathrm{tr}(SU(N_f)^3_+)\Big|_{N_f=N_c} = -N_f \ , \qquad \mathrm{tr}(SU(N_f)^3_+)\Big|_{N_f=N_c+1} = -N_f + 1 \ , \tag{3.57}$$

for $N_f = N_c$ or $N_f = N_c = 1$. From this and other 't Hooft anomalies, we see that the naive σ-model does *not* reproduce the 't Hooft anomalies of SQCD for generic N_f.

In fact, one can easily show that *all anomalies match if and only if* $N_f = N_c + 1$. In that case, the anomalies match almost miraculously. For instance, the cubic $U(1)_R$ 't Hooft anomaly in the σ-model is given by:

$$\mathrm{tr}(U(1)^3) = \frac{(N_f - 2N_c)^3}{N_f} + 2\frac{(N_c N_f - N_f - N_c^2)^3}{N_f^3}\binom{N_f}{N_c} \ . \tag{3.58}$$

This does match the corresponding SQCD anomaly in (3.55) for $N_f = N_c + 1$, namely:

$$\mathrm{tr}(U(1)^3)\Big|_{N_f=N_c+1} = -\frac{2N_c^4}{(N_c+1)^2} + N_c^2 - 1 \ . \tag{3.59}$$

3.5.4 $N_f = N_c$: deformed moduli space

For $N_f = N_c$, we just argued that the low energy theory at the origin of the moduli space cannot be simply a theory of massless mesons and baryons. To make a long story short, what happens in this case is that the origin of the moduli space *does not exist* quantum mechanically.

Recall that the classical moduli space is described as an hypersurface:

$$\det(M) - \widetilde{\mathcal{B}}\mathcal{B} = 0 \ . \tag{3.60}$$

Quantum mechanically, this relation is *deformed* to [3]:

$$\boxed{\det(M) - \widetilde{\mathcal{B}}\mathcal{B} = \Lambda^{b_0} \ ,} \qquad b_0 = 2N_c \ . \tag{3.61}$$

This is again a non-perturbative (one-instanton) effect. Note that the deformation is fully compatible with the (anomalous) symmetries from Table 2 with $N_f = N_c$.

Algebraically, this is an example of a *deformation* of a singularity—while the hypersurface (3.60) had a singularity at the origin, the deformed space (3.61) is smooth. [7]

3.5.5 $N_f = N_c + 1$: A σ-model

For $N_f = N_c + 1$, we saw above that we could saturate the 't Hooft anomalies with our naive σ-model of mesons and baryons. This cannot be the full description, however, since the moduli space has a lower dimension than the number of fields, $N_X = N_f^2 + 2N_f$. Instead, we should have a superpotential to impose some relations amongst the fields X. By symmetry, we can only have:

$$W = \alpha \frac{\det M}{\Lambda_0^b} + \beta \frac{\mathcal{B}^i M_i^j \widetilde{\mathcal{B}}_j}{\Lambda_0^b} \tag{3.62}$$

By various decoupling limits, one can fix $\alpha = -\beta = -1$. We thus claim that the correct superpotential is:

$$W = \frac{\mathcal{B}^i M_i^j \widetilde{\mathcal{B}}_j - \det M}{\Lambda_0^b} \ . \tag{3.63}$$

This is a rather strange result, since it does not seem to behave well in the classical limit, $\Lambda \to 0$. However, the numerator would also vanish in this limit, due to the classical constraints (3.37)—in particular, $\det M = 0$ classically since it is a matrix of rank $N_c < N_f$.

In the low-energy description, the F-term equations that follow from (3.63), when treating M and $\mathcal{B}$, $\widetilde{\mathcal{B}}$ as fundamental fields, give us:

$$\begin{aligned} \frac{\partial W}{\partial M_i^j} &= \mathcal{B}^i \widetilde{\mathcal{B}}_j - (M^{-1})_j^i \det M = 0 \ , \\ \frac{\partial W}{\partial \mathcal{B}^i} &= M_i^j \mathcal{B}^i = 0 \ , \\ \frac{\partial W}{\partial \widetilde{\mathcal{B}}_j} &= \widetilde{\mathcal{B}}_j M_i^j = 0 \ . \end{aligned} \tag{3.64}$$

These are precisely the constraints (3.37) that define the classical moduli space. Thus, for $N_f = N_c + 1$, the low-energy description seems to be in terms of a σ-model whose vacuum moduli space is exactly the same as in the UV description.

