Lecture notes on

Supersymmetry and supergravity

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Contents

Caveat emptor and recommended readings

These lecture notes are very much a work in progress, intended exclusively to accompany the MMathPhys lectures in Oxford during Hilary term 2020. They will be updated regularly during term. Similar comments hold for the lectures notes for Advanced Supersymmetry. Please let me know (by email or in person) of any mistake and confusion you may encounter.

The content of these lectures is standard. I have attempted to collect the most important aspects of supersymmetry that can be explained in a set of introductory lectures. This, of course, involved some arbitrary choices, and I would encourage you to read more broadly.

I also aimed to present the material in such a way that, once you understand supersymmetry in our own space-time (4d Minkowski space-time, to a good approximation), you can easily move on to other dimensions and other contexts, depending on your needs and interests.

Last updated on: April 22, 2020.

References:

The main sources for these lectures are:

- The Wess and Bagger book [1], the most widely used reference for 4d $\mathcal{N}=1$ supersymmetry. Chapters I to VIII are recommended reading.
- Weinberg's QFT book, Volume III [2]. A great book if you need more detail. Not the easiest read but always worth the effort.
- The lecture notes on 4d $\mathcal{N}=1$ supersymmetry by Philip Argyres, available on his website: `http://homepages.uc.edu/~argyrepc/cu661-gr-SUSY/index.html`.
- The original literature cited in the notes.

Other references you may find useful:

- The lectures by Joseph Conlon, here in Oxford: `https://www-thphys.physics.ox.ac.uk/people/JosephConlon/LectureNotes/SUSYLectures.pdf`.
- The supersymmetry lectures by Adel Bilal [3].
- The very detailed supersymmetry lecture notes by Matteo Bertolini: `https://people.sissa.it/~bertmat/susycourse.pdf`.
- A recent set of lectures by Yuji Tachikawa [4] dealing with more recent developments.
- ...

1 Supersymmetry: why and what?

1.1 Motivations for supersymmetry

We are accustomed to *symmetries* playing an important role in physics. This is especially true for the (so-called) fundamental physics of the XXth century, from special relativity to quantum mechanics to QFT.

In classical physics, continuous symmetries are associated to conservation laws, by Noether's theorem. More precisely, let us consider classical field theory in Minkowski space-time $\mathbb{R}^{1,d-1}$ (that includes ordinary mechanics for $d = 1$). For any *Lagrangian* mechanics with degrees of freedom $\phi(x)$, [1] and action:

$$S[\phi] = \int d^d x \, \mathscr{L}(\phi(x), \partial_\mu \phi(x)) \ , \tag{1.1}$$

a continuous Lie group symmetry is closely related to the existence of conserved currents. Let us have a symmetry group $\mathbf{G}$ with algebra $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$, with infinitesimal action:

$$\phi(x) = \phi(x) + \epsilon^a F_a(\phi(x), \partial_\mu \phi(x), \cdots) + O(\epsilon^2) \ . \tag{1.2}$$

Here, the parameters ϵ^a, with $a = 1, \cdots, \dim(\mathfrak{g})$, run over the group generators. (In other words, $\epsilon \in \mathfrak{g}$.) Let us assume the action (1.1) is left invariant by the infinitesimal transformation (1.2), in the sense that:

$$\delta_\epsilon S[\phi] = S[\phi + \epsilon^a F_a] - S[\phi] = 0 \ . \tag{1.3}$$

This means that the symmetry variation of the Lagrangian is a total derivative: [2]

$$\delta_\epsilon \mathscr{L}(\phi) = \epsilon^a \partial_\mu \Lambda_a^\mu \ . \tag{1.4}$$

Then, there exist conserved currents, the Noether currents, given by:

$$j_a^\mu(x) = \Lambda_a^\mu - \frac{\partial \mathscr{L}}{\partial \, \partial_\mu \phi} F_a \ , \qquad \partial_\mu j^\mu(x) = 0 \ . \tag{1.5}$$

Here, for simplicity, we assumed the Lagrangian is at most of second order in derivatives. We have also used the equations of motions. [3]

Symmetries play a similarly important role in quantum mechanics (QM) and in quantum field theory (QFT). In a QFT in space-time dimension d with continuous symmetries, the Noether currents $j_a^\mu(x)$ are now viewed as *local operators*, and the generator of the symmetry on the QFT Hilbert space is the charge operator:

$$Q_a = \int_{\Sigma_{d-1}} d^{d-1}x \, j_a^0(x) \ . \tag{1.6}$$

[1] We use the notation $x = (x^\mu) \in \mathbb{R}^{1,d-1}$.

[2] Here, we assume that the space-time measure $d^d x = dt dx^1 \cdots dx^{d-1}$ of Minkoswski space-time is left invariant by the symmetry. This is the case, in particular, for the Poincaré symmetries (translations, rotations and Lorentz boosts).

[3] We should note that the definition of any Noether current j^μ is slightly ambiguous, since one can always shift j_μ to $j^\mu + \partial_\nu B^{\mu\nu}$, with $B^{\mu\nu}$ an antisymmetric tensor.

Here, the integration is over a spatial slice $\Sigma_{d-1} \cong \mathbb{R}^d$ at constant time t. The conservation equation, $\frac{d}{dt}Q_a = 0$, follows from $\partial_\mu j^\mu = 0$, assuming appropriate boundary conditions for the fields at spatial infinity. [4]

1.1.1 A very brief history of supersymmetry

A bit more than fifty years ago (in the 1960's), Particle Physics was in state of creative confusion. There were many new "elementary" particles being discovered (all the many "mesons" and "baryons"...), and no obvious way to organise them and explain their interactions. Recall that, a little bit earlier, quantum electrodynamics (QED) was developed and gave us an extremely satisfactory quantum theory of the photon interacting with charged particles, [5] using the explicit Hamiltonian of a Maxwell potential A_μ coupled to matter, and relying on perturbation theory.

When it came to the interactions of hadrons—the baryons (proton, neutron,...) and mesons—, that success seemed hard to reproduce at the time, however. This led to a proliferation of new and clever ideas, many of which are still being pursued today. One general theme, at the time, was to try and "define" QFT by its observables only (as we should, in quantum mechanics), without relying on a particular "microscopic" Hamiltonian or Lagrangian. For this approach to work, one would have to rely heavily on symmetries (as well as on other more subtle but general principles, such as unitarity). The only observables in scattering experiments are the S-matrix elements, and therefore a natural question was: *What is the more general symmetry of the S-matrix?*

Coleman-Mandula theorem. An apparently definitive answer was given in a famous paper by Coleman and Mandula, in 1967, in the case of continuous symmetries [5]. The Coleman-Mandula (CM) theorem deals with a relativistic QFT in $\mathbb{R}^{1,3}$ such that (i) only a finite number of particle types [6] are associated with one-particle states of any given mass, (ii) there exists an energy gap between the vacuum and one-particle states, (iii) there is non-trivial scattering. Then, the most general symmetry of the S-matrix is:

$$\text{Poincaré} \times G_{\text{internal}} \; . \tag{1.7}$$

The Poincaré group is the invariance of the relativistic QFT, by assumption: [7]

$$\text{Poincaré} \equiv ISO(1, d-1) \equiv SO(1, d-1) \ltimes \mathbb{R}^{1,d-1} \; , \tag{1.8}$$

[4] Recall the one-line derivation:

$$\frac{d}{dt}Q_a = \int_\Sigma \partial_t j^t = -\int_\Sigma \partial_i j^i = 0 \; ,$$

where i are the space coordinate indices, $(x^\mu) = (x^0, x^i) = (t, x^i)$.

[5] The Nobel prize for QED was awarded to Schwinger, Feynman and Tomonaga in 1965.

[6] Here, by "particle type," we mean positive-energy representations of the Poincaré group.

[7] The Poincaré group is also know as the "inhomogeneous Lorentz group," hence the name $ISO(1, d-1)$. The translations, $\mathbb{R}^{1,d-1}$, form a normal subgroup.

and here we are specifically in the case $d = 4$. The theorem deals with the symmetry group component connected to the identity. In other words, it states that, in a non-trivial "massive" QFT, the most general space-time symmetry *Lie algebra* is the Poincaré algebra (the generators of rotations, boosts, translations), and any other "internal symmetries" G_{internal} must commute with it. (For instance, the historically important 'isospin,' and any similar 'flavor symmetries.')

Note that we assumed that the theory was "massive"—there is an energy gap in the spectrum. One way to evade the theorem is to consider theories without any built-in mass scales (for instance, with massless particles). Then, we can have a larger group of space-time symmetries, the conformal group $SO(2, d)$—this would lead us to the study of conformal field theories (CFT).

An important (implicit) assumption in the CM theorem, as stated above, is that the symmetry generators are *bosonic* operator, so that they satisfy *commutation relations.* That assumption can be relaxed. If we allow for *fermionic* generators, which satisfy *anti-commutation* relations, we can generalize the CM theorem, by replacing the Poincaré algebra with a so-called *super-Poincaré* algebra. This is the content of the Haag-Lopuszanski-Sohius theorem [6], from 1975. (The theorem was worked out after supersymmetry had been already discovered explicitly in QFTs.)

Historically, supersymmetry first appeared in the early days of string theory, in a paper of 1971 by Neveu and Schwarz [7]. This was supersymmetry on the string worlsheet, a $d = 2$ field theory—the first appearance of what we now call 2d superconformal theories. (See [8] for further details and references.)

The $d = 4$ (4d $\mathcal{N} = 1$, in modern language) *super-Poincaré* algebra was first introduced at the same time by Golfand and Likhtman [9], in 1971 in the USSR. That work went unnoticed by the larger community for several years, however. Then, 4d supersymmetry was independently discovered by Wess and Zumino in 1974 at CERN [10, 11]. This launched a sustained investigation of supersymmetric QFTs (and supergravity theories), which is still on-going 46 years later.

1.1.2 Motivations for the particle physicist

Once 4d supersymmetry was *theoretically* discovered, in the mid-1970s, it became natural to expect that it might play an important role in particle physics. Real-world particles do not form representations of the supersymmetry algebra, however. This leaves us with the tantalising possibility that supersymmetry might be *spontaneously broken* at the relatively low energies probed by particle accelerators—up to the TeV scale (so far).

Let us say from the get-go that we do not know of any airtight argument why supersymmetry should be required in Nature, and certainly not why it should be experimentally accessible in the near future. One can hope, however.

GUT and supersymmetry. One curious observation indirectly hints at supersymmetry at the TeV scale, however weakly. It has been a long-running idea in Particle Physics that the gauge group of the Standard Model (SM), $G_{\text{SM}} = SU(3) \times SU(2) \times U(1)$, might be "unified" by embedding into a larger GUT ('grand unified theory') gauge group G_{GUT}, such as $SU(5)$ or $SO(10)$. At some GUT scale M_{GUT}, the gauge group G_{GUT} would be spontaneously broken down to G_{SM}. The known renormalisation group (RG) running of the SM gauge coupling constants is approximately compatible with this scenario. When a particular *supersymmetric* version [8] of the Standard Model is considered instead, the agreement of the coupling-constant unification with a GUT scenario becomes much more impressive, with $M_{\text{GUT}} \sim 10^{16}$GeV.

The hierarchy problem. The other and main reason why many particle physicists believe (or believed, until recently) that supersymmetry might be discovered at the LHC is called *"naturalness,"* which is a rather theoretical—some would say, philosophical—construct. Naturalness already motivated searches for supersymmetry at LEP, the precursor of LHC at CERN. At the time of this writing, the LHC results from proton-proton collisions at 13TeV are all in perfect agreement with the Standard Model, and show no hint of supersymmetry. The jury is still out, but the mood is rather more somber than a few years ago for those who hoped for supersymmetry at the LHC.

What is naturalness, then? Roughly, it is the idea that numbers in physics that are very small must be small for a reason. The Standard Model, with its Higgs boson at 125GeV, does a great job at explaining the experiments so far. What is odd, however, is that the Higgs-potential parameters in the standard model seem "fine-tuned." The Higgs mass term in the SM Lagrangian is renormalised very strongly ("quadratically" in the RG scale, unlike the fermion masses, which only run logarithmically), and its "natural" scale should be at whatever threshold at which new physics appears. Since we know experimentally that the electroweak (EW) scale is at $\sim$100GeV, this means that the Higgs mass is fined-tuned—in QFT, one has to cancel two very large "bare" numbers against each other "by hand." This is, in a nutshell, the so-called hierarchy problem: what explains the apparent 'hierarchy' between the EW scale at 10^2GeV and whatever scales kicks in next—for instance the GUT scale at 10^{16}GeV, or the Planck scale of quantum gravity at 10^{19}GeV. The "natural" solution is that there is a scale with new Physics soon after the EW scale. Supersymmetry offers such models of particle physics beyond the SM which would "solve the hierarchy problem." However, even if supersymmetry were discovered tomorrow, there would still be some relative "unnaturalness" creeping in—a fully natural solution to the hierarchy problem has (likely) already been ruled out experimentally.

At this point, I should also point out that I am not an expert in Particle Physics, at all. I encourage you to directly ask experts in the Physics Department about what is the current status of supersymmetry experimentally. The situation is likely to keep evolving in the next few years.

[8] The MSSM with one Higgs doublet. See section 28.2 of Weinberg [2] for a detailed discussion.

1.1.3 Motivations for quantum field theorist and/or string theorist

However successful the Standard Model may be—and it is *very* successful—, we know it is not the full story. If anything, it does not incorporate gravity. Our best candidate for a *quantum theory of gravity,* to this day, is String Theory.

Supersymmetry historically appeared within string theory, and is intimately tied to it. In fact, supersymmetric QFT is part of the larger picture of (super)string theory in multiple, complementary ways. To name a few:

- The string worldsheet itself can be described by a 2d superconformal theory (2d SCFT). Interesting subsectors of string theory on non-trivial geometry are captured by observables in 2d supersymmetric theories, including 2d supersymmetric gauge theories.

- Supersymmetric quantum field theories (without gravity) in every possible dimensions ($d \leq 10$) appear naturally in the open-string sector of string theory; in particular, at low energy on so-called D-branes—and also on M-branes in 11-dimensional M-theory. The closed-string sector of string theory also contains *supergravity theories,* which tie supersymmetry with general relativity.

- The AdS/CFT duality, discovered by Maldacena in 1997 [12], is the statement that various QFT_d's with *conformal invariance* are *dual* to quantum gravity on an Anti-de-Sitter space-time in $d + 1$ dimensions (AdS_{d+1}). This means that, although a QFT looks very much different from a gravity theory in curved space-time, quantum-mechanically they are one and the same things: once we understand the proper dictionary between the two languages, all the observables agree! The best-studied instances of the AdS/CFT duality involve supersymmetric field theories dual to superstring theory in AdS. Thus, in the last 20 years, supersymmetric QFT (and, more precisely, superconformal theory) has become a tool to study quantum gravity.

Moreover, independently of string theory, supersymmetry is an important toolbox to better understand QFT more generally. Many problems that would be too hard in ordinary QFT, with our current technology, can be tackled analytically in the supersymmetric context.

Let us just give one outstanding example: We still have no analytic theoretical tools to study the confinement of quarks in QCD, because we lack the tools to address the strong-coupling regime quantitatively. On the other hand, the low-energy solution of 4d $\mathcal{N} = 2$ supersymmetric gauge theories by Seiberg and Witten in 1994 [13] provided a rather *explicit* derivation of confinement in some supersymmetric version of QCD (by monopole condensation, an idea that appeared in early work by Gerard 't Hooft).

This is arguably the one main motivation for many people who study supersymmetric QFT these days: to understand better Quantum Field Theory itself.

1.1.4 Motivations for the mathematician

In recent decades, there has been a very fruitful interplay between ideas in pure mathematics—especially, but not only, in geometry—and development in string theory and QFT. It works both ways: mathematical ideas inform and inspire the work of physicists, but results in theoretical physics have also often produced surprising mathematical conjectures, which are then studied by mathematicians in their own right.

Supersymmetry is the central beam on this bridge between Physics and Math. For instance, one can get a "physics proof" of the Atiyah-Singer index theorem in the context of supersymmetric quantum mechanics. More recently, there has been a rich interplay between Physics ideas in supersymmetric QFT (in particular, S-duality in 4d $\mathcal{N} = 4$ theory) and the geometric Langlands correspondence in algebraic geometry [14]. The list goes on and on.

1.2 Supersymmetry: a first definition

As mentioned above, supersymmetry is the only way to extend the Poincaré symmetry group by evading the Coleman-Mandula theorem. (In the case of QFTs with an energy gap.) It evades the CM theorem by introducing new generators which satisfy *anti-commutation* relations amongst themselves. The anti-commutator of two fermionic operators $\mathcal{A}$ and $\mathcal{B}$ is denoted by:

$$\{\mathcal{A}, \mathcal{B}\} \equiv \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} . \tag{1.9}$$

Aside: quantizing fermions. Recall that **fermions** in QFT obey Fermi-Dirac statistics: we pick a minus sign upon exchanging two identical particles. Let $|m\rangle$ denote the fermionic particle in some one-particle state indexed by m, which is created from the vacuum by some operator $b_m^\dagger$:

$$|m\rangle \equiv b_m^\dagger |0\rangle , \qquad |m; n\rangle \equiv b_n^\dagger b_m^\dagger |0\rangle , \qquad \cdots \tag{1.10}$$

Fermi statistics means that the creation operators anti-commute:

$$|m; n\rangle = -|n; m\rangle \qquad \leftrightarrow \qquad \{b_n^\dagger, b_m^\dagger\} = 0 . \tag{1.11}$$

This includes the Pauli exclusion principle:

$$|m; m\rangle = 0 \tag{1.12}$$

More generally, as we shall review below, quantization of a fermion leads to anti-commutation relations such as:

$$\{b_n, b_m^\dagger\} = \hbar\, \delta_{nm} . \tag{1.13}$$

The point, here, is just that: (1) Fermionic operators are nothing exotic. They exist necessarily in any theory with "fundamental fermions," such as *e.g* the real-world electron and quarks. (2) Fermionic operators satisfy anti-commutation relations amongst themselves, not commutation relations.