[7] Here, *deformation* is also a technical term; more precisely, we have a "complex structure deformation" of an algebraic variety.

3.5.6 $N_f > N_c + 1$: SCFTs, free theories, and duality

For $N_f > N_c + 1$, the low-energy physics is more interesting. It was elucidated in 1994 by Seiberg [4]. Here, we only give some brief summary of that beautiful subject, and we refer to the classic lecture notes by Intriligator and Seiberg [1] for further reading; see also Argyres' lecture notes (link in introduction) for a detailed and pedagogical account.

In order to describe what happens in SQCD with:

$$N_c + 1 < N_f \leq 3N_c \ , \tag{3.65}$$

It is perhaps easiest to first discuss what happens near the upper bound $N_f = 3N_c$. At precisely $N_f = 3N_c$, the Yang-Mills β-function vanishes at one-loop. We thus have have a fixed-point of the RG flow, at first order—a four-dimensional CFT. Consider the "exact" β function (2.46), which reads:

$$\beta\left(\frac{1}{g_c^2}\right) = \frac{3N_c - N_f(1-\gamma_\phi)}{8\pi^2 - N_c g_c^2} \ . \tag{3.66}$$

for SQCD; here, we used the global symmetries to equate all the anomalous dimensions,

$$\gamma_{Q_i} = \gamma_{\widetilde{Q}^j} \equiv \gamma_\phi \ . \tag{3.67}$$

For $N_f = 3N_c$, there is a fixed point, at all order in perturbation theory, if $\gamma_\phi = 0$. That is, if the squarks retain their classical dimensions, $\Delta = \frac{1}{2}$. Therefore, it seems that massless SQCD at $N_f = 3N_c$ is an "almost free" conformal field theory.

Now, for $N_f < 3N_c$, we could try to obtain a zero of the β function:

$$\beta\left(\frac{1}{g_c^2}\right) = 0 \qquad \Leftrightarrow \qquad \gamma_\phi = \frac{N_f - 3N_c}{N_f} \ , \tag{3.68}$$

where we used the global symmetries to equate all the anomalous dimensions. Consider, in particular, the case of large number of colors, $N_c \gg 1$ and $2N_c - N_f$ very small; then, the anomalous dimensions are arbitrarily small and one can understand the fixed point perturbatively. Such a fixed point is called a *Banks-Zaks fixed point;* it also occurs in QCD-like theories without supersymmetry [12]. All we need is a β-function of the form:

$$\beta\left(\frac{1}{g^2}\right) = \frac{b_0}{8\pi^2} - c_0 N_f g^2 + \mathcal{O}(g^4) \ , \tag{3.69}$$

at *two-loop order,* with $c_0 > 0$ a positive numerical constant. Then, we have a perturbative fixed point with a coupling constant:

$$g_*^2 = \frac{1}{8\pi^2}\frac{b_0}{c_0 N_f} \ll 1 \ , \tag{3.70}$$

if b_0 is smaller than $c_0 N_f$.

SCFTs. The claim is that there is a *non-trivial fixed point* in the IR of SQCD in the full range:

$$\frac{3N_c}{2} \le N_f \le N_c \ . \tag{3.71}$$

This is called the SQCD *conformal window.* The gauge coupling g_c^2 at the fixed point is small near the upper limit (for N_c and N_f sufficiently large, giving a Banks-Zaks fixed point), but becomes strong (with g_*^2 of order one) as we lower N_f, at fixed N_c. The lower bound on the conformal window comes about as follows. Any fixed point preserving supersymmetry necessarily enjoys a larger space-time symmetry algebra, called the $\mathcal{N}=1$ *superconformal algebra.* It has generators: [8]

$$\begin{array}{ccccc} & & P_\mu \ , & & \\ & Q_\alpha \ , & & \bar{Q}_{\dot\alpha} \ , & \\ \Delta \ , & & M_{\mu\nu} \ , & & R \ , \\ & S_\alpha \ , & & \bar{S}_{\dot\alpha} \ , & \\ & & K_\mu \ , & & \end{array} \tag{3.72}$$

generalising the super-Poincaré algebra. Here, Δ is the dilation operator, whose eigenvalues are the *quantum dimensions* (or just "conformal dimensions") of the operators, by definition, and R is the $U(1)_R$ charge. The R-charge is now a non-trivial part of the algebra. In 4d $\mathcal{N}=1$ superconformal field theories (SCFTs), the *scalar chiral operators* Φ satisfy a BPS-type relation tying up their R-charges and dimensions:

$$R[\Phi] = \frac{2}{3}\Delta[\Phi] \ . \tag{3.73}$$

Note that this relation is compatible with a classically-marginal (that is, conformally-invariant) superpotential. Another general fact about 4d CFTs (with or without supersymmetry) is that the dimension of any well-defined (gauge-invariant) operator must satisfy:

$$\Delta(\mathcal{O}) \ge 1 \ , \tag{3.74}$$

and the operator is free if and only if this so-called *unitary bound* is satisfied.

Consider the quantum dimension (3.68) for the squarks chiral superfields. Using (3.73), that implies:

$$R[Q] = R[\widetilde{Q}] = \frac{2}{3}\left(1+\frac{1}{2}\gamma_\phi\right) = 1 - \frac{N_c}{N_f} = r \ , \tag{3.75}$$

precisely the anomaly-free R-charge of SQCD. Then, for the gauge-invariant meson operators $M = \widetilde{Q}Q$ to satisfy the unitarity bound, we must have:

$$\Delta(\widetilde{Q}Q) = 3 - 3\frac{N_c}{N_f} \ge 1 \qquad \Leftrightarrow \qquad N_f \ge \frac{3}{2}N_c \ . \tag{3.76}$$

That explains the lower-bound on the conformal window.

[8] Here the generators are organised according to their conformal dimensions, from $\Delta[P_\mu]=1$ to $\Delta[K_\mu]=-1$. The Poincaré supercharges $Q_\alpha, \bar{Q}_{\dot\alpha}$ have dimension $\frac{1}{2}$, and the special conformal supercharges $S_\alpha, \bar{S}_{\dot\alpha}$ have dimension $-\frac{1}{2}$.

IR-free phase. Finally, we should discuss what happens in the window:

$$N_c + 1 < N_f < \frac{3}{2} N_c \ . \tag{3.77}$$

We saw that the naive mesons and baryons cannot give a good description of the origin of the moduli space. One heuristic reason is that the only superpotential term allowed for the mesons is of the form:

$$W \sim (\det M)^{\frac{1}{N_f - N_c}} \ , \tag{3.78}$$

which is singular at the origin. Such singularities in the effective action typically hint at the presence of light particles, which we forgot to take into account in the Wilsonian action. The extraordinary claim, due to Seiberg, is that one should describe the low energy physics in terms of some *IR-free gauge theory* with gauge group:

$$SU(N_f - N_c) \tag{3.79}$$

and N_f flavors in chiral multiplets q^i and $\widetilde{q}_j$, coupled to some additional N_f^2 gauge-singlets caled M_i^j, with a cubic superpotential:

$$W = \text{tr}\ \widetilde{q}_j M_i^j q^i \ . \tag{3.80}$$

This SQCD-like theory, which we call the "Seiberg-dual theory" of SQCD, has a β-function coefficient:

$$b_0^D = 3(N_f - N_c) - N_f = 2N_f - 3N_c \ , \tag{3.81}$$

which is *negative* in the window (3.77). Thus, indeed, it becomes a free theory in the infrared (and needs to be defined with a UV cut-off, at the scale Λ).

Let us repeat the claim: the low-energy theory for asymptotically-free SQCD in the window (3.77) is given in terms of an *IR-free gauge theory,* with gauge group $SU(N_f - N_c)$. The gauge bosons and matter fields of this "dual theory" have nothing to do with the original fundamental fields of SQCD in the UV. Nonetheless, one can check that all 't Hooft anomalies match between SQCD and the proposed IR description! The proposal, in fact, passes many other consistency checks, which we will not discuss here.

Seiberg duality. Even more amazingly, this relation between two different gauge theories, known as "Seiberg duality," extends all the way into the conformal window, where both the $SU(N_c)$ and the "dual" $SU(N_f - N_c)$ gauge group are asymptotically free. In that case, we have two well-defined asymptotically free gauge theories in the UV, written schematically as:

$$\boxed{SU(N_c)\ , \quad N_f\ , \quad W = 0 \quad \leftrightarrow \quad SU(N_f - N_c)\ , \quad N_f\ , \quad W = \widetilde{q} M q\ .} \tag{3.82}$$

They are certainly two different theories, with different numbers of degrees of freedom in the UV. The claim is that, in the conformal window, *they both flow to the same SCFT in the infrared.* Moreover, when one description is strongly coupled, the other is weakly coupled—that fact, as you can imagine, can be very useful.