1.2.1 Mathematical definition

We can be a bit more formal:

Definition: A *superalgebra* (over $\mathbb{C}$) is a $\mathbb{Z}_2$ graded vector space:

$$A = A_0 \oplus A_1 \tag{1.14}$$

with a bilinear multiplication $A \times A \to A$ such that:

$$a_0 a_0' \in A_0 \ , \quad a_0 a_1 \in A_1 \ , \quad a_1 a_1' \in A_0 \ , \qquad \text{if } a_0, a_0' \in A_0 \ , \ a_1, a_1' \in A_1 \ . \tag{1.15}$$

We call the elements of A_0 *bosonic* (or 'even') and the elements of A_1 *fermionic* (or 'odd'). It is obvious that the bosonic algebra A_0 is a sub-algebra of A.

Let $a, b \in A$, and let $|a|, |b|$ denote their $\mathbb{Z}_2$ degree. The super-commutator is simply:

$$[a, b\} = ab - (-1)^{|a||b|} ba \ . \tag{1.16}$$

Thus, we arrive at our first formal definition of supersymmetry:

Definition: A *supersymmetry algebra* over the space-time $\mathbb{R}^{1,d-1}$ is a superalgebra that contains the d-dimensional Poincaré symmetry algebra $iso(1, d-1)$ as a subalgebra of its bosonic subalgebra.

A *super-Poincaré algebra* is a supersymmetry algebra whose bosonic subalgebra is the Poincaré algebra. In these lectures, unless otherwise specified, we will use the term "supersymmetry algebra" to mean a super-Poincaré algebra, in keeping with common usage.

1.2.2 Schematic form of the supersymmetry algebra

Let X denote either Poincaré symmetry generators, or internal symmetry generators. Then, a general *supersymmetry algebra* will have generators:

$$X \in A_0 \ , \qquad Q \in A_1 \ , \tag{1.17}$$

and it will take the schematic form:

$$[X, X'] = X'' \ , \qquad [X, Q] = Q'' \ , \qquad \{Q, Q'\} = X \ . \tag{1.18}$$

This *a priori* form can be constrained further by studying the closure of the algebra. In particular, Jacobi identities impose strong constraints.

The fermionic generators Q are the **supersymmetry generators,** by definition.

Depending on the space-time dimension, we can have slightly different forms of the supersymmetry algebra. As we will see, we always have some relations of the form:

$$\{Q, Q'\} \sim P_\mu \ , \tag{1.19}$$

with P_μ the momentum operator, generator of translations. Thus, the supersymmetries can be thought of as a "square root" of space-time momentum.

But first, let us study an interesting 'toy model' of supersymmetry, in quantum mechanics.

1.3 Supersymmetric quantum mechanics (a first look)

It is sometimes useful to view QM as a 'QFT in $d = 1$' (the "fields" depend only on time, not space). The 1d Poincaré algebra is simply generated by $E = -i\frac{d}{dt}$. We introduce $\mathcal{N}$ supersymmetry generators:

$$Q^I , \qquad I = 1, \cdots , \mathcal{N} . \tag{1.20}$$

The 1d supersymmetry algebra takes the form:

$$[Q^I, H] = 0 , \qquad \{Q^I, Q^J\} = 2H\delta^{IJ} + Z^{IJ} , \tag{1.21}$$

where $Z^{IJ} = Z^{JI}$ $(I \neq J)$ are some real central charges (that is, which commute with Q^I and H).

Relativistic massless spinning particle. In the Feynman path integral language for quantum mechanics, a "1d fermion" is simply a Grassmann-valued field $\psi = \psi(t)$: [9]

$$\{\psi(t), \psi(t')\} = 0 . \tag{1.22}$$

A free fermion has a Lagrangian:

$$L = i\psi\frac{d\psi}{dt} + m\psi\psi , \tag{1.23}$$

corresponding to the 1d "Dirac equation" $(i\frac{d}{dt} + m)\psi = 0$.

Possibly the simplest 1d $\mathcal{N} = 1$ supersymmetric model is a theory of D free 1d bosons $X^\mu(t)$ and D free 1d fermions $\psi^\mu(t)$, of vanishing mass (with the index $\mu = 1, \cdots , D$). It is defined by the Lagrangian:

$$L = \frac{1}{2}\dot{X}_\mu\dot{X}^\mu + i\psi_\mu\dot{\psi}^\mu . \tag{1.24}$$

Here, we use the standard notation $\dot{x} \equiv \frac{dx}{dt}$. The indices μ, ν are lowered with a "target-space metric" $g_{\mu\nu}$—for simplicity, we choose $g_{\mu\nu} = \eta_{\mu\nu}$ the Minkowski metric in D space-time dimensions. This system describes a relativistic massless *spinning* particle in $\mathbb{R}^{1,D-1}$. The 1d "worldline" fields $X^\mu(t)$ are the coordinates of the point particle in $\mathbb{R}^{1,D-1}$, and the 1d fermions ψ^μ are the spin degrees of freedom [15]. Classically, they satisfy:

$$\{\psi^\mu, \psi^\nu\} = 0 . \tag{1.25}$$

[9] Here, ψ denotes a classical field, not an operator.

We have the conjugate momenta:

$$\Pi_X^\mu = \dot{X}^\mu \ , \qquad \Pi_\psi^\mu = i\psi^\mu \ , \tag{1.26}$$

so that a 1d fermion is conjugate to itself. Quantum mechanically, we then have:

$$[X^\mu, \Pi_X^\nu] = ig^{\mu\nu} \ , \qquad \{\psi^\mu, \psi^\nu\} = g^{\mu\nu} \ , \tag{1.27}$$

in canonical quantisation. Thus, the classical Grassmann algebra (1.25) becomes the Clifford algebra quantum mechanically.

The Lagrangian (1.24) enjoys 1d $\mathcal{N} = 1$ supersymmetry, which acts on the fields as:

$$\boxed{\delta X = 2i\epsilon\psi \ , \qquad \delta\psi = -\epsilon\dot{X} \ ,} \tag{1.28}$$

with ϵ a constant supersymmetry parameter. Indeed, we can check that:

$$\delta L = i\epsilon \frac{d}{dt}\left(\psi_\mu \dot{X}^\mu\right) \ . \tag{1.29}$$

The Noether charge for supersymmetry is then the Hermitian operator:

$$Q = \psi_\mu \dot{X}^\mu \ . \tag{1.30}$$

We also have the energy operator:

$$H = -P_0 = \frac{1}{2}\dot{X}^2 \ , \tag{1.31}$$

which is associated to translations in time. The operators Q and H are conserved, since $\dot{Q} = 0$ and $\dot{H} = 0$ upon using the 1d equations of motions. One also sees that:

$$\{Q, Q\} = \{\psi_\mu, \psi_\nu\}\dot{X}^\mu \dot{X}^\nu = 2H \ , \tag{1.32}$$

using the canonical commutation relation for the operator ψ in (1.27).

On quantising a real fermion. The careful reader must have wondered how one obtained the canonical commutation relations (1.27) for the real fermions. One apparent issue is that the phase space variables ψ and $\Pi_\psi = i\psi$ are linearly related:

$$\psi + i\Pi_\psi = 0 \ . \tag{1.33}$$

This is an example of a *(second class) constraint,* in the Dirac formalism for constrained dynamical systems. We also find that the Hamiltonian of the free fermion is identically zero:

$$H_\psi = 0 \ . \tag{1.34}$$

To quantize this theory in the canonical formalism, one should use the so-called Dirac bracket instead of the Poisson bracket, and then replace the Dirac bracket by the (anti)-commutator quantum-mechanically. One can then eliminate Π_ψ from the description using the constraint, to finds:

$$\{\psi, \psi\} = 1 \ . \tag{1.35}$$

We refer to section 7.6 of Weinberg volume I [16] for a general discussion. The case of interest here is treated explicitly in chapter 7 (section 7.1.1) of [17].

1.4 Supermultiplets (a first look)

A *supermultiplet* is a representation of the supersymmetry algebra.

For instance, in the 1d example above, the fields X^μ and ψ^μ form the supermultiplets:

$$\mathbf{X}^\mu = (X^\mu, \psi^\mu) \ . \tag{1.36}$$

The supersymmetry transformations (1.28) realise the supersymmetry algebra:

$$\{Q, Q\} = 2E \ , \tag{1.37}$$

on fields, with $\delta \equiv \epsilon Q$ and $H \equiv E$, since:

$$\delta^2 X = -2i\epsilon^2 \frac{d}{dt} X \ , \qquad \delta^2 \psi = -2i\epsilon^2 \frac{d}{dt}\psi \ . \tag{1.38}$$

1.4.1 General properties

Consider the general supersymmetric algebra in QM,

$$\{Q^I, Q^J\} = 2E\delta^{IJ} + Z^{IJ} \ , \qquad [E, Q^I] = 0 \ . \tag{1.39}$$

The trivial commutator $[E, Q^I] = 0$ implies all the states in a given super-multiplet have the *same energy.* Moreover, in a supersymmetry theory, the energy of any state is non-negative:

$$\boxed{\langle\Omega|E|\Omega\rangle = \frac{1}{2}\langle\Omega|\{Q^1, Q^1\}|\Omega\rangle = |Q^1|\Omega\rangle|^2 \geq 0 \ .} \tag{1.40}$$

We define the fermion number operator $(-1)^F$, which acts as:

$$(-1)^F|b\rangle = |b\rangle \ , \qquad (-1)^F|f\rangle = -|f\rangle \ , \tag{1.41}$$

on bosonic and fermionic one-particle states, respectively. Since Q sends bosons to fermions, and vice-versa, we have:

$$(-1)^F Q^I = -Q^I(-1)^F \ , \tag{1.42}$$

for any Q^I. We then find:

$$\mathrm{Tr}\left((-1)^F\{Q^I, Q^J\}\right) = \mathrm{Tr}\left((-1)^F Q^I Q^J + (-1)^F Q^J Q^I\right) = 0 \ , \tag{1.43}$$

using (1.42) and the cyclicity of the trace. This holds for any *finite-dimensional* representation of the supersymmetry algebra (so that the trace is well-defined). Using (1.39), we find:

$$\mathrm{Tr}\left((-1)^F(2E\delta^{IJ} + Z^{IJ})\right) = 0 \ , \tag{1.44}$$

and, in particular, we have:

$$\boxed{\mathrm{Tr}\left((-1)^F E\right) = \langle E\rangle\, \mathrm{Tr}\left((-1)^F\right) = 0 \ ,} \tag{1.45}$$

where $\langle E\rangle$ is the fixed energy of the supermultiplet. If $\langle E\rangle$ is non-zero, we must have as many bosonic as fermionic states, so that the trace vanishes. In other words, *finite-energy* supermultiplets contain *as many bosons as fermions.*

1.4.2 1d $\mathcal{N} = 1$ supermultiplet

The simplest case is $\mathcal{N} = 1$, with the supersymmetry algebra (1.37). Consider Q acting on some set of states of energy E. We define the rescaled operator:

$$b = \frac{1}{\sqrt{2E}} Q \ , \tag{1.46}$$

so that the supersymmetry algebra (on those states) takes the simple form:

$$\{b, b\} = 1 \ . \tag{1.47}$$

This supersymmetry algebra is isomorphic to the anti-commutator of a single *real fermion*, as in (1.35). (That is, a 1d Clifford algebra.) It can be represented on a two-dimensional Hilbert space $\mathcal{H}$, where b is realized as the matrix:

$$b = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{1.48}$$

We then simply have a two-component supermultiplet:

$$\{|E\rangle \ , \ b|E\rangle\} \ , \tag{1.49}$$

as in (1.36). Here, the boson and the fermion in the supermultiplet can be represented by the vectors:

$$|E\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{H} \ , \qquad |E\rangle' = \sqrt{2} b |E\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{H} \ ,$$

respectively. Note that this obviously is in agreement with the discussion in terms of fields; in particular, acting twice with Q gives back the same state times its energy, $Q^2|E\rangle = E|E\rangle$.

Caveat: The two-dimensional representation of the 1d Clifford algebra (1.47) that we just gave is not irreducible. Instead, (1.47) has two irreducible one-dimensional representations, which can be given as:

$$b = \pm \frac{1}{\sqrt{2}} \ . \tag{1.50}$$

However, such irreducible representations correspond to the linear combinations:

$$|E\rangle \pm |E\rangle' \ , \tag{1.51}$$

which mix a bosonic and a fermionic state (corresponding to the fields X and ψ). The physical 1d $\mathcal{N} = 1$ supermultiplets are the two-dimensional ones, as given above.

1.4.3 1d $\mathcal{N} = 2n$ supermultiplets

For applications to higher-dimensional field theories, it is convenient to focus on the case $\mathcal{N} = 2n$ an even integer. We then introduce the complex supercharges:

$$\mathcal{Q}^i = Q^i + iQ^{n+i} \ , \qquad \bar{\mathcal{Q}}^i = Q^i - iQ^{n+i} \ , \qquad i = 1, \cdots, n \ , \tag{1.52}$$

in terms of which the supersymmetry algebra takes the form:

$$\{\mathcal{Q}^i, \bar{\mathcal{Q}}^j\} = \delta^{ij} 4E \ , \qquad \{\mathcal{Q}^i, \mathcal{Q}^j\} = 2i\mathcal{Z}^{ij} \ , \qquad \{\bar{\mathcal{Q}}^i, \bar{\mathcal{Q}}^j\} = -2i\mathcal{Z}^{ij} \ , \tag{1.53}$$

with $\mathcal{Z}^{ij} = Z^{i,n+j}$.

The "massive" (Clifford) supermultiplet. For simplicity, let us assume that $\mathcal{Z}^{ij} = 0$. Then, we define:

$$a_i = \frac{1}{2\sqrt{E}} \mathcal{Q}_i \ , \qquad a_i^\dagger = \frac{1}{2\sqrt{E}} \bar{\mathcal{Q}}_i \ , \tag{1.54}$$

so that we have a Clifford algebra:

$$\{a_i, a_j^\dagger\} = \delta_{ij} \ , \qquad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \ . \tag{1.55}$$

This is isomorphic to the algebra obtained upon quantising n *complex* fermions. Given a Clifford vacuum, $|\Omega\rangle$ such that $a_i|\Omega\rangle = 0$, we have the states:

$$[\Omega_{i_1 i_2 \cdots i_k}\rangle = a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_k}^\dagger |\Omega\rangle \ . \tag{1.56}$$

Since the $a^\dagger$'s anti-commute, we have $\binom{n}{k}$ such states and the highest state is the unique state at $k = n$. The total number of states in the supermultiplet is then:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \ . \tag{1.57}$$

In later lectures, we will see that this "massive" multiplet structure arises in higher-dimensional QFTs in the case of a massive one-particle state with $P_\mu = (E, 0, \cdots, 0)$, where $2n$ is the number of real supercharges of the QFT_d. We will also come back to the case when the 1d central charges might be non-trivial (that might be the momentum operator in QFT).

1.5 The Witten index

Consider a supersymmetric quantum mechanics with (by assumption) a discrete spectrum of $H = E$ and a finite number of vacua. One defines its *Witten index* as:

$$\boxed{\mathbf{I}_W = \text{Tr}_{\mathcal{H}} \left((-1)^F e^{-\beta H} \right) \ ,} \tag{1.58}$$

where the trace is over the full Hilbert space of the theory. However, any states with $E > 0$ contribute trivially to the index, by the property (1.45). Thus, the Witten index is really a property of the vacuum only:

$$\mathbf{I}_W = \mathrm{Tr}_{\mathcal{H},H=0}\left((-1)^F\right) = n_B - n_F \ , \tag{1.59}$$

and it is given by the number of bosonic vacua minus the number of fermionic vacua. This quantity plays an important role in supersymmetric QFT—it was first introduced by Edward Witten to explore the possibility of spontaneous supersymmetry breaking [18], as we will explain in a later lecture, but it also plays a central role in many more recent developments.

The main interest of the Witten index is that it is invariant under (appropriate) deformations of the theory. As we deform various parameters, the spectrum might change, but the index does not. This is because non-zero energy states come in a boson-fermion pairs. As we vary the parameters, states can leave or hit the ground state, $H = 0$, but only in boson-fermion pairs, therefore (1.59) is invariant.

2 Spinors: a review

2.1 Spinors in various dimensions

In any space-time dimensions, the supercharges transform like spinors under the Lorentz group $SO(1, d-1)$. Moreover, to discuss supersymmetry at all, we need to be very familiar with fermions in QFT, which are also spinors (by the spin-statistic theorem).