We should point out that the above intricate picture of the quantum vacuum structure of SQCD has no definite proof for $N_f \geq N_c$, to this day, but it passes so many highly non-trivial consistency checks that its correctness is beyond any reasonable doubt.

Part II
Dynamics of 4d $\mathcal{N} = 2$ gauge theories

We now turn our attention to gauge theories with minimally extended ($\mathcal{N} = 2$) supersymmetry. Our aim will be to introduce the most important elementary facts about $\mathcal{N} = 2$ supersymmetric gauge theories, and then to discuss the Seiberg-Witten solution for their low-energy dynamics. For simplicity, we shall mainly focus on the simplest case of a single $SU(2)$ $\mathcal{N} = 2$ vector multiplet.

4 Gauge theories with $\mathcal{N} = 2$ supersymmetry

By way of introduction, let us consider an $\mathcal{N} = 1$ gauge theory consisting of an $\mathcal{N} = 1$ vector multiplet $\mathcal{V}$ for a gauge group G, and of a single chiral superfield Φ in the *adjoint representation* of G.

In $\mathcal{N} = 1$ superspace notation, the Lagrangian reads:

$$\mathscr{L} = \text{Im}\left[-\frac{\tau}{8\pi}\int d^2\theta\, \text{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha\right] + \frac{1}{g^2}\int d^2\theta d^2\bar{\theta}\ \text{tr}\ \bar{\Phi} e^{-2[V,-]}\Phi\ , \tag{4.1}$$

where V acts on Φ in the adjoind representation, as denoted by the commutator, and the holomorphic gauge coupling τ is as defined in (1.39). Here, we introduced a seemingly arbitrary normalization factor $1/g^2$ in front of the kinetic term for Φ. This is because of the (non-obvious) fact that this action actually preserves $\mathcal{N} = 2$ supersymmetry, and the normalization is the natural one in that case. To see some evidence for the presence of the extended supersymmetry, let us write down the action in components:

$$\begin{aligned}\mathscr{L}\ &= \frac{1}{g^2}\,\text{tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda + \frac{1}{2}D^2 - D_\mu\bar{\phi}D^\mu\phi \\ &\quad - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + \bar{F}F - \bar{\phi}[D,\phi] - i\sqrt{2}\bar{\phi}[\lambda,\psi] + i\sqrt{2}[\bar{\lambda},\bar{\psi}]\phi\Big) \\ &\quad - \frac{\theta}{64\pi^2}\,\text{tr}\left(\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\ .\end{aligned} \tag{4.2}$$

We claim that this is the action of a $\mathcal{N}=2$ vector multiplet. The $\mathcal{N}=2$ vector multiplet contains a complex scalar ϕ, two gauginos λ^I ($I=1,2$), a vector field A_μ, and three real auxiliary fields $D^{IJ}=D^{JI}$:

$$\mathcal{V}_{\mathcal{N}=2} = \left(\phi \ , \ \bar{\phi} \ , \ A_\mu \ , \ \lambda^I \ , \ \bar{\lambda}_I \ , \ D^{IJ}\right) \ , \tag{4.3}$$

for a total of $8+8$ off-shell degrees of freedom, with all fields transforming in the adjoint representation of $\mathfrak{g}=\mathrm{Lie}(G)$. In the $\mathcal{N}=1$ notation above, we have:

$$(\lambda^I) = (\lambda, \psi) \ , \qquad D^{IJ} \ \sim (D, F, \bar{F}) \ . \tag{4.4}$$

Let us integrate out the auxiliary fields. Then, the Lagrangian (4.2) becomes:

$$\begin{aligned} \mathscr{L}_{\mathcal{N}=2\,\mathrm{SYM}} &= \frac{1}{g^2}\,\mathrm{tr}\,\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\bar{\phi}D^\mu\phi - i\bar{\lambda}_I\bar{\sigma}^\mu D_\mu\lambda^I \\ &\quad -\frac{1}{2}[\bar{\phi},\phi]^2 - \frac{i}{\sqrt{2}}\bar{\phi}\epsilon_{IJ}[\lambda^I,\lambda^J] + \frac{i}{\sqrt{2}}\epsilon^{IJ}[\bar{\lambda}_I,\bar{\lambda}_J]\phi\Big) \ , \end{aligned} \tag{4.5}$$

plus the topological term (with coupling constant θ), which is separately supersymmetric. This Lagrangian is classically invariant under a $SU(2)_R\times U(1)_r$ R-symmetry, with $I=1,2$ the $SU(2)_R$ index. We will explore this important point momentarily.

Note also the form of the scalar potential that appears in (4.5):

$$V_0(\phi,\bar{\phi}) = \frac{1}{2g^2}\,\mathrm{tr}\left([\bar{\phi},\phi]^2\right) \ . \tag{4.6}$$

If the theory were only $\mathcal{N}=1$ supersymmetric, we could have considered adding a superpotential term $W(\Phi)$ for the adjoint scalar ϕ, and that would contribute additional terms to the scalar potential. The simple form (4.6) is dictated by the requirement of extended supersymmetry.

4.1 $\mathcal{N}=2$ supersymmetry and R-symmetry

The $\mathcal{N}=2$ supersymmetry algebra takes the form:

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} &= 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu\,\delta^I{}_J \ , \\ \{Q^I_\alpha, Q^J_\beta\} &= 2\varepsilon_{\alpha\beta}\epsilon^{IJ}\bar{Z} \ , \\ \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= 2\varepsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}Z \ , \end{aligned} \tag{4.7}$$

with $I,J=1,2$ and $\epsilon^{12}=\epsilon_{12}=1$ for the $SU(2)_R$ indices. The maximal R-symmetry of the $\mathcal{N}=2$ supersymmetry algebra is:

$$U(2) \cong SU(2)_R\times U(1)_r \ , \tag{4.8}$$

with the charge assignements:

$$
\begin{array}{c|c|c|c}
 & SU(2)_R \times U(1)_r & T_3^{SU(2)_R} & U(1)_R^{\mathcal{N}=1} \\
\hline
Q^I_\alpha & (\mathbf{2})_{-1} & \pm\frac{1}{2} & -\delta^I{}_1 \\
\bar{Q}_{\dot\alpha J} & (\bar{\mathbf{2}})_1 & \mp\frac{1}{2} & +\delta^I{}_1 \\
\hline
Z & (\bar{\mathbf{1}})_2 & 0 &
\end{array}
\tag{4.9}
$$

The presence of a supercharge $Z \neq 0$ breaks the $U(1)_r$ symmetry explicitly. Regardless, one can keep track of the symmetry by assigning $r(Z) = 2$ to the central charge, as indicated.

Here we also considered an $\mathcal{N} = 1$ subalgebra of the $\mathcal{N} = 2$ superalgebra, corresponding to the supercharges $Q^{I=1}, \bar{Q}_{I=1}$. The corresponding the $\mathcal{N} = 1$ $U(1)_R$ symmetry is simply:

$$
R_{\mathcal{N}=1} = \frac{1}{2} r - T_3^{(R)} \ , \tag{4.10}
$$

where r is the $U(1)_r$ charge, and T_3 is the generator in the Cartan of $SU(2)_R$.

4.1.1 The massless vector multiplet

We can easily build the massless one-particle multiplets (see section 3.3.1 of 'Supersymmetry and Supergravity'). The $\mathcal{N} = 2$ vector multiplet takes the form:

$$
\begin{aligned}
&\Big|\lambda = -1;\, (\mathbf{1})_0\Big\rangle \xrightarrow{a_I^\dagger} \Big|\lambda = -\frac{1}{2};\, (\bar{\mathbf{2}})_1\Big\rangle \xrightarrow{a_I^\dagger} \Big|\lambda = 0;\, (\mathbf{1})_2\Big\rangle \ , \\
&\Big|\lambda = 0;\, (\mathbf{1})_{-2}\Big\rangle \xrightarrow{a_I^\dagger} \Big|\lambda = \frac{1}{2};\, (\bar{\mathbf{2}})_{-1}\Big\rangle \xrightarrow{a_I^\dagger} \Big|\lambda = 0;\, (\mathbf{1})_0\Big\rangle \ ,
\end{aligned}
\tag{4.11}
$$

where we indicated the helicities and the R-charges of the states. The gauge field A_μ corresponds to the $|\lambda| = 1$ states, which must be neutral under the R-symmetry. This then determines the R-charges of the other states. Note $\mathbf{2} \cong \bar{\mathbf{2}}$, so that the fermions, with helicity $|\lambda| = \frac{1}{2}$, form a CPT-invariant set. Their free-field realization is in term of the $\mathcal{N} = 2$ gaugino $\lambda^I_\alpha, \bar\lambda_{\dot\alpha I}$:

$$
\begin{array}{c|ccccc}
 & A_\mu & \lambda^I & \bar\lambda_I & \phi & \bar\phi\ , \\
\hline
SU(2)_R \times U(1)_r & (\mathbf{1})_0 & (\mathbf{2})_1 & (\bar{\mathbf{2}})_{-1} & (\mathbf{1})_2 & (\mathbf{1})_{-2}
\end{array}
\tag{4.12}
$$

Note that the complex scalar ϕ has $U(1)_R$ charge 2.