Consider first the Lorentz group in space-time d dimension. At first, we may consider the general case $SO(p, q)$ with $p + q = d$. This is the invariance of the metric:

$$(\eta_{\mu\nu}) = \mathrm{diag}\big(\underbrace{-1, \cdots, -1}_{p \text{ times}}, \underbrace{1, 1, \cdots, 1}_{q \text{ times}}\big) \ . \tag{2.1}$$

We focus on the Minkowski signature, $(p, q) = (1, d-1)$. For many purposes, it is also useful to consider the Euclidean signature, $(p, q) = (0, d)$.

The generators of $SO(p, q)$ are designated by $M_{\mu\nu} = -M_{\nu\mu}$. They satisfy the $so(p, q)$ algebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right) \ . \tag{2.2}$$

The fundamental representation of $SO(p, q)$ is the vector representation, of dimension d. The spinor representations, on the other hand, are not strictly speaking representation of the group $SO(p, q)$ but rather of its double cover, $\mathrm{Spin}(p, q)$. [10]

[10] In particular, $\mathrm{Spin}(d)$ is simply-connected for $d > 2$.

For low d, it is useful to keep in mind the (accidental) isomorphisms:

$$\begin{aligned} &\mathrm{Spin}(2) \cong U(1)\ , && \mathrm{Spin}(1,1) \cong Gl(2,\mathbb{R})\ , \\ &\mathrm{Spin}(3) \cong Sp(1) \cong SU(2)\ , && \mathrm{Spin}(1,2) \cong Sl(2,\mathbb{R})\ , \\ &\mathrm{Spin}(4) \cong SU(2)\times SU(2)\ , && \mathrm{Spin}(1,3) \cong Sl(2,\mathbb{C})\ , \\ &\mathrm{Spin}(5) \cong Sp(2)\ , && \mathrm{Spin}(1,4) \cong Sp(1,1)\ . \end{aligned} \tag{2.3}$$

Of course, $SO(p,q)$ and $\mathrm{Spin}(p,q)$ have the same Lie algebra (2.2), denoted by $so(p,q)$.

The (Dirac) spinor representation of $so(p,q)$ can be constructed explicitly as follows. First, let us introduce the gamma matrices γ^μ which satisfy the *Clifford algebra*:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\ . \tag{2.4}$$

Then, the spinor representation matrices are given explicitly by: [11]

$$\mathbf{M}_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]\ . \tag{2.5}$$

The gamma matrices can be constructed explicitly in any dimensions. They are of dimension:

$$D = 2^n \qquad \text{if} \qquad d = 2n \qquad \text{or} \qquad d = 2n+1\ . \tag{2.6}$$

Thus, $D = 2^n$ is also the dimension of the Dirac spinor. It particular, we have $D = 4$ in $d = 4$.

The Dirac spinor representation is not necessarily an *irreducible* representation of $so(p,q)$, however. Which are the actual irreducible spinor representations depends non-trivially on p and q. We see this below, in the physically-important examples in $d \leq 4$.

2.1.1 Lorentzian signature

Consider the case $(p,q) = (1, d-1)$. Suppose we are given a set of gamma matrices $\{\gamma^\mu\}$ in $d = 2n$ dimensions, with $\mu = 0, \cdots, 2n-1$. Then, we directly have the γ matrices in $d = 2n+1$ dimensions, by introducing additional matrix:

$$\gamma^{2n+1} \equiv (-i)^{n+1}\gamma^0\gamma^1\cdots\gamma^{2n-1}\ . \tag{2.7}$$

Since μ runs from 0 to $d-1$, we should really denote γ^{2n+1} by "γ^{2n}," but the notation is customary. (In particular, for $d = 4$, we have γ^5 used to define chirality.)

Given γ^μ in $d = 2n$, we can also construct the matrices Γ^M in $d = 2n+2$, with:

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^1\ , \qquad \Gamma^{2n+1} = \gamma^{2n+1}\otimes\sigma^1\ , \qquad \Gamma^{2n+2} = \mathbf{1}_{2^n\times 2^n}\otimes\sigma^2\ . \tag{2.8}$$

Therefore, we can build the γ matrices inductively, for any d. Let us now discuss the first few cases, by making some convenient choices:

[11] Be mindful of what is a representation matrix—here, denoted $\mathbf{M}$—and what is an "abstract" generator $M_{\mu\nu}$ of the algebra. This should always be clear from context.

d = 2 and d = 3: In the case $n = 1$, it is useful to introduce the Pauli matrices σ^i $(i = 1, 2, 3)$:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.9}$$

Recall that:

$$\sigma^i \sigma^j = \delta^{ij} \mathbf{1}_{2\times 2} + i\epsilon^{ijk}\sigma^k . \tag{2.10}$$

We can then choose:

$$\gamma^0 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.11}$$

d = 4 and d = 5: For $n = 2$, we take the 4×4 gamma matrices of $so(1, 3)$ to be:

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} , \tag{2.12}$$

with $i = 1, 2, 3$, and with $\mathbf{1}$ the 2×2 identity matrix. We then have:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} . \tag{2.13}$$

Note that (2.12) is not the choice of the 4d γ-matrices that would follow from the general construction (2.8) starting from (2.11). Instead, this is a particularly convenient choice of 4d γ matrices for later purposes. It corresponds to an equivalent general construction with:

$$\Gamma^0 = \mathbf{1}_{2^n\times 2^n} \otimes (-i\sigma^2) , \qquad \Gamma^i = \gamma^i \otimes \sigma^1 , \qquad \Gamma^{2n+2} = i\gamma^0 \otimes \sigma^1 , \tag{2.14}$$

up to a permutation of the spatial coordinates. Here, the index i runs over $1, \cdots , 2n$, where $i = 2n$ corresponds to the γ^{2n+1} matrix.

2.1.2 Euclidean signature

In the case $(p, q) = (0, d)$ with $d = 2n$, with indices $\mu = 1, \cdots , 2n$, we similarly define:

$$\gamma^{2n+1} \equiv (-i)^n \gamma^1\gamma^1 \cdots \gamma^{2n} . \tag{2.15}$$

This is the *same matrix* as in (2.7) provided that we take:

$$\gamma^{2n} = i\gamma^0 . \tag{2.16}$$

Consider again the cases $n = 1$ and $n = 2$:

d = 2 and d = 3. In Euclidean signature, we now simply have:

$$\gamma^i = \sigma^i . \tag{2.17}$$

d = 4 and d = 5. In this case, we have the $so(4)$ gamma matrices:

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} , \qquad \gamma^4 = \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix} \tag{2.18}$$

and the γ^5 matrix is:

$$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} . \tag{2.19}$$

2.2 Spinors in 4d

In these lectures, we will focus on $d = 4$ in Lorentzian signature.

2.2.1 Weyl spinors

From the explicit gamma matrices (2.12), it is easy to see that the Dirac spinor representation is reducible. We use the projector:

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5) , \tag{2.20}$$

which commutes with the 4d gamma matrices, to decompose Dirac spinors $\Psi = (\Psi_a)$, with Dirac indices $a = 1, \cdots, 4$, into Weyl spinors:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} . \tag{2.21}$$

Here, ψ_α, with $\alpha = 1, 2$, denotes a so-called two-component *left-handed Weyl spinor.* [12] It it a two-component complex vector, $\psi \in \mathbb{C}^2$, which sits in the fundamental representation $\mathbf{2}$ of $Sl(2, \mathbb{C})$. The dotted index $\dot\alpha$ corresponds to the conjugate representation $\bar{\mathbf{2}}$; the corresponding *right-handed Weyl spinor* is denoted by $\bar\psi^{\dot\alpha}$, with:

$$(\psi_\alpha)^* = \bar\psi^{\dot\alpha} , \qquad (\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha} . \tag{2.22}$$

Note that undotted (α) lower indices are row index, while undotted upper indices are column index, while the dotted ($\dot\alpha$) index follow the *opposite* convention—upper dotted indices are row index, as is apparent in (2.21), and lower dotted indices are column indices.

It is useful to note that $Sl(2, \mathbb{C})$ can be viewed as:

$$SL(2, \mathbb{C}) \cong SU(2) \times SU(2)^* , \tag{2.23}$$

where the two $SU(2)$ factors are exchanged under complex conjugation, as we will discuss further in the exercises. A spinorial representation of $so(1, 3)$ can then be denoted by (j, k), with $j, k \in \frac{1}{2}\mathbb{Z}$ the $SU(2)$ spins, and with $(j, k)^* = (k, j)$ the

[12] The terminology "left-handed" or "right-handed" in this context in somewhat imprecise, albeit standard. The more correct terminology is to simply call $P_\pm\Psi$ the (left- or right-) *chiral* fermions.

conjugate representation. A left-handed Weyl spinor ψ sits in a representation $(\frac{1}{2}, 0)$ and its conjugate, the right-handed Weyl spinor $\bar{\psi}$, sits in $(0, \frac{1}{2})$. The vector representation corresponds to:

$$(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) \ . \tag{2.24}$$

Any other representation of $so(1,3)$ can be obtained by tensor products, which follow from the tensor products of $SU(2)$ representations. For instance,

$$(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (1, 0) \oplus (0, 0) \ , \tag{2.25}$$

where $(1, 0)$ corresponds to a self-dual anti-symmetric tensor of $so(1,3)$.

The isomorphism between $\mathrm{Spin}(1,3)$ and $Sl(2,\mathbb{C})$ can be made explicit by introducing the σ-matrices:

$$(\sigma^\mu) = (\sigma^0, \sigma^i) \ , \qquad \text{with} \quad \sigma^0 = -\sigma_0 = -\mathbf{1} \ . \tag{2.26}$$

We have:

$$\sigma^\mu X_\mu = \begin{pmatrix} -X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_0 - X_3 \end{pmatrix} \ , \tag{2.27}$$

for any 4-covector X_μ. We have the following index structure, following the conventions of Wess and Bagger [1]:

$$\sigma^\mu_{\alpha\dot{\alpha}} \ . \tag{2.28}$$

In particular, we have the bi-spinor $X_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} X_\mu$ in (2.27), in agreement with (2.24).

We lower and raise the α and $\dot{\alpha}$ indices with the anti-symmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$, defined by:

$$\varepsilon^{12} = -\varepsilon^{21} = 1 \ , \qquad \varepsilon_{12} = -\varepsilon_{21} = -1 \ , \tag{2.29}$$

such that $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$. Then:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \ , \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \ , \tag{2.30}$$

and similarly for the dotted indices. We then define the $\bar{\sigma}$-matrices, with raised indices:

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} \ . \tag{2.31}$$

We can check that:

$$(\bar{\sigma}^\mu) = (\sigma^0, -\sigma^i) \ . \tag{2.32}$$

Finally, we choose the following implicit notation for contraction of spinor indices:

$$\psi\chi \equiv \psi^\alpha \chi_\alpha \ , \qquad \bar{\chi}\bar{\psi} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} \ . \tag{2.33}$$

Note that, in these conventions,

$$(\chi\psi)^\dagger = \bar{\psi}\bar{\chi} \ . \tag{2.34}$$

We also have $\chi\psi = \psi\chi$, for anticommuting spinors.

2.2.2 Lorentz symmetry generators.

Note that the gamma matrices (2.12) read:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\bar\sigma^\mu & 0 \end{pmatrix} , \tag{2.35}$$

and therefore:

$$M^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = -i \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix} , \tag{2.36}$$

where we defined the matrices:

$$\sigma^{\mu\nu} \equiv \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) , \qquad \bar\sigma^{\mu\nu} \equiv \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) , \tag{2.37}$$

with indices:

$$(\sigma^{\mu\nu})_\alpha{}^\beta , \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} . \tag{2.38}$$

Therefore, $-i\sigma^{\mu\nu}$ and $-i\bar\sigma^{\mu\nu}$ are the Lorentz group generators on left- and right-handed Weyl spinors, respectively. Note the relations:

$$\begin{aligned} (\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta &= -2\eta^{\mu\nu}\delta_\alpha{}^\beta , \\ (\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)_\alpha{}^\beta &= -2\eta^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta} , \end{aligned} \tag{2.39}$$

which ensure that γ^μ satisfy the Clifford algebra. Finally, we should point out that:

$$i\sigma^{12} = \frac{i}{4}(\sigma^1\bar\sigma^2 - \sigma^2\bar\sigma^1) = \frac{1}{2}\sigma^3 , \tag{2.40}$$

which has eigenvalues $\pm\frac{1}{2}$. That corresponds to the usual $J^3 = -M_{12}$ spin in the $\{x^1, x^2\}$ plane.

2.2.3 Fierz identities

Given some Weyl spinors, we can write down various bilinears by contracting the Weyl indices, such as:

$$\psi\chi , \qquad \bar\psi\bar\sigma^\mu\chi , \qquad \psi\sigma^\mu\bar\chi , \qquad \cdots . \tag{2.41}$$

In explicit computations, it is often necessary to use some non-obvious-looking identities amongst spinor bilinears, such as, for instance:

$$(\psi\eta)\bar\chi_{\dot\alpha} = -\frac{1}{2}(\eta\sigma^\mu\bar\chi)(\psi\sigma_\mu)_{\dot\alpha} . \tag{2.42}$$

Here, we assumed that the Weyl spinors $\psi, \eta, \bar\chi$ are also fermionic (that is, $\psi_\alpha\eta_\beta = -\eta_\beta\psi_\alpha$, etc.). Such identities are known as *Fierz identities.* We will study them in a problem sheet.

2.2.4 Majorana spinors

A Majorana spinor is a "real Dirac" spinor, of the form:

$$(\Psi_a) = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} . \tag{2.43}$$

Recall that, by definition, *Majorana fermions* are their own anti-particle.

2.3 Spinors in 2d

The $d = 2$ case is useful for string theory. More generally, 2d QFTs, with or without supersymmetry, are very interesting toy-models of more general QFT phenomena.

Two-dimensional Dirac spinors are two-component vectors, which we denote by ψ_α. Using the projector:

$$P_\pm = \frac{1}{2}(1 \pm \gamma^3) , \tag{2.44}$$

we can decompose them into two one-component Weyl spinors, denoted by $\psi_\pm$:

$$(\psi_\alpha) = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} . \tag{2.45}$$

Note that, unlike in 4d, the two Weyl spinors are not related by complex conjugation. In Euclidean signature, Spin(2) $= U(1)$, with the spinors having half-integer charges. In this language, the 2d Weyl spinors $\psi_\mp$ have spin $\pm\frac{1}{2}$, respectively. *Further discussion of 2d spinors is left for the problem sheets.*

3 Supersymmetry in various dimensions (but mostly $d = 4$)

Given the above discussion of spinors, we are ready to write down super-Poincaré algebras in various dimensions. Recall the Poincaré algebra, in any space-time dimension d:

$$\begin{aligned} &[P_\mu, P_\nu] = 0 , \\ &[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) , \\ &[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}\right) . \end{aligned} \tag{3.1}$$

The supersymmetry generators, which we will denote by "Q" or "$\mathcal{Q}$," transform as spinors under the Lorentz group. Given a supercharge $\mathcal{Q}$ in some irreducible D-dimensional spinor representation $\mathcal{S}$ of $so(1, d-1)$, [13] this determines the commutators:

$$[P_\mu, \mathcal{Q}_a] = 0 , \qquad [M_{\mu\nu}, \mathcal{Q}_a] = -(\mathbf{M}_{\mu\nu})_a{}^b \mathcal{Q}_b . \tag{3.2}$$

[13] Or $so(d)$, if we are interested in Euclidean signature.

d	spinor	$\mathcal{N}$	max R-sym.	N_Q	$\mathcal{N}_{\text{max}}^{\text{SUSY}}$	$\mathcal{N}_{\text{max}}^{\text{SUGRA}}$
1	"Dirac"	$\mathcal{N}=n$	$SO(\mathcal{N})$	$\mathcal{N}$	16	32
2	Weyl	$\mathcal{N}=(n_L,n_R)$	$SO(n_L)\times SO(n_R)$	n_L+n_R	$(8,8)$	$(16,16)$
3	Dirac	$\mathcal{N}=n$	$SO(\mathcal{N})$	$2\mathcal{N}$	8	16
4	**Weyl**	$\boldsymbol{\mathcal{N}=n}$	$\boldsymbol{U(\mathcal{N})}$	$\boldsymbol{4\mathcal{N}}$	**4**	**8**

Table 1: Supersymmetry in $d \leq 4$. Here, "spinor" denotes the irreducible $so(1, d-1)$ spinor representation in each dimension, and $n \in \mathbb{N}$. We highlighted the case $d = 4$, which will be our main object of study. We will briefly discuss the higher-dimensional case, $d > 4$, at the end of this section.

Here, we wrote down explicitly the spinor indices $a, b = 1, \cdots, D$. The constant $D \times D$ matrix $\mathbf{M}_{\mu\nu}$ is the generator $M_{\mu\nu}$ in the representation $\mathcal{S}$. For a Dirac spinor, it is given by (2.5).

The super-Poincaré algebras in d dimensions are labelled by a number:

$$\mathcal{N} \in \mathbb{N} , \tag{3.3}$$

the number of distinct irreducible spinors $\mathcal{Q}^I$ $(I = 1, \cdots, \mathcal{N})$ amongst the generators. In dimensions 1 to 4, the types of supersymmetry are summarized in Table 1. Note that, for $d = 2$, there can exist an independent number n_L and n_R of "left-chiral" and "right-chiral" Weyl spinor supercharges, Q_-^I with $I = 1, \cdots, n_L$ and Q_+^K with $K = 1, \cdots, n_R$.