4.1.2 Massive multiplets and BPS bound

Consider massive particle states. A generic massive one-particle multiplet contains $8+8$ on-shell degrees of freedom. On the other hand, in the presence of non-trivial central charges, there can short representations for massive particle, called the *BPS states.* [9] Let us see how this comes about.

[9] Note that 'BPS' stands for Bogomolnyi-Prasad-Sommerfield.

On massive one-particle states, with $P_\mu = (-M,0,0,0)$, the supersymmetry algebra takes the form:

$$\begin{aligned}
&\{Q^I_1, \bar{Q}_{\dot{1}J}\} = \{Q^I_2, \bar{Q}_{\dot{2}J}\} = 2M\delta^I{}_J \ , \\
&\{Q^I_1, Q^J_2\} = -2\bar{Z}\epsilon^{IJ} \ , \\
&\{\bar{Q}_{\dot{1}I}, \bar{Q}_{\dot{2}J}\} = -2Z\epsilon_{IJ} \ ,
\end{aligned} \tag{4.13}$$

with all the other anticommutators vanishing. Let us define the operators:

$$\begin{aligned}
a^I &= \frac{1}{\sqrt{2}}(Q^I_1 + \alpha_{(I)}\epsilon^{IJ}\bar{Q}_{\dot{2}J}) \ , \qquad & a^\dagger_I &= \frac{1}{\sqrt{2}}(\bar{Q}_{\dot{1}I} + \bar{\alpha}_{(I)}\epsilon_{IJ}\bar{Q}^J_2) \ , \\
b^I &= \frac{1}{\sqrt{2}}(Q^I_1 - \alpha_{(I)}\epsilon^{IJ}\bar{Q}_{\dot{2}J}) \ , \qquad & b^\dagger_I &= \frac{1}{\sqrt{2}}(\bar{Q}_{\dot{1}I} - \bar{\alpha}_{(I)}\epsilon_{IJ}\bar{Q}^J_2) \ ,
\end{aligned} \tag{4.14}$$

with α_1, α_2 some pure phases. One can check that the only non-zero commutators amongst these operators are:

$$\begin{aligned}
&\{a^I, a^\dagger_J\} = \delta^I{}_J\big(2M - \alpha_{(I)}\bar{Z} - \bar{\alpha}_{(I)}Z\big) \ , \\
&\{b^I, b^\dagger_J\} = \delta^I{}_J\big(2M + \alpha_{(I)}\bar{Z} + \bar{\alpha}_{(I)}Z\big) \ , \\
&\{a^I, b^\dagger_J\} = \delta^I{}_J\big(-\alpha_{(I)}\bar{Z} + \bar{\alpha}_{(I)}Z\big) \ , \\
&\{a^\dagger_I, b^J\} = \delta^I{}_J\big(\alpha_{(I)}\bar{Z} - \bar{\alpha}_{(I)}Z\big) \ .
\end{aligned} \tag{4.15}$$

For any fixed Z, we can choose $\alpha_{(I)} = e^{i\arg(Z)}$, so that the RHS of the last two lines in (4.15) vanish. We then find the interesting conditions:

$$\{a^I, a^\dagger_I\} = 2\,(M + |Z|) \geq 0 \ , \qquad \{b^I, b^\dagger_I\} = 2\,(M - |Z|) \geq 0 \ , \tag{4.16}$$

(no summation on I here), since $\{a^I, a^\dagger_I\}$ and $\{b^I, b^\dagger_I\}$ are positive-definite. These so-called BPS inequalities are very important in the study of 4d $\mathcal{N}=2$ quantum field theories.

We can easily study the supermultiplets of one-particle states, as before. For $M \neq \pm|Z|$, we have an ordinary massive multiplet, also known as a *long multiplet.* It has $2^4 = 16$ states, 8 bosonic and 8 fermionic. When the so-called *BPS condition:*

$$M = |Z| \ , \tag{4.17}$$

is satisfied, on the other hand, we have a *short multiplet*, with half the number of components, since $b^I, b^\dagger_I$ in (4.16) are then realised trivially on one-particle states.

4.2 The classical Coulomb branch

The $\mathcal{N}=2$ SYM theory (4.5) has a classical vacuum moduli space determined by (4.6), namely:

$$[\phi, \bar{\phi}] = 0 \ , \tag{4.18}$$

modulo G gauge transformations. Semi-classically, this can be described simply, as follows. The vanishing of the commutator (4.18) implies that ϕ and $\bar{\phi}$ can be diagonalized simultaneously, by a $\mathfrak{g}$ transformation—that is, ϕ can be conjugated to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This fixes the gauge up to gauge transformations in the Weyl group W_G of G. This classical moduli space thus takes the form

$$\mathcal{M}_C = \mathfrak{h}_{\mathbb{C}}/W_G \ . \tag{4.19}$$

It is called the *Coulomb branch*, for reason that will be clear momentarily. Its complex dimension is the rank of G.

We will focus on $G = SU(2)$. Then, the scalar ϕ can be conjugated to:

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \ , \qquad a \in \mathbb{C} \ . \tag{4.20}$$

The Weyl group of $SU(2)$ is $S_2 = \mathbb{Z}_2$, and it acts on a as a sign invertion:

$$W_{SU(2)} = \mathbb{Z}_2 \ : \ a \to -a \ . \tag{4.21}$$

It is clear that the VEV (4.20) with $a \neq 0$ breaks the gauge group according to the pattern:

$$SU(2) \to U(1) \ , \tag{4.22}$$

by the Higgs mechanism. For a general G, we have:

$$G \to U(1)^{r_0} \ , \qquad r_0 \equiv \text{rank}(G) \ . \tag{4.23}$$

at a generic point on the Coulomb branch. Thus, classically, the low-energy dynamics is dictated by the Higgs mechanism. The strict IR limit contains only massless abelian gauge fields $A_\mu^{U(1)}$, and their $\mathcal{N} = 2$ superpartners in abelian vector multiplets $\mathcal{V}_{\mathcal{N}=2}^{U(1)}$. Any vacuum moduli space of a supersymmetric theory with this property is called a *Coulomb branch,* because of the presence of long-range interactions between charged particle in the IR through the 'Coulombic' interactions mediated by the abelian gauge fields. [10]

On the Coulomb branch (4.20) of the $SU(2)$ theory, we also have a perturbative massive vector field, the W-boson, of mass:

$$M_W = 2|a| \ . \tag{4.24}$$

At large distance on the Coulomb branch, that is with $|a| \to \infty$, the semi-classical Higgs mechanism gives the full physics. On the other hand, we will see that, at finite distance on the Coulomb branch, strong coupling effects will change the low energy dynamics significantly.

[10] In addition, we also have attractive forces from the exchange of the massless scalars. For a BPS particle, the repulsive Coulombic forces and the scalar forces cancel out exactly.

4.3 't Hooft-Polyakov monopoles on the Coulomb branch

In our discussion below, we will need a key fact about the classical $SU(2)$ gauge theory spontaneously broken to $U(1)$, which is that it allows for topologically non-trivial *solitons*, called 't Hooft-Polyakov monopoles. For further reference, see the lectures [13]. [11]

First, some basic definitions, to avoid any possible confusions. In any (classical) relativistic field theory:

- an *instanton* is a non-trivial solution of the classical equations of motion, $\varphi = \varphi_0(x)$, that has finite action, $S[\varphi_0] < 0$.
- a *soliton* is a non-trivial solution of the classical equations of motions, $\varphi = \varphi_0(x)$, that has finite energy, $E < 0$.

We discussed Yang-Mills instantons before; they are solutions of the Yang-Mills equations, setting all other fields in the theory to zero. By contrast, monopoles are solitons that exist in theories with both gauge fields and scalar. Consider the $G = SU(2)$ theory with an adjoint hermitian scalar φ (that is, $\varphi^\dagger = \varphi$), with action:

$$S = \frac{1}{g^2}\int d^4x \, \mathrm{tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\varphi D^\mu\varphi\Big) \; . \tag{4.25}$$

The energy of any classical field configuration reads:

$$E[B,\varphi] = \frac{1}{2g^2}\int d^3x \, \mathrm{tr}\Big(B_iB_i + D_i\varphi D_i\varphi\Big) \tag{4.26}$$