In Table 1, N_Q denotes the number of "real supercharges"—that is, effectively, the number of independent supersymmetries. Many general aspects of supersymmetric QFTs in various dimensions depend on N_Q, essentially because the size of the supermultiplets is determined by N_Q, not $\mathcal{N}$.

We also indicated $\mathcal{N}_{\text{max}}$ in the Table, corresponding to $N_Q = 16$. This is the "maximal supersymmetry" in each dimension. This means the maximal amount of supersymmetry that can be realised by interacting *quantum field theories.* For higher $\mathcal{N}$, we necessarily need to include gravity. Indeed, even with gravity, we can only consider $N_Q \leq 32$. [14] Thus, for any d, there is a finite list of "physical" supersymmetry algebras.

In addition to the commutators (3.1) and (3.2), the d-dimensional super-Poincaré algebra consists of the anti-commutators amongst supercharges. *Schematically,* we have:

$$\{\mathcal{Q}_a, \mathcal{Q}_b\} = \mathcal{C}^{\mu}_{ab} P_\mu + \mathcal{Z}_{ab} \ . \tag{3.4}$$

The second term in (3.4), $\mathcal{Z}_{ab}$, is the *central charge*—that is, a generator of an "internal symmetry" which commutes with the full Poincaré symmetry. In particular, $\mathcal{Z}^{ab}$ is a Lorentz scalar.

[14] *More on this later, and on the problem sheets.*

For any d and $\mathcal{N}$, the right-hand-side of (3.4) can be fixed by consistency with Lorentz invariance and the super-Jacobi identities. Let us define the super-commutator:

$$[\mathcal{O}_a, \mathcal{O}_b\} \equiv \mathcal{O}_a\mathcal{O}_b - (-1)^{\epsilon_a\epsilon_b}\mathcal{O}_b\mathcal{O}_a \;, \tag{3.5}$$

where $\epsilon_a \in \{0, 1\}$ is the $\mathbb{Z}_2$ grading of $\mathcal{O}_a$, the Jacobi identities of a super-algebra read:

$$(-1)^{\epsilon_c\epsilon_a}[[\mathcal{O}_a, \mathcal{O}_b\}, \mathcal{O}_c\} + (-1)^{\epsilon_a\epsilon_b}[[\mathcal{O}_b, \mathcal{O}_c\}, \mathcal{O}_a\} + (-1)^{\epsilon_b\epsilon_c}[[\mathcal{O}_c, \mathcal{O}_a\}, \mathcal{O}_b\} = 0 \;.$$

This can be used to constraint the form of (3.4). Indeed, using (3.2) and the Jacobi identity, we must have:

$$[P_\mu, \{\mathcal{Q}_a, \mathcal{Q}_b\}] = \{[P_\mu, \mathcal{Q}_a], \mathcal{Q}_b\} + \{[P_\mu, \mathcal{Q}_b], \mathcal{Q}_a\} = 0 \;. \tag{3.6}$$

and

$$\begin{aligned} [M_{\mu\nu}, \{\mathcal{Q}_a, \mathcal{Q}_b\}] &= \{[M_{\mu\nu}, \mathcal{Q}_a], \mathcal{Q}_b\} + \{[M_{\mu\nu}, \mathcal{Q}_b], \mathcal{Q}_a\} \\ &= -\,(\mathbf{M}_{\mu\nu})_a{}^c\{\mathcal{Q}_b, \mathcal{Q}_c\} - (\mathbf{M}_{\mu\nu})_b{}^c\{\mathcal{Q}_a, \mathcal{Q}_c\} \;. \end{aligned} \tag{3.7}$$

This implies that the commutator $\{\mathcal{Q}_a, \mathcal{Q}_b\}$ can only give a linear combination of the momentum operator P_μ and (possibly) a central charge—and the last identity can be used to completely fix the structure constants $\mathcal{C}^\mu_{ab}$.

3.1 R-symmetry

In a supersymmetric QFT, almost any internal continuous Lie group symmetry commute with the super-Poincaré algebra. In particular, in any supermultiplet—to be discussed below—all states must transform in the same representation (possibly trivial) of any such internal symmetry. Such internal symmetries are often called “flavor symmetries.” There is one important exception to this, known as an *R-symmetry.*

Definition: An R-symmetry is an *automorphism of the super-Poincaré algebra.*

The maximal possible R-symmetry, for a given $d \leq 4$ and $\mathcal{N}$, is shown in Table 1. This corresponds to the R-symmetry *without central charges.* Central charges will generally break the R-symmetry explicitly to a subgroup of the maximal R-symmetry.

Consider the supersymmetry generators $\mathcal{Q}^I$, $I = 1, \cdots, \mathcal{N}$. An R-symmetry acts on the supercharges as:

$$[R, \mathcal{Q}^I_a] = -\mathbf{R}^I{}_J\, \mathcal{Q}^I_a \;, \tag{3.8}$$

where $\mathbf{R}$ is a representation matrix of the R-symmetry group, while leaving the supersymmetry algebra invariant. Note that R must commute with the Poincaré algebra (and, in particular, with $so(1, d-1)$), in agreement with the Coleman-Mandula theorem.

The *actual* R-symmetry of a given supersymmetric QFT depends on the details of the theory. In general, only a subgroup (possibly trivial) of the maximal R-symmetry is actually realised in a given QFT. For instance, in a theory defined by a Lagrangian, the Lagrangian might not be invariant under the R-symmetry. Even if the classical Lagrangian is R-symmetry invariant, one still has to check whether the symmetry still holds quantum-mechanically. We will see examples of this phenomenon in later lectures.

Since R has a non-trivial commutator with $\mathcal{Q}$, the different components of a supermultiplet will necessarily span different representations of the R-symmetry group.

3.2 Minimal supersymmetry in 4d

Let us now, finally, consider in detail the supersymmetry algebra we will most study: minimal supersymmetry in four dimensions, also known as 4d $\mathcal{N}=1$ supersymmetry. There are four real supercharges, $N_Q = 4$, which fill out one complex Weyl spinor Q_α, and its conjugate $\bar{Q}^{\dot{\alpha}}$. The supersymmetry algebra reads:

$$\boxed{\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \;, \qquad \{Q_\alpha, Q_\beta\} = 0 \;, \qquad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \;,} \tag{3.9}$$

Of course, we also have:

$$[P_\mu, Q_\alpha] = 0 \;, \qquad [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0 \;, \tag{3.10}$$

and:

$$[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu} Q)_\alpha \;, \qquad [M_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = -i(\bar{Q}\bar{\sigma}_{\mu\nu})_{\dot{\alpha}} \;. \tag{3.11}$$

This, together with the bosonic Poincaré algebra (3.1)ï¿½itself, gives the full 4d $\mathcal{N}=1$ supersymmetry algebra.

Note that the structure of the RHS of (3.9) is consistent with the decomposition (2.24) . Note that we also have (2.25), which imply that $\{Q_\alpha, Q_\beta\}$ could be given by the sum of a spin-$(1,0)$ (self-dual antisymmetric tensor) and of a scalar. This could only be:

$$\{Q_\alpha, Q_\beta\} = M_{\alpha\beta} Y + \epsilon_{\alpha\beta} Z \;, \tag{3.12}$$

where $M_{\alpha\beta}$ is the $so(1,3)$ generator written as a bi-spinor, and Y and Z are constants. From (3.6), one can see that $Y=0$; we must also have $Z=0$ since the LHS is symmetric in $(\alpha \leftrightarrow \beta)$.

3.2.1 R-symmetry $U(1)_R$

The maximal R-symmetry of 4d $\mathcal{N}=1$ supersymmetric theories is the abelian group $U(1)_R$. It acts on the supercharges as:

$$Q_\alpha \to e^{-i\alpha} Q_\alpha \;, \qquad \bar{Q}_{\dot{\alpha}} \to e^{i\alpha} \bar{Q}_{\dot{\alpha}} \;. \tag{3.13}$$

This clearly leaves the supersymmetry algebra (3.9) invariant. In other words, we assign $U(1)_R$ charges -1 and $+1$ to Q_α and $\bar{Q}_{\dot{\alpha}}$, respectively. We have:

$$[R, Q_\alpha] = -Q_\alpha \ , \qquad [R, \bar{Q}_{\dot{\alpha}}] = \bar{Q}_{\dot{\alpha}} \ . \tag{3.14}$$

Thus, acting on a state $|\Omega, r\rangle$ of non-zero R-charge $r \in \mathbb{R}$ with Q_α, we decrease the R-charge by 1 unit:

$$Q_\alpha |\Omega; r\rangle \sim |\Omega'; r-1\rangle \ , \tag{3.15}$$

and similarly $\bar{Q}_{\dot{\alpha}}$ increases the R-charge by 1 unit.

At this point, the R-symmetry might look like a curiosity, but it will play an important role in the discussion of the dynamics of actual 4d $\mathcal{N} = 1$ supersymmetric theories, later on.

3.2.2 Supermultiplets: Massive representations

In QFT, essentially by definition, a "particle" is an irreducible finite-dimensional unitary representation of the Poincaré algebra. Using Wigner's induced representation method, one labels particle states by their mass and spin (for a "massive particle") or by their energy and helicity (for a "massless particle").

Let us define:

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \ , \tag{3.16}$$

the Pauli-Ljubanski pseudovector. One can show that:

$$C_1 = P_\mu P^\mu \ , \qquad C_2 = W_\mu W^\mu \ , \tag{3.17}$$

are Casimir operators of the Poincaré algebra—that is, they commute with every Poincaré generator. In fact, they are the only two Casimir operators of $ISO(1,3)$. For massive particles, they define the mass and spin, respectively.

Consider then a *massive* particle of energy-momentum $P_\mu = p_\mu$, for which we have $C_1 = -M^2 < 0$. We can go to the rest frame,

$$p^\mu = (M, 0, 0, 0) \ . \tag{3.18}$$

Then, particle are classified in terms of finite-dimensional unitary representations of the "little group" that leaves (3.18) invariant, namely $SO(3)$. We then have:

$$W^\mu = (0, W^i) \ , \qquad W^i = -M J^i \ , \qquad \text{with} \quad J^i \equiv -\frac{1}{2} \epsilon^{ijk} M_{jk} \ , \tag{3.19}$$

where J^i are the $SO(3)$ spin operator, satisfying:

$$[J^i, J^j] = i \epsilon^{ijk} J^k \ . \tag{3.20}$$

Therefore, we have the massive particles classified by their mass M and their spin $j \in \frac{1}{2}\mathbb{Z}$:

$$C_1 = -M^2 \ , \qquad C_2 = M^2 J_i J^i = M^2 j(j+1) \ . \tag{3.21}$$

Let us now build the *supermultiplets,* which are collections of one-particle states that form representation of the super-Poincaré algebra. Since we obviously have:

$$[C_1, Q_\alpha] = [P_\mu P^\mu, Q_\alpha] = 0 \ , \qquad [C_1, \bar{Q}_{\dot\alpha}] = [P_\mu P^\mu, \bar{Q}_{\dot\alpha}] = 0 \ , \tag{3.22}$$

all particles in a supermultiplet have the *same invariant mass*, M^2. In other words, C_1 is a Casimir the full super-Poincaré algebra. This is not the case of C_2, on the other hand.

Note that J^i acts as:

$$[J^i, Q_\alpha] = -\frac{1}{2}(\sigma^i)_\alpha{}^\beta Q_\beta \ , \qquad [J^i, \bar{Q}_{\dot\alpha}] = \frac{1}{2}\bar{Q}_{\dot\beta}(\sigma^i)^{\dot\beta}{}_{\dot\alpha} \ , \tag{3.23}$$

on the supercharges, in term of the Pauli matrices σ^i. In particular:

$$[J^3, Q_\alpha] = \frac{(-1)^\alpha}{2} Q_\alpha \ , \qquad [J^3, \bar{Q}_{\dot\alpha}] = -\frac{(-1)^\alpha}{2}\bar{Q}_{\dot\alpha} \ , \tag{3.24}$$

for the four supercharges, where $\alpha = 1, 2$ and $\dot\alpha = 1, 2$.

At fixed mass M^2, a representation of the Poincaré group is a representation of $SO(3)$—more precisely, of the spin group $\mathrm{Spin}(3) = SU(2)$, since the spin can be half-integer. The spin-j representation consists of $2j + 1$ states:

$$|j\rangle = \Big\{ |j, m\rangle \ , \ m = -j, -j+1, \cdots, j-1, j \Big\} \ . \tag{3.25}$$

In the rest frame (3.18), the supersymmetry algebra (3.9) reads:

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2M\delta_{\alpha\dot\beta} \ , \tag{3.26}$$

with all other anticommutators vanishing. Defining the annihilation and creation fermionic operators $a_\alpha = \frac{1}{\sqrt{2M}} Q_\alpha$ and $a_\alpha^\dagger = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot\alpha}$ (with $\alpha = \dot\alpha$), respectively, we have:

$$\{a_\alpha, a_\beta^\dagger\} = \delta_{\alpha\beta} \ . \tag{3.27}$$

Note that:

$$[J^3, a_1^\dagger] = \frac{1}{2} a_1^\dagger \ , \qquad [J^3, a_2^\dagger] = -\frac{1}{2} a_2^\dagger \ , \tag{3.28}$$

so that, while $J^3|j, m\rangle = m|j, m\rangle$,

$$J^3 a_1^\dagger |j, m\rangle = (m + \frac{1}{2}) a_1^\dagger |j, m\rangle \ , \qquad J^3 a_2^\dagger |j, m\rangle = (m - \frac{1}{2}) a_2^\dagger |j, m\rangle \ . \tag{3.29}$$

In fact, acting on a spin-j set of states $|j\rangle$ with $a_\alpha^\dagger$, we obtain the spin $j + \frac{1}{2}$ and $j - \frac{1}{2}$ representations, since $a_\alpha^\dagger$ carries spin $\frac{1}{2}$:

$$j \otimes \frac{1}{2} = (j + \frac{1}{2}) \oplus (j - \frac{1}{2}) \ , \tag{3.30}$$

assuming $j > 0$. (In the spin-zero case, $j = 0$, we simply have a single spin-$\frac{1}{2}$ representation.) More explicitly, we can build the states:

$$\begin{aligned}\left|j \pm \frac{1}{2}, m\right\rangle &= C_{\frac{1}{2},j}\big(j \pm \frac{1}{2}, m; \frac{1}{2}, m - \frac{1}{2}\big)\, a_1^\dagger \left|j, m - \frac{1}{2}\right\rangle \\ &\quad + C_{\frac{1}{2},j}\big(j \pm \frac{1}{2}, m; -\frac{1}{2}, m + \frac{1}{2}\big)\, a_2^\dagger \left|j, m + \frac{1}{2}\right\rangle\ ,\end{aligned} \tag{3.31}$$

with $C_{\frac{1}{2},j}$ the Clebsh-Gordan coefficients for coupling a spin j to a spin $\frac{1}{2}$. Acting with *two* creation operator, we obtain the states:

$$\varepsilon^{\alpha\beta} a_\alpha^\dagger a_\beta^\dagger |j, m\rangle\ , \tag{3.32}$$

of the same spin that we started with. Thus, the massive supermultiplet has the schematic form:

$$\begin{array}{cccc} & |j\rangle\ , & a^\dagger|j\rangle \sim |j + \frac{1}{2}\rangle \oplus |j - \frac{1}{2}\rangle & a^\dagger a^\dagger |j\rangle \\ \mathrm{dof}: & 2j+1\ , & (2j+2) \oplus (2j)\ , & 2j+1\ . \end{array} \tag{3.33}$$

Here we indicated the number of degree of freedom at each level. As expected, we have as many bosons as fermion (namely, $4j + 2$). Let us consider some examples:

Massive chiral multiplet. Consider the spin-zero case, $j = 0$. Then, we have two scalar bosons and a spin-$\frac{1}{2}$ fermion:

$$\begin{array}{ll} \text{bosons:} & |0\rangle\ ,\ \ a_1^\dagger a_2^\dagger |0\rangle\ , \\ \text{fermions:} & a_\alpha^\dagger|0\rangle \sim |\frac{1}{2}\rangle\ , \end{array} \tag{3.34}$$

giving us two bosonic and two fermionic one-particle states. There is thus 4 states in total, in agreement with the discussion around (1.57).

Massive vector multiplet. This multiplet starts with $j = \frac{1}{2}$, which is fermionic. We then have:

$$\begin{array}{ll} \text{bosons:} & a_\alpha^\dagger|\frac{1}{2}\rangle \sim |1\rangle \oplus |0\rangle\ , \\ \text{fermions:} & |\frac{1}{2}\rangle\ ,\ \ a_1^\dagger a_2^\dagger |\frac{1}{2}\rangle\ . \end{array} \tag{3.35}$$

Thus, this multiplet contains an $SO(3)$ vector and a scalar, as well as two spin-$\frac{1}{2}$ fermions.

3.2.3 Supermultiplets: Massless representations

Let us now consider *massless* representations of the supersymmetry algebra, such that $C_1 = -P_\mu P^\mu = 0$. We can choose a frame such that:

$$p^\mu = (E, 0, 0, E)\ . \tag{3.36}$$

Recall that massless particles are indexed by their energy E, which we assume to be positive, and by their *helicity*, which is a representation of the little group $SO(2)$. More precisely, the helicity $\lambda \in \frac{1}{2}\mathbb{Z}$ is a representation Spin(2), a double cover of $SO(2) = U(1)$. [15]

Plugging in $P_\mu = (-E, 0, 0, E)$ in the supersymmetry algebra (3.9), we obtain:

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 4E \ , \qquad \{Q_2, \bar{Q}_{\dot{2}}\} = 0 \ , \tag{3.37}$$

and all other anticommutators vanishing. Thus, we define $a_1 = \frac{1}{2\sqrt{E}} Q_1$, $a_1^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}}$, and we have a single pair of fermionic creation and annihilation operators:

$$\{a_1, a_1^\dagger\} = 1 \ . \tag{3.38}$$

The helicity operator corresponds to $J^3 = -M_{12}$, the rotation in the (x^1, x^2) plane, with:

$$J^3 |E, \lambda\rangle = \lambda |E, \lambda\rangle \ , \tag{3.39}$$

by definition. We have:

$$[J^3, a_1^\dagger] = \frac{1}{2} a_1^\dagger \ , \qquad [J^3, a_1] = -\frac{1}{2} a_1 \ , \tag{3.40}$$

Using the fact that $\{Q_2, \bar{Q}_{\dot{2}}\} = 0$, one can check that:

$$Q_2 |E, \lambda\rangle = 0 \ , \tag{3.41}$$

on any state. Thus, the supermultiplets consists of pairs of states:

$$|E; \lambda\rangle \ , \qquad a_1^\dagger |E; \lambda\rangle = |E; \lambda + \frac{1}{2}\rangle \ . \tag{3.42}$$

Massless chiral multiplet. One of the most important example is the chiral multiplet, for $\lambda = 0$. It consists of a scalar boson and a massless $\lambda = \frac{1}{2}$ fermion:

$$\text{boson}: \quad |E; 0\rangle \ , \qquad\qquad \text{fermion}: \quad |E; \frac{1}{2}\rangle = a_1^\dagger |E; 0\rangle \ . \tag{3.43}$$

A $\lambda = \pm\frac{1}{2}$ particle is a massless Weyl fermion, which is left-chiral (ψ_α) or right-chiral ($\bar{\psi}^{\dot{\alpha}}$), respectively. In a QFT, every particle must be accompanied by its CPT conjugate, by CPT invariance. We will often write "chiral multiplet" for the $\lambda = \frac{1}{2}$ supermultiplet that contains ψ_α, while the CPT conjugate that contains $\bar{\psi}^{\dot{\alpha}}$ is called "the anti-chiral multiplet:"

$$\text{boson}: \quad |E; 0\rangle \ , \qquad\qquad \text{fermion}: \quad |E; -\frac{1}{2}\rangle = a_1 |E; 0\rangle \ . \tag{3.44}$$

On the other hand, it is also common to just refer to the CPT-invariant pair of both multiplets as "the chiral multiplet."

[15]The full little group on the light-cone is the double cover of $ISO(2)$, but the finite-dimensional representations just correspond to representations of the compact subgroup Spin(2).

Massless vector multiplet (a.k.a. gauge multiplet). Starting with $\lambda = \frac{1}{2}$, we obtain the pair:

$$\text{boson}: \quad |E;1\rangle = a_1^\dagger |E; \frac{1}{2}\rangle \ , \qquad \qquad \text{fermion}: \quad |E; \frac{1}{2}\rangle \ . \tag{3.45}$$

The $\lambda = 1$ particle, together with its CPT conjugate with $\lambda = -1$, give us a massless vector, which necessarily has some *gauge invariance.* Thus, a massless vector multiplet (generally just called "vector multiplet") contains a four-dimensional *gauge field* A_μ. Its fermionic superpartner, generally denoted by $\lambda_\alpha, \bar{\lambda}^{\dot{\alpha}}$, is called the *"gaugino".*

Supergravity multiplet. If we take $\lambda = \frac{3}{2}$ and its CPT conjugate, we get the states:

$$\text{fermion}: \quad |E; \pm\frac{3}{2}\rangle \ , \qquad \qquad \text{boson}: \quad |E; \pm 2\rangle \ . \tag{3.46}$$

A massless particle of helicity $|\lambda| = 2$—in other words, a spin-2 massless particle—is a *graviton*—it can only appear in a supersymmetric theory of gravity, known as a *supergravity.* The superpartner of the graviton, of helicity $\frac{3}{2}$, is a fermion called the *gravitino.*

3.3 Non-minimal supersymmetry in 4d

Non-minimal supersymmetry in 4d means that we have $\mathcal{N} > 1$ Weyl spinors Q^I_α, and their complex conjugates $\bar{Q}^I_{\dot{\alpha}}$. The supersymmetry algebra takes the form:

$$\begin{aligned}
\{Q^I_\alpha, \bar{Q}_{\dot{\beta} J}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \, \delta^I{}_J \ , \\
\{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta} \, \bar{Z}^{IJ} \ , \\
\{\bar{Q}_{\dot{\alpha} I}, \bar{Q}_{\dot{\beta} J}\} &= \varepsilon_{\dot{\alpha}\dot{\beta}} \, Z_{IJ} \ ,
\end{aligned} \tag{3.47}$$

with $I = 1, \cdots, \mathcal{N}$. Here, there is the possibility of a non-trivial complex *central charges* Z_{IJ}, with:

$$Z_{IJ} = -Z_{JI} \ . \tag{3.48}$$

By a Hermitian transformation of the supercharges, we can always bring Z_{IJ} to a canonical block-diagonal form, where each block is a 2×2 anti-symmetric matrix. For $\mathcal{N} = 2$, we take:

$$Z^{IJ} = 2\epsilon_{IJ} Z \ , \tag{3.49}$$

with a single complex central charge, denoted by Z.

3.3.1 Massless multiplets

With $\mathcal{N} > 1$, let us focus on the massless multiplets, for simplicity. From the fact that Q^I_2 and $Q^I_{\dot{2}}$ are realised trivially on such massless states, as in the $\mathcal{N} = 1$

case, we find that the central charges must also vanish, *i.e.* $Z^{IJ} = 0$, on massless multiplets. We then simply have:

$$\{Q^I_1, \bar{Q}_{\dot{1}J}\} = 4E\delta^I{}_J \ , \tag{3.50}$$

with all the other anticommutators vanishing. We thus have $\mathcal{N}$ pairs of helicity-lowering and raising operators:

$$a_I = \frac{1}{2\sqrt{E}} Q^I_1 \ , \qquad a^\dagger_I = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}I} \ , \qquad \{a_I, a^\dagger_J\} = \delta_{IJ} \ . \tag{3.51}$$

Starting from a state $|E;\lambda\rangle$ of helicity λ, we obtain the states:

$$a^\dagger_{I_1} \cdots a^\dagger_{I_k} |E;\lambda\rangle \sim |E;\lambda + \frac{k}{2}\rangle \ , \tag{3.52}$$

with k running from 0 to $\mathcal{N}$. Thus, an extended supersymmetry multiplet takes the form:

$$|E;\lambda\rangle \ , \quad \mathcal{N}\times|E;\lambda+\frac{1}{2}\rangle \ , \quad \cdots \quad \binom{\mathcal{N}}{k}\times|E;\lambda+\frac{k}{2}\rangle \ , \quad \cdots \ , \quad |E;\lambda+\frac{\mathcal{N}}{2}\rangle \ . \tag{3.53}$$

Any such extended massless multiplet spans helicities λ to $\lambda + \frac{1}{2}\mathcal{N}$. If $-\lambda \neq \lambda + \frac{\mathcal{N}}{2}$, we also need to add the CPT conjugate supermultiplet.

In the case $\mathcal{N}$ *even*, $\mathcal{N} = 2n$, the minimal helicity in a supermultiplet is:

$$|\lambda_{\rm max}| = \frac{1}{4}\mathcal{N} = \frac{n}{2} \ , \tag{3.54}$$

with helicities $\lambda = \{-\frac{\mathcal{N}}{4}, \frac{\mathcal{N}}{4} + \frac{1}{2} \ , \cdots \ , \frac{\mathcal{N}}{4} - \frac{1}{2} \ , \frac{\mathcal{N}}{4}\}$, which is CPT invariant. For any other multiplets, for any $\mathcal{N}$, we also need to add the CPT conjugate multiplet in any physical theory.

Rigid supersymmetry and supergravity. Massless particles of helicity $|\lambda| > \frac{1}{2}$ are associated to *gauge symmetries*:

$$\begin{aligned}
|\lambda| = 1 \qquad &\leftrightarrow \qquad \text{Lie group gauge theory (“local” symmetry } \mathbf{G}) \ , \\
|\lambda| = \frac{3}{2} \qquad &\leftrightarrow \qquad \text{local } \textit{supersymmetry} \ , \\
|\lambda| = 2 \qquad &\leftrightarrow \qquad \text{gravity (local Poincaré)} \ , \\
|\lambda| > 2 \qquad &\leftrightarrow \qquad \text{higher-spin theories—always free.}
\end{aligned} \tag{3.55}$$

For gauge theories, this is the statement that the Lagrangian of a free massless vector field has a invariance:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \ , \tag{3.56}$$

for any function $\alpha(x)$—here, we consider the Abelian case, $\mathbf{G} = U(1)$, for simplicity. Equivalently, the so-called *gauge field* A_μ couples to a *conserved current*:

$$S_{Aj} = \int d^4x A_\mu j^\mu \quad : \qquad S \to S + \int d^4x \partial_\mu \alpha j^\mu = S - \int d^4x \alpha \partial_\mu j^\mu = S \ . \quad (3.57)$$

We will come back to gauge theories, and to the $|\lambda| > 1$ generalization, later in the lectures. For now, we would just like to point out that theories with massless particles of helicity $|\lambda| = \frac{3}{2}$ or $|\lambda| = 2$ necessarily involve gravity. Indeed, an helicity 2 massless particle is a graviton; moreover, we claim that an helicity $\frac{3}{2}$ massless particle, called a gravitino, is necessarily paired with a graviton in a consistent field theory. The rough idea is that the gravitino couples to a supersymmetry current, while the graviton couples to the energy-momentum tensor; but, per the structure of the supersymmetry algebra, we cannot have "local supersymmetry" without also having "local super-Poincaré"—that is, supergravity.

Supersymmetric theories without massless particles of helicity $|\lambda| > 1$ are called *rigid supersymmetric theories*—we can just keep calling them *supersymmetric theories,* for short.

Any supersymmetric theory with a graviton and gravitinos is called a *supergravity.*

Theories with massless particles of helicities $|\lambda| > 2$ are called *higher-spin theories.* It is known that such theories are either free, or must contain an infinite tower of particles of essentially every spin. This is a rather esoteric subject, and we will not mention it again.

Thus, if we are interested in rigid supersymmetry only, there is a finite list of 4d supersymmetries:

$$\mathcal{N} = 1, 2, 3, 4 \ . \qquad (3.58)$$

Four-dimensional 4d $\mathcal{N} = 4$ theory is known as maximally supersymmetric Yang-Mills (SYM) theory. From our discussion above, we easily understand that it must be a *gauge theory—i.e. a Yang-Mills (YM) theory*, since there is only one possible multiplet in the case, with $|\lambda_{\max}| = 1$.

If we are interested in supergravity, we can consider supergravities with:

$$\mathcal{N} = 1, 2, \cdots, 7, 8 \ . \qquad (3.59)$$

In particular, 4d $\mathcal{N} = 8$ supersymmetry is called *maximal supergravity* in four dimensions. It has is a unique massless multiplet with $|\lambda_{\max}| = 2$.

We will study extended supermultiplets further in a problem sheet.

3.3.2 R-symmetry

The maximal R-symmetry of 4d extended supersymmetry is $U(\mathcal{N})$, with (Q^I) transforming in the fundamental representation, and $(\bar{Q}_I)$ transforming in the anti-fundamental representation.

In particular, in the rigid-supersymmetry case, the maximal R-symmetry of 4d $\mathcal{N} = 2$ theories is $U(2)_R \cong U(1)_r \times SU(2)_R$. The maximal R-symmetry of 4d $\mathcal{N} = 4$ supersymmetry is $U(4)$, but the actually realized symmetry in $\mathcal{N} = 4$ SYM is $SU(4) \cong SO(6)$.

3.4 Supersymmetry in 3d

Supersymmetry in 3d, in Lorentzian signature, can be discussed similarly. The irreducible spinor of $SO(1,2)$ is the Dirac spinor ψ_α, where the spinor index takes values $\alpha = 1, 2$.

Consider the γ matrices as in (2.11), namely:

$$(\gamma^\mu)_\alpha{}^\beta = (-i\sigma^2, \sigma^1, \sigma^3)_\alpha{}^\beta \,, \tag{3.60}$$

with $\mu = 0, 1, 2$. We may raise and lower the spinor indices with $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$, as for Weyl spinors in 4d. They satisfy:

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho \,, \tag{3.61}$$

with $\epsilon^{012} = 1$. Therefore, the $so(1,2)$ matrices in the spinor representation are:

$$\mathbf{M}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\rho}\gamma^\rho \,. \tag{3.62}$$

The supersymmetry generators are *real* Dirac spinor supercharges Q^I_α, and the supersymmetry algebra must take the form:

$$\{Q^I_\alpha, Q^J_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu \delta^{IJ} + \varepsilon_{\alpha\beta} Z^{IJ} \,. \tag{3.63}$$

We leave it as an exercise for the reader to check this, using the Jacobi identity; the overall constant can be chosen at will, by rescaling the Q^I's. The possible central charges are antisymmetric, $Z^{IJ} = -Z^{IJ}$.

The most well-studied supersymmetric $d = 3$ QFTs are with $\mathcal{N}$ is even. Then, we can organise the supercharges into *complex* Dirac spinors, similarly to (1.52). In particular, for $\mathcal{N} = 2$, we define:

$$\mathcal{Q}_\alpha = Q^1_\alpha + iQ^2_\alpha \,, \qquad \bar{\mathcal{Q}}_\alpha = Q^1_\alpha - iQ^2_\alpha \,. \tag{3.64}$$

The 3d $\mathcal{N} = 2$ superalgebra is closely related to the 4d $\mathcal{N} = 1$ superalgebra. (We will make this more precise in an exercise.)

3.5 Supersymmetry in 2d

Supersymmetry in 2d can be discussed similarly. We leave this as an exercise.

d	spinor	$\mathcal{N}$	max R-sym.	N_Q	$\mathcal{N}^{\text{SUSY}}_{\text{max}}$	$\mathcal{N}^{\text{SUGRA}}_{\text{max}}$
1	"Dirac"	$\mathcal{N} = n$	$SO(\mathcal{N})$	$\mathcal{N}$	16	32
2	Weyl	$\mathcal{N} = (n_L, n_R)$	$SO(n_L)\times SO(n_R)$	$n_L + n_R$	$(8, 8)$	$(16, 16)$
3	Dirac	$\mathcal{N} = n$	$SO(\mathcal{N})$	$2\mathcal{N}$	8	16
4	**Weyl**	$\boldsymbol{\mathcal{N} = n}$	$\boldsymbol{U(\mathcal{N})}$	$\boldsymbol{4\mathcal{N}}$	**4**	**8**
5	Dirac	$\mathcal{N} = n$	$Sp(\mathcal{N})$	$8\mathcal{N}$	2	4
6	Weyl	$\mathcal{N} = (n_L, n_R)$	$Sp(n_L)\times Sp(n_R)$	$8(n_L+n_R)$	$(1, 1)$ or $(2, 0)$	$n_L+n_R = 4$
7	Dirac	$\mathcal{N} = n$	$Sp(\mathcal{N})$	$16\mathcal{N}$	1	2
8	Weyl	$\mathcal{N} = n$	$U(\mathcal{N})$	$16\mathcal{N}$	1	2
9	Dirac	$\mathcal{N} = n$	$SO(\mathcal{N})$	$16\mathcal{N}$	1	2
10	Weyl	$\mathcal{N} = (n_L, n_R)$	$SO(n_L)\times SO(n_R)$	$16(n_L+n_R)$	$(1, 0)$	$(1, 1)$ or $(2, 0)$
11	Dirac	$\mathcal{N} = n$	$SO(\mathcal{N})$	$64\mathcal{N}$	0	1

Table 2: Supersymmetry in $1 \leq d \leq 11$. There is no rigid supersymmetry beyond $d = 10$, and no supergravity beyond $d = 11$.

3.6 Supersymmetry in higher dimensions

Let us give a very brief overview of supersymmetry in space-time dimensions $d > 4$. We list the types of supersymmetry up to $d = 11$ in Table 2.

We may focus on the general structure of the *massless supermultiplets,* similarly to the $d = 4$ case discussed above. Let us just state two important results:

- There is a maximal dimension in which one can have *rigid supersymmetry*, namely $d = 10$. The corresponding 10d $\mathcal{N} = (1, 0)$ SYM is closely related to 4d $\mathcal{N} = 4$ SYM.

- The maximal dimension in which one can have a *supergravity* ($\lambda \leq 2$) is $d = 11$. The corresponding unique supergravity theory is simply called 11d SUGRA.

- There are also two types of maximal superrgavities in $d = 10$, called type IIA and type IIB, which have $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ supersymmetry, respectively.