Here, we defined the electric and magnetic field in the usual way:

$$F_{0i} = E_i \; , \qquad F_{ij} = \epsilon_{ijk}B^k \; . \tag{4.27}$$

Now, we have the following trick:

$$\begin{aligned} E[B,\varphi] &= \frac{1}{2g^2}\int d^3x \left(\mathrm{tr}(B_i \mp D_i\varphi)^2 \pm 2\,\mathrm{tr}(B_iD_i\varphi)\right) \\ &\geq \pm\frac{1}{g^2}\int d^3x \, \mathrm{tr}\, B_iD_i\varphi = \pm\frac{1}{g^2}\int_{S^2_\infty} dn^i \, \mathrm{tr}(B^i\varphi) \end{aligned} \tag{4.28}$$

In the second line, we used the fact that the term $|B_i \mp D_i\varphi|^2$ is positive definite, and that the remaining term is a total derivative, so that the final integral is over the sphere at spatial infinity. In this way, we learn that the energy of any field configuration is bounded by the magnetic charge of the gauge field times the VEV of the scalar φ at spatial infinity.

[11] Like for our discussion of instantons in previous lectures, we cannot do justice to this beautiful subject here. I encourage you to read more broadly about solitons in classical field theories.

The lower bound (4.28) on the soliton energy is also called 'the BPS bound.' The bound is saturated by solutions that satisfy the Bogolmonyi equations:

$$B_i = D_i\varphi \ . \tag{4.29}$$

This is essentially the condition for the solution for the bosonic background (A_μ, ϕ) of an $\mathcal{N} = 2$ $SU(2)$ vector multiplet to preserve half of the supersymmetry.

The 't Hooft-Polyakov monopole is a specific solution for an adjoint scalar with boundary conditions set at spatial infinity, and the solution can be chosen to saturates the BPS bound. Let us pick the VEV (4.20) for the complex scalar ϕ. By a $U(1)_r$ rotation, we can choose $a = |a| \in \mathbb{R}$, for simplicity. This is thus equivalent to:

$$\varphi = \begin{pmatrix} |a| & 0 \\ 0 & -|a| \end{pmatrix} \ , \tag{4.30}$$

up to some unimportant constant factor. This breaks $SU(2)$ to $U(1)$, and the profiles for the $SU(2)$ gauge field compatible with the VEV are then of the form:

$$A_\mu = \begin{pmatrix} A_\mu^{U(1)} & 0 \\ 0 & -A_\mu^{U(1)} \end{pmatrix} \ , \tag{4.31}$$

up to gauge transformations. The monopole solution is then indexed by the the magnetic flux of this abelian gauge field at spatial infinity:

$$\mathfrak{m} = \frac{1}{2\pi} \int_{S^2_\infty} dn^i B_i^{U(1)} \ . \tag{4.32}$$

and the VEV (4.30) for the scalar field. It has energy: [12]

$$E = M_{\text{monopole}} = \frac{4\pi}{g^2} |a| \mathfrak{m} \ . \tag{4.33}$$

In conclusion, on the Coulomb branch of the $SU(2)$ $\mathcal{N} = 2$ SYM theory, there exists monopoles in the classical theory. Note that their mass goes like $\frac{1}{g^2}$, and therefore they are very massive in the perturbative regime, $g^2 \ll 1$ (much more massive than the W-boson, whose mass is given by (4.24)).

In the quantum theory, we should view the monopoles as particle excitations, on par with the W-boson. Due to the BPS bound, the monopole with the lowest magnetic charge is a stable particle. Note that, while the W-boson is electrically charged under the $U(1)$ gauge symmetry in the IR, the monopole is magnetically charged.

Simply comparing the semi-classical masses $M_W \sim |a|$ and $M_{\text{monopole}} \sim \frac{|a|}{g^2}$, we may suspect that the W-bosons and the monopoles become much more similar in the strong coupling regime. This is indeed the case. As we will see next, in the full quantum field theory, the gauge coupling becomes strong near the origin of the Coulomb branch, and the monopoles play a role very similar to the W-bosons in the full quantum theory.

[12] Up to some numerical factor we did not keep track of.

5 Theories of abelian vector multiplets

5.1 Stating the problem: "solving $\mathcal{N}$=2 SYM"?

5.2 $U(1)$ gauge fields and electro-magnetic duality

5.3 The $\mathcal{N}$=2 prepotential for an abelian vector multiplet

5.4 Semi-classical dynamics: anomalies and one-loop running

5.5 Abelian vector multiplet coupled to hypermultiplet

6 The Seiberg-Witten solution for pure $SU(2)$

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