These maximally supersymmetric theories appear prominently in string theory—in particular, IIA/B supergravity is the low-energy limit of the ten-dimensional type-IIA/B superstring. On the other hand, 11d SUGRA is thought to be the low-energy limit of a conjectured theory of quantum gravity in 11 dimensions, called M-theory.

4 Supermultiplets, superfields, and superspace

4.1 Representing supersymmetry on fields

In a QFT, we would like to realise supersymmetry explicitly *on fields*—namely, some functions of the space time coordinates, $\varphi(x)$, which may also transform in some non-trivial representations of the Lorentz group—, not only on one-particle states. Moreover, if at all possible, we would like to realise the supersymmetry algebra *off-shell*—that is, without the need to impose the equations of motion of fields (in fact, without the need of *specifying* the equations of motions, or equivalently the Lagrangian).

We are familiar with the way the Poincaré algebra is realised on fields. In particular, the momentum operator P_μ is simply realised as:

$$\mathbf{P}_\mu = -i\partial_\mu \ , \tag{4.1}$$

acting on fields of any spin. On a classical field, it acts as:

$$e^{ia^\mu \mathbf{P}_\mu}\varphi(x) = \varphi(x+a) \ . \tag{4.2}$$

This is equivalent to:

$$e^{-ia^\mu P_\mu}\varphi(x)e^{ia^\mu P_\mu} = \varphi(x+a) \qquad \Leftrightarrow \qquad [P_\mu, \varphi] = -\mathbf{P}_\mu \varphi \ , \tag{4.3}$$

for the corresponding field *operator.*

Similarly, we would like to realise the supersymmetry algebra on fields, with explicit transformations such as:

$$[Q_\alpha, \phi] = \psi_\alpha \ , \qquad [Q_\alpha, \psi_\beta] = \varphi_{\alpha\beta} \ , \qquad \cdots \ , \tag{4.4}$$

for instance, for 4d $\mathcal{N}=1$ supersymmetry. (In this schematic example, ϕ and $\varphi_{\alpha\beta}$ are bosonic fields, and ψ_α is a fermionic field.) We would like to find supersymmetry transformations as in (4.4) which close on a set of fields $(\phi, \psi, \varphi, \cdots)$—the *supermultiplet*—and which realise explicitly the supersymmetry algebra (by definition).

In any dimension d, we denote the action of supersymmetry on the fields by:

$$\delta = i\epsilon^a \mathcal{Q}_a \ , \tag{4.5}$$

where ϵ are *supersymmetry parameters,* which we choose to be anti-commuting. For 4d $\mathcal{N}=1$ supersymmetry, we have:

$$\delta = \delta_\epsilon + \delta_{\bar\epsilon} \ , \tag{4.6}$$

with:

$$\boxed{\delta_\epsilon \equiv i\epsilon Q = i\epsilon^\alpha Q_\alpha \ , \qquad \delta_{\bar\epsilon} \equiv i\bar\epsilon\bar Q = -i\bar\epsilon^{\dot\alpha}\bar Q_{\dot\alpha} \ ,} \tag{4.7}$$

where the supersymmetry parameters are constant, anti-commuting Weyl spinors, ϵ^α and $\bar\epsilon^{\dot\alpha}$. Since the supersymmetry parameters are chosen to be fermionic, the variations themselves are bosonic operators.

(*Note:* The variations δ_ϵ and $\delta_{\bar\epsilon}$ are not independent, since ϵQ and $\bar\epsilon\bar Q$ are Hermitian conjugate of each other. Nonetheless, for all intents and purposes, we can treat the parameters ϵ and $\bar\epsilon$ as independent. This is effectively what we do below.)

The supersymmetry variations should satisfy the supersymmetry algebra, namely:

$$\boxed{[\delta_\epsilon, \delta_{\bar\epsilon}] = -2(\epsilon\sigma^\mu\bar\epsilon)\, P_\mu \ , \qquad [\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 0 \ , \qquad [\delta_{\bar\epsilon_1}, \delta_{\bar\epsilon_2}] = 0 \ .} \tag{4.8}$$

In principle, one can try and work out the supersymmetry transformations for the most general supermultiplet, by writing down the most general transformations on fields allowed by Lorentz covariance, and then fixing all the coefficients by requiring that the variations satisfy (4.8). In the following, we present the result of this procedure for the off-shell *chiral multiplet*, the simplest supermultiplet of 4d $\mathcal{N}=1$ supersymmetry.

4.1.1 The chiral multiplet, off-shell

A 4d $\mathcal{N}=1$ *chiral multiplet* consists of the fields:

$$\Phi = (\phi, \psi_\alpha, F) \ , \qquad \bar\Phi = (\bar\phi, \bar\psi^{\dot\alpha}, \bar F) \ . \tag{4.9}$$

Here, Φ and $\bar\Phi$ are CPT-conjugate of each other. They are often called the 'chiral multiplet' and the 'anti-chiral multiplet,' respectively. The corresponding *on-shell* multiplets were discussed in the previous section. In the massive case, we have a spin-$\frac{1}{2}$ fermion and its complex conjugate, giving two real degrees of freedom; in the massless case, we have an helicity $|\lambda| = \frac{1}{2}$ fermion, again giving us two real degrees of freedom. In all cases, we also have two bosonic particles which are Lorentz scalars. Those bosons are accounted for by the *complex* scalar field ϕ in (4.9). [16] On the other hand, the fermionic particles should correspond to the Weyl spinor field ψ, $\bar\psi$. Since a Weyl spinor ψ_α has *four* real degrees of freedom off-shell (but only two real degrees of freedom on-shell!), we should add (by hand) two real bosonic degrees of freedom in the off-shell description, to maintain the fermion-boson degeneracy. These so-called *auxiliary fields* are scalar fields denoted by F, $\bar F$ in (4.9). They are called 'auxiliary' because their equations of motion are algebraic (and therefore they can be eliminated from the description by imposing those equations of motions).

[16] As a side comment, we should note that the two real scalars A and B in the complex scalar $\phi = A + iB$ are really a scalar and a pseudo-scalar, respectively. (That B is a pseudo-scalar means that it changes sign under a parity transformation.) Indeed, the scalar $\varepsilon^{\alpha\beta} a_1^\dagger a_2^\dagger |0\rangle$ in (3.34) is really a pseudo-scalar, if $|0\rangle$ is a proper scalar (the telltale sign, as always, is the presence of the ε symbol). In other words, parity maps the multiplet Φ to $\bar\Phi$, as is clear at the level of the fermions. We refer to Weinberg's book [2] for a detailed account of parity in supersymmetric theories; in these lectures, we are not keeping track explicitly of the discrete symmetries.

The supersymmetry transformations of the chiral multiplet Φ reads:

$$\begin{aligned}
&\delta\phi = \sqrt{2}\epsilon\psi \ , \\
&\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar\epsilon)_\alpha\partial_\mu\phi + \sqrt{2}\epsilon_\alpha F \ , \\
&\delta F = i\sqrt{2}\bar\epsilon\bar\sigma^\mu\partial_\mu\psi \ .
\end{aligned} \tag{4.10}$$

One can easily check that this realises the supersymmetry algebra (4.8). For future convenience, we take the following convention for two successive variations of a field φ (of any spin):

$$\delta_1\delta_2\varphi = \delta_2(\delta_1\varphi) \ . \tag{4.11}$$

For instance:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi = \sqrt{2}(\epsilon_1\delta_{\epsilon_2}\psi - \epsilon_2\delta_{\epsilon_1}\psi) = 2(\epsilon_1\epsilon_2 - \epsilon_2\epsilon_1)F = 0 \ . \tag{4.12}$$

Similarly, we can check:

$$[\delta_\epsilon, \delta_{\bar\epsilon}]\phi = \sqrt{2}\epsilon\delta_{\bar\epsilon}\psi = 2i\epsilon\sigma^\mu\bar\epsilon\partial_\mu\phi \ . \tag{4.13}$$

This indeed realises (4.8), with $P_\mu = -i\partial_\mu$. We also have:

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar\epsilon}]\psi_\alpha &= \sqrt{2}\epsilon_\alpha\delta_{\bar\epsilon}F - i\sqrt{2}(\sigma^\mu\bar\epsilon)_\alpha\partial_\mu(\delta_\epsilon\phi) \\
&= 2i\epsilon_\alpha(\bar\epsilon\bar\sigma^\mu\partial_\mu\psi) - 2i(\sigma^\mu\bar\epsilon)_\alpha\,\epsilon\partial_\mu\psi \\
&= 2i\epsilon\sigma^\mu\bar\epsilon\partial_\mu\psi_\alpha \ .
\end{aligned} \tag{4.14}$$

Here, to go from the second to the third line, we used the non-trivial Fierz identity:

$$\epsilon_\alpha(\bar\epsilon\bar\sigma^\mu\psi) - (\sigma^\mu\bar\epsilon)_\alpha\epsilon\psi = (\epsilon\sigma^\mu\bar\epsilon)\psi_\alpha \ .$$

One can check the remaining anti-commutators (4.8), acting on the fields ϕ, ψ and F, in a similar manner. (The reader is encouraged to do so.)

Similarly, the transformations rules for the anti-chiral multiplet, $\bar\Phi$, are:

$$\begin{aligned}
&\delta\bar\phi = \sqrt{2}\bar\epsilon\bar\psi \ , \\
&\delta\bar\psi^{\dot\alpha} = i\sqrt{2}(\bar\sigma^\mu\epsilon)^{\dot\alpha}\partial_\mu\bar\phi + \sqrt{2}\bar\epsilon^{\dot\alpha}\bar F \ , \\
&\delta\bar F = i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\bar\psi \ .
\end{aligned} \tag{4.15}$$

One can then write down *supersymmetric Lagrangians* for chiral multiplets. The simplest one is:

$$\mathscr{L}_{\text{kin}} = -\partial_\mu\bar\phi\partial^\mu\phi - i\bar\psi\bar\sigma^\mu\partial_\mu\psi + \bar F F \ . \tag{4.16}$$

This is the sum of the canonical kinetic terms for the scalar and Weyl fermion, respectively, plus a trivial "kinetic term" for the auxiliary field F. *(Exercise: check explicitly that this is supersymmetric!)* More interestingly, we can introduce *interaction terms* coupling the bosons to the fermions. For instance, the term:

$$\mathscr{L}_{\text{cubic}} = g_0(F\phi^2 - \psi\psi\phi) + \bar g_0(\bar F\bar\phi^2 - \bar\psi\bar\psi\bar\phi) \ , \tag{4.17}$$

is supersymmetric, for any complex coupling constant g_0. Indeed, using (4.10) we easily check that:

$$\delta(F\phi^2 - \psi\psi\phi) = \partial_\mu\left(i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\psi\phi^2\right) \ , \tag{4.18}$$

and similarly for the complex conjugate term. Considering:

$$\mathscr{L} = \mathscr{L}_{\text{kin}} + \mathscr{L}_{\text{cubic}} \ , \tag{4.19}$$

and "integrating out" the auxiliary fields F by using their equations of motion, we find:

$$\mathscr{L} = -\partial_\mu\bar{\phi}\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\psi - V \ , \tag{4.20}$$

with the potential:

$$\boxed{V = |g_0|^2|\phi|^4 + g_0\psi\psi\phi + \bar{g}_0\bar{\psi}\bar{\psi}\bar{\phi} \ .} \tag{4.21}$$

Thus, we find a simple *supersymmetric* ϕ^4 *model* with Yukawa couplings, where the coupling constants are related as shown—this exact relation between *a priori* very different coupling constants is the hallmark of supersymmetry, and one technical reason why it is so powerful. We will come back to this class of models (known as Wess-Zumino models, historically the first 4d $\mathcal{N} = 1$ models to be studied) after we introduce some more powerful technology for constructing supersymmetric Lagrangians.

4.2 Superspace (4d $\mathcal{N} = 1$)

The procedure just outlined to realise supersymmetry on fields, known as "supersymmetry in components," is perfectly fine, but one can do a bit better. The formalism of *superspace*, which we will now describe, allows us to work with supermultiplets more efficiently, in a way which is essentially *covariant* with respect to the super-Poincaré algebra.

4.2.1 Coset manifolds

Let us start with a mathematical digression, by reviewing the general construction of *coset manifolds* for Lie groups, and the resulting induced action of the group on the coset.

Consider a Lie group G with a subgroup $H \subset G$. At the level of the Lie algebra, $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{h} = \text{Lie}(H)$, we have the direct sum:

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{K} \ , \tag{4.22}$$

where $\mathfrak{K}$ is the complement of $\mathfrak{h}$ inside $\mathfrak{g}$. Let T_A denote the generators of $\mathfrak{g}$, with:

$$[T_A, T_B] = iC_{AB}{}^C T_C \ , \tag{4.23}$$

and the indices $A, B, \cdots = 1, \cdots, \dim(\mathfrak{g})$. Let us split the generators of T^A into generators of $\mathfrak{h}$ and $\mathfrak{K}$, respectively:

$$(T_A) = (M_I, K_a) \ , \qquad M_I \in i\mathfrak{h} \ , \qquad K_a \in i\mathfrak{K} \tag{4.24}$$

with the indices $I = 1, \cdots, \dim(\mathfrak{h})$ and $a = 1, \cdots, \dim(\mathfrak{K})$. We then write general group elements $g \in G$ and $h \in H$ as:

$$g = e^{i\epsilon^A T_A} = e^{i\omega^I M_I + i\alpha^a K_a} \;, \qquad h = e^{i\widetilde{\omega}^I M_I} \;, \tag{4.25}$$

with $\epsilon^A = (\omega^I, \alpha^a)$ and $\widetilde{\omega}^I$ some real coefficients. The coset manifold:

$$\mathcal{M} = G/H \;, \tag{4.26}$$

is defined as the set of equivalence classes under right multiplication by H:

$$\mathcal{M} \cong G/\sim \;, \qquad \text{with} \qquad g \sim g' \quad \text{iff} \quad \exists h \in H \mid gh = g' \;. \tag{4.27}$$

To obtain a good manifold, we assume that the coset is *reductive*, meaning that:

$$[\mathfrak{h}, \mathfrak{K}] \subset \mathfrak{K} \;. \tag{4.28}$$

Furthermore, if we also have:

$$[\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{h} \;, \tag{4.29}$$

the coset manifold is a symmetric space. In terms of the Lie algebra generators, we thus have:

$$\begin{aligned} &[M_I, M_J] = iC_{IJ}{}^K M_K \;, \\ &[M_I, K_a] = iC_{Ia}{}^b K_b \;, \\ &[K_a, K_b] = iC_{ab}{}^c K_c + iC_{ab}{}^I M_I \;, \end{aligned} \tag{4.30}$$

with $C_{ab}{}^c = 0$ for a symmetric space. Let us denote by $\mathbf{x}(y)$ the points in (4.27), with y some local coordinates on the manifold $\mathcal{M}$. In the following, we will consider:

$$\mathbf{x}(y) = e^{-iy^a K_a} \in G \;, \tag{4.31}$$

as representatives of the equivalence classes that define $\mathcal{M}$; in the examples we consider, y^a form a natural set of coordinates on the coset. In general, one could choose any convenient set of coordinates y on $\mathcal{M}$.

Given this coset construction, there is an induced action of G on $\mathcal{M}$ (with G acting from the left). Indeed, for any $g \in G$, we should have: [17]

$$\boxed{g^{-1}\mathbf{x}(y) = \mathbf{x}(y')h(g, y) \;,} \tag{4.32}$$

for some $h(g, y) \in H$, which generally depends on the coordinate y and the group element g. In explicit computations, we will need to use the Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots} \;, \tag{4.33}$$

for the product of exponentiated generators (there is an infinity of terms on the RHS, but we will just need the first non-trivial one).

[17] We act with g^{-1} instead of g for later convenience; this is just a convention.

The relation (4.32) implies a transformation $g : y \to y'$ on the coordinates on $\mathcal{M}$. At first order around the identity in g, we find:

$$g \cong 1 + i\epsilon^A T_A \ , \qquad y'^a \cong y^a + \epsilon^A k^a_A(y) \ , \tag{4.34}$$

for some functions $k^a_A(y)$. Then, the differential operators:

$$\mathbf{T}_A \equiv -ik^a_A(y)\frac{\partial}{\partial y^a} \ , \tag{4.35}$$

realise the algebra $\mathfrak{g}$ on scalar fields $\Phi(y)$—that is, functions—on the coset manifold $\mathcal{M}$. [18] Note that the infinitesimal action of:

$$g^{-1} = g_1^{-1} g_2^{-1} g_1 g_2 \ , \tag{4.36}$$

on y^a induced by (4.32), using the expansion (4.34), is:

$$g^{-1}\cdot y^a \equiv y'^a = y^a + \epsilon_1^A \epsilon_2^B (k^b_A \partial_b k^a_B - k^b_B \partial_b k^a_A) = y^a - \epsilon_1^A\epsilon_2^B [\mathbf{T}_A, \mathbf{T}_B]^a + \cdots \ , \tag{4.37}$$

where we used the short-hand notation $\partial_a \equiv \frac{\partial}{\partial y^a}$.

Note on conventions. We are using conventions in which the action of G on (classical) fields is defined to be:

$$U(g)\Phi(y) = (1 + i\epsilon^A \mathbf{T}_A + \cdots)\Phi(y) = \Phi(y') \ , \tag{4.38}$$

which is sometimes called a "passive transformation," with:

$$U(g) = e^{i\epsilon^A \mathbf{T}_A} \ , \tag{4.39}$$

some explicit representation of the group. Correspondingly, for a quantum-mechanical *operator* $\Phi(y)$ on $\mathcal{M}$, we have:

$$U(g)^\dagger \Phi(y) U(g) = \Phi(y') \ , \tag{4.40}$$

with $U(g) = e^{i\epsilon^A T_A}$ a G-valued operator. We then have:

$$U(g)^\dagger \Phi(y) U(g) = \Phi(y)(y) - i[T_A, \Phi(y)] + \cdots \ , \tag{4.41}$$

and:

$$[T_A, \Phi] = -\mathbf{T}_A \Phi \ , \tag{4.42}$$

where $\mathbf{T}_A$ is a particular representation of T_A on the field Φ in terms of differential operators. The minus sign in the RHS (4.42) is necessary for consistency with the Jacobi identity:

$$[[T_A, T_B], \Phi] = [T_A, [T_B, \Phi]] - [T_B, [T_A, \Phi]] \ , \tag{4.43}$$

since $[T_A, [T_B, \Phi]] = \mathbf{T_B T_A}\Phi$.

[18] Moreover, the coset manifold has a natural metric $g_\mathcal{M}$ induced from the Killing metric on G, and one can show that the real vector fields:

$$\mathbf{K}_A = i\mathbf{T}_A = k^a_A(y)\frac{\partial}{\partial y^a} \ ,$$

are Killing vectors of the pseudo-Riemannian manifold $(\mathcal{M}, g_\mathcal{M})$.

4.2.2 Minkowski space as a coset manifold

Consider the 4d Poincaré *group:*

$$ISO(1,3) \cong SO(1,3) \ltimes \mathbb{R}^{1,3} \ . \tag{4.44}$$

Any group element (connected to the identity) can be written as:

$$g = e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}+ix^\mu P_\mu} \in ISO(1,3) \ , \tag{4.45}$$

for some parameters $\omega^{\mu\nu}$ and x^μ. Then, Minkowski space-time itself can be thought of as a coset manifold with $G = ISO(1,3)$ and $H = SO(1,3)$. That is:

$$\mathbb{R}^{1,3} \cong ISO(1,3)/SO(1,3) \ , \tag{4.46}$$

where $SO(1,3)$ acts from the right. Here, the generators K_a are simply the translation generators P_μ. The quotient gives a symmetric space, since (4.29) trivially holds.

Let us parameterise the coset $ISO(1,3)/SO(1,3)$ with the coordinates $y = (x^\mu)$, with:

$$\mathbf{x}(x) = e^{-ix^\mu P_\mu} \ , \tag{4.47}$$

where the minus sign is introduced for convenience. Let us see how translations and rotations act on G. Consider the left-multiplication (4.32) by either a translation or a rotation:

$$g_T \equiv e^{ia^\mu P_\mu} \ , \qquad g_R \equiv e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}} \ . \tag{4.48}$$

For the translation, we have:

$$g_T^{-1}\mathbf{x}(x) = \mathbf{x}(x+a) = \mathbf{x}(x') \ , \tag{4.49}$$

trivially, therefore we find:

$$x'^\mu = x^\mu + a^\mu \ , \tag{4.50}$$

and the differential operator constructed in (4.35) is simply:

$$\mathbf{P}_\mu = -i\frac{\partial}{\partial x^\mu} \ , \tag{4.51}$$

as expected. Similarly, for a rotation, we find:

$$\begin{aligned} g_R^{-1}\mathbf{x}(x) &= e^{-ix^\mu P_\mu - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} - \frac{1}{4}\omega^{\rho\sigma}x^\mu[M_{\rho\sigma},P_\mu]+\cdots} \\ &= \mathbf{x}(x')h = e^{-ix'^\mu P_\mu}e^{-\frac{i}{2}\widetilde{\omega}^{\rho\sigma}M_{\rho\sigma}} \\ &= e^{-ix'^\mu P_\mu - \frac{i}{2}\widetilde{\omega}^{\mu\nu}M_{\mu\nu} - \frac{1}{4}\omega^{\rho\sigma}x'^\mu[P_\mu,M_{\rho\sigma}]+\cdots} \ . \end{aligned} \tag{4.52}$$

At first order, this gives $\omega^{\mu\nu} = \widetilde{\omega}^{\mu\nu}$ and:

$$x'^\mu = x^\mu + \frac{1}{2}\omega^{\rho\sigma}x^\nu(-\eta_{\nu\rho}\delta_\sigma{}^\mu + \eta_{\nu\sigma}\delta_\rho{}^\mu) \ . \tag{4.53}$$

This gives us the generators:

$$\mathbf{M}_{\rho\sigma} = i\left(x_\rho \frac{\partial}{\partial x^\sigma} - x_\sigma \frac{\partial}{\partial x^\rho}\right) , \tag{4.54}$$

where $x_\mu \equiv \eta_{\mu\nu} x^\nu$. The differential operators (4.51) and (4.54) satisfy the Poincaré algebra (3.1) on *scalar* fields $\varphi(y)$ in $\mathbb{R}^{1,3}$, as one can readily check.

4.2.3 4d $\mathcal{N} = 1$ superspace as a coset super-manifold

The four-dimensional $\mathcal{N} = 1$ *superspace* is defined similarly to (4.46), as the coset:

$$\mathbb{R}^{1,3|4} \cong ISO(1,3|4)/SO(1,3) . \tag{4.55}$$

Here, $ISO(1,3|1)$ denotes the 4d $\mathcal{N} = 1$ super-Poincaré "group" (or supergroup) obtained by exponentiating the super-Poincaré algebra generators:

$$g = e^{\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} + i x^\mu P_\mu + i\theta^\alpha Q_\alpha + i\bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}} \in ISO(1,3|1) . \tag{4.56}$$

The parameters θ^α and $\bar\theta_{\dot\alpha}$ are *Grassmanian numbers*—that is, they anti-commute:

$$\{\theta^\alpha, \theta^\beta\} = 0 , \qquad \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0 , \qquad \{\bar\theta_{\dot\alpha}, \theta^\beta\} = 0 . \tag{4.57}$$

We parameterise the quotient (4.55) by:

$$\mathbf{x}(y) = e^{-i(x^\mu P_\mu + i\theta Q + i\bar\theta\bar Q)} , \tag{4.58}$$

with the *superspace coordinates*:

$$y = (x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}) , \tag{4.59}$$

bosonic and fermionic—the central new players are the four Grasmmanian coordinates θ and $\bar\theta$.

The key idea of superspace is that we can view supersymmetry *geometrically*, as a "translation" along the fermionic coordinates of superspace. Indeed, consider the supersymmetry "group element:"

$$g_{\text{SUSY}} = e^{i\eta Q + i\bar\eta\bar Q} , \tag{4.60}$$

for some arbitrary Grassmanian parameters η^α, $\bar\eta_{\dot\alpha}$. We have:

$$g^{-1}_{\text{SUSY}}\mathbf{x}(y) = e^{-ix^\mu P_\mu - i(\theta+\eta)Q - i(\bar\theta+\bar\eta)\bar Q - \frac{1}{2}[\eta Q, \bar\theta\bar Q] - \frac{1}{2}[\bar\eta\bar Q, \theta Q] + \cdots} . \tag{4.61}$$

Using the supersymmetry algebra, we get:

$$[\eta Q, \bar\theta\bar Q] = \eta^\alpha\bar\theta^{\dot\alpha}\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2\eta\sigma^\mu\bar\theta P_\mu , \tag{4.62}$$

for any two anti-commuting spinors η and $\bar{\theta}$. Therefore, we find the induced action of *supersymmetry* on the superspace coordinates:

$$\begin{aligned} x'^{\mu} &= x^{\mu} - i\eta\sigma^{\mu}\bar{\theta} + i\theta\sigma^{\mu}\bar{\eta} \ , \\ \theta' &= \theta + \eta \ , \\ \bar{\theta}' &= \bar{\theta} + \bar{\eta} \ , \end{aligned} \tag{4.63}$$

Since supersymmetry should act on classical fields as:

$$U(g_{\text{SUSY}})\Phi(x, \theta, \bar{\theta}) = e^{i\eta\mathbf{Q} + i\bar{\eta}\bar{\mathbf{Q}}}\, \Phi(x, \theta, \bar{\theta}) \ , \tag{4.64}$$

we obtain the differential operators *on superspace*:

$$\boxed{\mathbf{Q}_{\alpha} = -i\left(\frac{\partial}{\partial\theta^{\alpha}} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}\right) \ , \qquad \bar{\mathbf{Q}}_{\dot{\alpha}} = i\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}\right) \ .} \tag{4.65}$$

The Grassmanian derivatives are defined as:

$$\frac{\partial}{\partial\theta^{\alpha}}\theta^{\beta} = \delta^{\beta}_{\alpha} \ , \qquad \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} \ . \tag{4.66}$$

We refer to subsection 4.2.4 for further discussion. The differential operators (4.65) realise the supersymmetry algebra, since:

$$\{\mathbf{Q}_{\alpha}, \bar{\mathbf{Q}}_{\dot{\alpha}}\} = -2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} = 2\sigma^{\mu}_{\alpha\dot{\alpha}}\mathbf{P}_{\mu} \ , \quad \{\mathbf{Q}_{\alpha}, \mathbf{Q}_{\beta}\} = 0 \ , \quad \{\bar{\mathbf{Q}}_{\dot{\alpha}}, \bar{\mathbf{Q}}_{\dot{\beta}}\} = 0 \ . \tag{4.67}$$

Note that the field $\Phi(y)$ is a function of superspace, not only space-time–this, almost by definition, is a *superfield.* (We'll give a more precise definition of what is a superfield momentarily.)

Exercise: How are the $SO(1, 3)$ *generators* $M_{\mu\nu}$ *represented on superspace? It should be clear that it cannot be simply as in* (4.54)*. (Explain why.)*

4.2.4 On manipulating the superspace coordinates

Since θ and $\bar{\theta}$ are Grassman numbers, we have to be a bit careful with signs. Let us introduce the short-hand notation:

$$\partial_{\alpha} \equiv \frac{\partial}{\partial\theta^{\alpha}} \ , \qquad \partial^{\alpha} \equiv \frac{\partial}{\partial\theta_{\alpha}} \ , \qquad \bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \ , \qquad \bar{\partial}^{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \ . \tag{4.68}$$

As mentioned above, we define:

$$\partial_{\alpha}\theta^{\beta} = \delta^{\beta}_{\alpha} \ , \qquad \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} \ . \tag{4.69}$$

This implies:

$$\partial^{\alpha}\theta_{\beta} = -\delta^{\alpha}_{\beta} \ , \qquad \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\delta^{\dot{\alpha}}_{\dot{\beta}} \ . \tag{4.70}$$

All derivative are taken from the left; this means that we first need to move any θ^α to the left (incurring whatever signs) before using the definition $\partial_\beta \theta^\alpha = \delta^\alpha_\beta$, and similarly for the $\bar\partial_{\dot\alpha}$ derivative. Note that, in particular, we have:

$$\partial_\alpha \,\theta\theta = 2\theta_\alpha \ , \qquad \bar\partial_{\dot\alpha}\,\bar\theta\bar\theta = -2\bar\theta_{\dot\alpha} \ , \tag{4.71}$$

and therefore:

$$\epsilon^\alpha \partial_\alpha \,\theta\theta = 2\eta\theta \ , \qquad \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha}\,\bar\theta\bar\theta = 2\bar\epsilon\bar\theta \ . \tag{4.72}$$

Because Grassman numbers are anti-commuting, we can always expand any function into a polynomial in θ^α and $\bar\theta^{\dot\alpha}$—the Taylor expansion truncates. For instance, a function of only x^μ and a single Grassman number, θ^1, is given by:

$$F(x,\theta^1) = f_0(x) + \theta^1 f_1(x) \ , \tag{4.73}$$

where f_0 and f_1 are arbitrary functions of x. The expansion truncates because $(\theta^1)^2 = 0$. Thus, derivation is always a purely algebraic operation. For instance, in this example:

$$\frac{\partial}{\partial\theta^1} F(x,\theta^1) = f_1(x) \ . \tag{4.74}$$

In the following, we will also use a notion of *integration* over Grassman numbers—known as Berezin integration. For a single Grassman number, say θ^1, the integration can be defined as:

$$\int d\theta^1 \theta^1 = 1 \ , \qquad \int d\theta^1 g(x,\theta^2,\bar\theta) = 0 \ . \tag{4.75}$$

In other words, it acts just like a derivation. One can check that this is a linear operation. We also have a fermionic "Stoke's theorem:"

$$\int d\theta^1 \frac{\partial}{\partial\theta^1} F = 0 \ . \tag{4.76}$$

Integration over superspace. In 4d $\mathcal{N}=1$ superpace, we define:

$$\int d^2\theta \equiv \frac{1}{2}\int d\theta^1 d\theta^2 \ , \qquad \int d^2\bar\theta \equiv \frac{1}{2}\int d\bar\theta^2 d\bar\theta^1 \ . \tag{4.77}$$

Since $\theta\theta = 2\theta^2\theta^1$ and $\bar\theta\bar\theta = 2\bar\theta^1\bar\theta^2$, this is such that:

$$\int d^2\theta \ \theta\theta = 1 \ , \qquad \int d^2\bar\theta \ \bar\theta\bar\theta = 1 \ . \tag{4.78}$$

In particular, an integral over the four Grassman coordinates is equivalent to collecting the $\theta\theta\bar\theta\bar\theta$ coefficient in the Taylor expansion of the integrand:

$$\int d^2\theta d^2\bar\theta \, F(x,\theta,\bar\theta) = F(x,\theta,\bar\theta)\big|_{\theta\theta\bar\theta\bar\theta} \ . \tag{4.79}$$

One can also check that:

$$\int d^2\theta = \varepsilon^{\alpha\beta}\partial_\alpha\partial_\beta \ , \qquad \int d^2\bar\theta = -\varepsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\alpha}\bar\partial_{\dot\beta} \ . \tag{4.80}$$

4.3 Superfields

A *superfield* is a function over superspace,

$$\mathcal{S}(x,\theta,\bar{\theta}) \ , \tag{4.81}$$

which transforms according to:

$$\begin{aligned} e^{ia^\mu \mathbf{P}_\mu + i\epsilon\mathbf{Q} + i\bar{\epsilon}\bar{\mathbf{Q}}}\, \mathcal{S}(x,\, \theta,\, \bar{\theta}) &= \mathcal{S}(x',\, \theta',\, \bar{\theta}') \\ &= \mathcal{S}(x + a - i\epsilon\sigma\bar{\theta} + i\theta\sigma\bar{\epsilon},\, \theta + \epsilon,\, \bar{\theta} + \bar{\epsilon}) \ , \end{aligned} \tag{4.82}$$

under translations and supersymmetry (that is, any superspace translations). It is clear that linear combinations of superfields are superfields, and products of superfields are superfields.

We can expand an arbitrary superfield $\mathcal{S}$ in the Grassman coordinates, to obtain:

$$\begin{aligned} \widetilde{\mathcal{S}}(x,\theta,\bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta\theta M(x) - \frac{i}{2}\bar{\theta}\bar{\theta}\bar{M}(x) \\ &\quad - \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\eta}(x) - i\bar{\theta}\bar{\theta}\theta\eta(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\,\widetilde{D}(x) \ . \end{aligned} \tag{4.83}$$

The coefficients in the expansions are ordinary fields in Minkowski space-time, [19] the *components* of the superfield. Assuming that the fields $C, M, \bar{M}, v_\mu$ and $\widetilde{D}$ are bosonic, the fields $\chi, \bar{\chi}, \eta, \bar{\eta}$ are fermionic. The simplest example of a general superfield is when C is a scalar field—then, $\chi_\alpha, \bar{\chi}^{\dot{\alpha}}$ are spinor fields, and so on and so forth.

If C is a real field, the superfield $\mathcal{S}$ contain 8 bosonic and 8 bosonic degrees of freedom. This is too many to furnish an *irreducible* representation of the supersymmetry algebra. For instance, we saw in section 4.1.1 that the off-shell chiral multiplet has $4+4$ degrees of freedom. To obtain irreducible supersymmetry multiplet in superspace, therefore, we will need to impose *superspace constraints.*

4.4 Superspace for other dimensions and/or $\mathcal{N}$'s?

Before we specialise to 4d $\mathcal{N}=1$ superspace in most of the following, we should briefly mention how the superspace formalism can be generalised to other space-time dimensions and other amounts of supersymmetry.

In four dimensions, it turns out that there is no *useful* superspace formalism (of the simple type we just discussed) for $\mathcal{N} > 1$. [20] More precisely, for $\mathcal{N}=2$ there exists a superspace formalism for pure gauge theories, but not for matter fields (in so-called hypermultiplets). In the $\mathcal{N}=4$ case, there is no superspace formalism at all.

[19] The numerical coefficients in (4.83) are just a matter of conventions, of course.

[20] There exists more sophisticated approaches to superspace for extended supersymmetry, but they are rather more complicated, and it is fair to say that none has been particularly useful in explicit computations. One important exception is the superspace description of the 4d $\mathcal{N}=2$ vector multiplet.

In other space-time dimensions, the rule of thumb is that there exists a useful superspace formalism for less than or equal to 4 real supercharges, $N_Q \leq 4$; in particular, only in dimensions $d \leq 4$. The basic reason can be understood as follows. In a general supersymmetric theory with N_Q supercharges, we would introduce N_Q superspace coordinates θ^a, $a = 1, \cdots, N_Q$, and a general superfield would take the form:

$$F(x,\theta) = f_0(x) + \theta^a f_a(x) + \frac{1}{2}\theta^a\theta^b f_{ab}(x) + \cdots + \theta^1\theta^2\cdots\theta^{N_Q} f_{\text{top}}(x) \ . \qquad (4.84)$$

That gives a total of:

$$\sum_{k=0}^{N_Q} \binom{N_Q}{k} = 2^{N_Q} = 2^{N_Q-1} \text{ bosonic} + 2^{N_Q-1} \text{ fermionic} \qquad (4.85)$$

components. In general, this is much larger than the number of degrees of freedom expected in an irreducible representation of supersymmetry. For instance, for 4d $\mathcal{N}=4$ rigid supersymmetry, the vector multiplet contains 4 Weyl spinors, for a total of 16 off-shell fermionic components, while $2^{N_Q-1} = 2^{15}$, which is much larger than $16 = 2^4$. In general, there might not exist any consistent sets of constraints that give rise to the correct off-shell supermultiplets. There are also no-go theorems to that effect. For instance, there does not exist any off-shell formulation of 4d $\mathcal{N}=4$ SYM with a finite number of auxiliary fields [19].

5 4d $\mathcal{N}=1$ supersymmetry, part I: chiral multiplets

In this section, we write down supersymmetric field theories explicitly. More precisely, we construct 4d $\mathcal{N}=1$ supersymmetric Lagrangians systematically, restricting ourselves to theories of *chiral multiplets*. These are theories that contain only scalar fields and Weyl fermions.

(Later on in the lectures, we will discuss in detail the final ingredient necessary to describe anything resembling the "real world" of Particle Physics: the gauge fields.)

5.1 The SUSY-covariant derivatives

Since P_μ commutes with the supercharges, space-time derivatives ∂_μ commute with the supercharge operators $\mathbf{Q}$ and $\bar{\mathbf{Q}}$. This is also clear from their explicit expressions,

$$\mathbf{Q}_\alpha = -i\left(\partial_\alpha - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\right) \ , \qquad \bar{\mathbf{Q}}_{\dot\alpha} = i\left(\bar\partial_{\dot\alpha} - i(\theta^\alpha\sigma^\mu)_{\dot\alpha}\partial_\mu\right) \ . \qquad (5.1)$$

This is necessary so that various differential constraints, such as for instance the on-shell condition $P^2 = -M^2$ for a free scalar field, commute with supersymmetry.

On the other hand, the naive superspace derivatives ∂_α and $\bar{\partial}_{\dot{\alpha}}$ do *not* commute with supersymmetry. For instance:

$$[\delta, \partial_\alpha] = [i\bar{\epsilon}\bar{Q}, \partial_\alpha] = -i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu \neq 0 \ . \tag{5.2}$$

Then, the derivative $\partial_\alpha\mathcal{S}$ of a superfield $\mathcal{S}$ is not itself a superfield. There exists, however, *supersymmetry-covariant derivatives.* They are given by:

$$\boxed{\mathrm{D}_\alpha = \partial_\alpha + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \ , \qquad \bar{\mathrm{D}}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \ .} \tag{5.3}$$

The same way that the supercharges (5.1) are constructed by looking as left multiplication in (4.61), one can define (5.3) in the coset construction by *right multiplication*—one can readily check that:

$$\mathbf{x}(y)g_{\mathrm{SUSY}}^{-1} = e^{-ix^\mu P_\mu - i\theta Q - i\bar{\theta}\bar{Q}}\, e^{-i\eta Q - i\bar{\eta}\bar{Q}} = \mathbf{x}(y') \ , \tag{5.4}$$

gives a superspace translation generated by (5.3). The differential operators (5.3) anti-commute with the supercharges:

$$\{\mathrm{D}_\alpha, \mathbf{Q}_\beta\} = \{\mathrm{D}_\alpha, \bar{\mathbf{Q}}_{\dot{\beta}}\} = 0 \ , \qquad \{\bar{\mathrm{D}}_{\dot{\alpha}}, \mathbf{Q}_\beta\} = \{\bar{\mathrm{D}}_{\dot{\alpha}}, \bar{\mathbf{Q}}_{\dot{\beta}}\} = 0 \ . \tag{5.5}$$

They also satisfy the supersymmetry algebra with the opposite sign:

$$\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}\} = -2\sigma^\mu_{\alpha\dot{\alpha}}\mathbf{P}_\mu \ , \quad \{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = 0 \ , \quad \{\bar{\mathrm{D}}_{\dot{\alpha}}, \bar{\mathrm{D}}_{\dot{\beta}}\} = 0 \ . \tag{5.6}$$

Using D, $\bar{D}$ and ∂_μ, we can build an arbitrary superfield by taking product of superfields and their covariant derivatives. Note the curious property:

$$\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \ . \tag{5.7}$$

In the language of differential geometry, this means that the connection:

$$\nabla = (\partial_\mu, \mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}) \tag{5.8}$$

on "flat" superspace has a non-trivial *torsion,* with $T_{\alpha\dot{\alpha}}{}^\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}}$ being the superspace torsion tensor. For future reference, we also compute the commutator:

$$\begin{aligned}[\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}] &= 2\partial_\alpha\bar{\partial}_{\dot{\alpha}} - 2i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\alpha\partial_\mu + 2i(\sigma^\mu\bar{\theta})_\alpha\partial_{\dot{\alpha}}\partial_\mu \\ &\quad - \theta\sigma^\mu\theta(\sigma_{\mu\alpha\dot{\alpha}}\partial^\rho\partial_\rho - 2\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\partial_\mu) \ .\end{aligned} \tag{5.9}$$

(*Exercise: Check this!*)

5.2 General multiplet and real multiplet

The "general multiplet" is a *long multiplet* with components:

$$\widetilde{\mathcal{S}} = \Big(C, \chi \ , \bar{\chi} \ , M \ , \bar{M} \ , v_\mu \ , \eta \ , \bar{\eta} \ , \widetilde{D}\Big) \ , \tag{5.10}$$

as introduced above. One often considers the case when the bottom component C is a *scalar*, but it could also transform in any representation of the Lorentz group. Let us again write down the component expansion of the corresponding *superfield*:

$$\begin{aligned}\widetilde{\mathcal{S}}(x,\theta,\bar\theta) &= C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \frac{i}{2}\theta\theta M(x) - \frac{i}{2}\bar\theta\bar\theta\bar M(x) \\ &\quad - \theta\sigma^\mu\bar\theta v_\mu(x) + i\theta\theta\bar\theta\bar\eta(x) - i\bar\theta\bar\theta\theta\eta(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\,\widetilde{D}(x)\ .\end{aligned} \tag{5.11}$$

It is a tedious but straightforward exercise to derive the supersymmetry transformations in components, using the definition:

$$\delta = \delta_\epsilon + \delta_{\bar\epsilon} = i\epsilon\mathbf{Q} + i\bar\epsilon\mathbf{Q}\ . \tag{5.12}$$

We simply write down:

$$\delta\mathcal{S}(\varphi) = \mathcal{S}(\delta\varphi)\ , \tag{5.13}$$

where φ denote the component fields, and match the result term by term in the $\theta,\bar\theta$ expansion. One then finds:

$$\begin{aligned}&\delta C = i\epsilon\chi - i\bar\epsilon\bar\chi\ ,\\ &\delta\chi_\alpha = \epsilon_\alpha M + (\sigma^\mu\bar\epsilon)_\alpha(\partial_\mu C + iv_\mu)\ ,\\ &\delta\bar\chi_{\dot\alpha} = \bar\epsilon_{\dot\alpha}\bar M + (\epsilon\sigma^\mu)_{\dot\alpha}(\partial_\mu C - iv_\mu)\ ,\\ &\delta M = i\bar\epsilon\bar\sigma^\mu\partial_\mu\chi + 2\bar\epsilon\bar\eta\ ,\\ &\delta\bar M = i\epsilon\sigma^\mu\partial_\mu\bar\chi + 2\epsilon\eta\ ,\\ &\delta v_\mu = i\epsilon\sigma_\mu\bar\eta + i\bar\epsilon\bar\sigma_\mu\eta - \frac{1}{2}\epsilon\sigma^\nu\bar\sigma_\mu\partial_\nu\chi - \frac{1}{2}\bar\epsilon\bar\sigma^\nu\sigma_\mu\partial_\nu\bar\chi\ ,\\ &\delta\eta_\alpha = i\epsilon_\alpha\widetilde{D} - \frac{1}{2}(\sigma^\nu\sigma^\mu\epsilon)_\alpha\partial_\mu v_\nu + \frac{i}{2}(\sigma^\mu\bar\epsilon)_\alpha\partial_\mu M\ ,\\ &\delta\bar\eta_{\dot\alpha} = -i\bar\epsilon\widetilde{D} - \frac{1}{2}(\bar\epsilon\bar\sigma^\mu\sigma^\nu)_{\dot\alpha}\partial_\mu v_\nu - \frac{i}{2}(\epsilon\sigma^\mu)_{\dot\alpha}\partial_\mu M\ ,\\ &\delta\widetilde{D} = -\epsilon\sigma^\mu\partial_\mu\bar\eta + \bar\epsilon\bar\sigma^\mu\partial_\mu\eta\ .\end{aligned} \tag{5.14}$$

These transformations law look rather cumbersome. Thankfully, we can find a simpler-looking form by a simple field redefinition. Let us introduce the new fields $\lambda,\bar\lambda$ and D to replace $\eta,\bar\eta$ and $\widetilde{D}$, respectively:

$$\begin{aligned}&\lambda_\alpha = \eta_\alpha - \frac{i}{2}(\sigma^\mu\partial_\mu\bar\chi)_\alpha\ ,\\ &\bar\lambda^{\dot\alpha} = \bar\eta^{\dot\alpha} - \frac{i}{2}(\bar\sigma^\mu\partial_\mu\chi)^{\dot\alpha}\ ,\\ &D = \widetilde{D} - \frac{1}{2}\partial^2 C\ ,\end{aligned} \tag{5.15}$$

where $\partial^2 \equiv \partial_\mu \partial^\mu$. Then, the supersymmetry variations are simplified to:

$$\begin{aligned}
\delta C &= i\epsilon\chi - i\bar{\epsilon}\bar{\chi} \ , \\
\delta\chi_\alpha &= \epsilon_\alpha M + (\sigma^\mu\bar{\epsilon})_\alpha(\partial_\mu C + iv_\mu) \ , \\
\delta\bar{\chi}_{\dot{\alpha}} &= \bar{\epsilon}_{\dot{\alpha}}\bar{M} + (\epsilon\sigma^\mu)_{\dot{\alpha}}(\partial_\mu C - iv_\mu) \ , \\
\delta M &= 2i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\chi + 2\bar{\epsilon}\bar{\lambda} \ , \\
\delta\bar{M} &= 2i\epsilon\sigma^\mu\partial_\mu\bar{\chi} + 2\epsilon\lambda \ , \\
\delta v_\mu &= i\epsilon\sigma_\mu\bar{\lambda} + i\bar{\epsilon}\bar{\sigma}_\mu\lambda + \epsilon\partial_\mu\chi + \bar{\epsilon}\partial_\mu\bar{\chi} \ , \\
\delta\lambda_\alpha &= i\epsilon_\alpha D + 2(\sigma^{\mu\nu}\epsilon)_\alpha\partial_\mu v_\nu \ , \\
\delta\bar{\lambda}_{\dot{\alpha}} &= -i\bar{\epsilon}_{\dot{\alpha}} D - 2(\bar{\epsilon}\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}\partial_\mu v_\nu \ , \\
\delta D &= -\epsilon\sigma^\mu\partial_\mu\bar{\lambda} + \bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\lambda \ ,
\end{aligned} \tag{5.16}$$

as one can readily check. This is the most convenient parameterisation of a *general multiplet:*

$$\mathcal{S} = \big(C, \chi \ , \bar{\chi} \ , M \ , \bar{M} \ , v_\mu \ , \lambda \ , \bar{\lambda} \ , D\big) \ . \tag{5.17}$$

When C is real, this is called a *real multiplet*—in that case, the fields χ, M, λ and $\bar{\chi}, \bar{M}, \bar{\lambda}$ are complex conjugate of each other, as implied by the notation. In the general case, however, all the fields in (5.17) can be complex and unrelated to each other. The corresponding general superfield is given by:

$$\begin{aligned}
\mathcal{S}(x,\theta,\bar{\theta}) \ &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta\theta M(x) - \frac{i}{2}\bar{\theta}\bar{\theta}\bar{M}(x) \\
&\quad - \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}(\bar{\sigma}^\mu\partial_\mu\chi(x))\right) \\
&\quad - i\bar{\theta}\bar{\theta}\theta\left(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) + \frac{1}{2}\partial^2 C(x)\right) \ ,
\end{aligned} \tag{5.18}$$

when expanded in components.

By direct computation, one can show that the supersymmetry variations (5.16) indeed satisfy the supersymmetry algebra (4.8)—we know this has to be true, by construction, but it never hurts to check it explicitly.

5.3 Chiral multiplet

The chiral multiplet contains a complex boson and a left-chiral Weyl fermion, but no right-chiral Weyl fermion. Looking at the general multiplet (5.17), we should therefore set to zero the right-chiral fermions:

$$\bar{\chi}_{\dot{\alpha}} = 0 \ , \qquad \bar{\lambda}_{\dot{\alpha}} = 0 \ . \tag{5.19}$$

For this constraints to be *consistent* with supersymmetry, the supersymmetry variations of $\bar{\chi}$ and $\bar{\lambda}$ should vanish, too. We easily see that $\delta\bar{\chi} = 0$ implies $\bar{M} = 0$ and $v_\mu = -i\partial_\mu C$. Then, $\delta\bar{\lambda} = 0$ just implies that $D = 0$, and $\delta\bar{M} = 0$ (for consistency with $\bar{M}=0$) fixes $\lambda=0$ as well. In short, the chiral multiplet Φ is obtained from the general multiplet $\mathcal{S}$ by setting:

$$\begin{aligned}\mathcal{S}^{\Phi} \;&= \Big(C^{\Phi}=\phi,\, \chi^{\Phi}=-i\sqrt{2}\psi,\, \bar{\chi}^{\Phi}=0,\, M^{\Phi}=-2iF,\, \bar{M}^{\Phi}=0,\\ &\qquad v_{\mu}^{\Phi}=-i\partial_{\mu}\phi,\, \lambda^{\Phi}=0,\, \bar{\lambda}^{\Phi}=0,\, D^{\Phi}=0\Big)\;.\end{aligned} \tag{5.20}$$

The numerical factors are a matter of convention, of course. One can readily check that plugging (5.20) into the supersymmetry transformations (5.16) reproduces the chiral multiplet transformations (4.10). Similarly, we can construct the anti-chiral multiplet as:

$$\begin{aligned}\mathcal{S}^{\bar{\Phi}} \;&= \Big(C^{\bar{\Phi}}=\bar{\phi},\, \chi^{\bar{\Phi}}=0,\, \bar{\chi}^{\bar{\Phi}}=i\sqrt{2}\bar{\psi},\, M^{\bar{\Phi}}=0,\, \bar{M}^{\bar{\Phi}}=2i\bar{F},\\ &\qquad v_{\mu}^{\bar{\Phi}}=i\partial_{\mu}\bar{\phi},\, \lambda^{\bar{\Phi}}=0,\, \bar{\lambda}^{\bar{\Phi}}=0,\, D^{\bar{\Phi}}=0\Big)\;.\end{aligned} \tag{5.21}$$

Such supersymmetry-preserving constraints can be imposed more elegantly by using the supersymmetry-covariant derivative. By definition, a *chiral superfield* Φ is one that satisfy the differential constraint:

$$\boxed{\bar{\mathrm{D}}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta})=0\;.} \tag{5.22}$$

Similarly, an *anti-chiral superfield* $\bar{\Phi}$ satisfies:

$$\boxed{\mathrm{D}_{\alpha}\bar{\Phi}(x,\theta,\bar{\theta})=0\;.} \tag{5.23}$$

One can readily solve these constraints by the following trick. Introduce the "chiral coordinates:"

$$z^{\mu}=x^{\mu}+i\theta\sigma^{\mu}\bar{\theta}\;,\qquad\qquad \bar{z}^{\mu}=x^{\mu}-i\theta\sigma^{\mu}\bar{\theta}\;. \tag{5.24}$$

They satisfy:

$$\bar{\mathrm{D}}_{\dot{\alpha}}z^{\mu}=0\;,\qquad\qquad \mathrm{D}_{\alpha}\bar{z}^{\mu}=0\;. \tag{5.25}$$

A chiral superfield is simply a function of $(z^{\mu},\theta^{\alpha})$ only, while an anti-chiral superfield is a function of $(\bar{z}^{\mu},\bar{\theta}^{\dot{\alpha}})$ only:

$$\Phi=\Phi(z,\theta)\;,\qquad\qquad \bar{\Phi}=\bar{\Phi}(\bar{z},\bar{\theta})\;, \tag{5.26}$$

since $\bar{\mathrm{D}}_{\dot{\alpha}}\theta=0$ and $\mathrm{D}_{\alpha}\bar{\theta}=0$. Thus, a chiral superfield in chiral coordinates has the simple expansion:

$$\Phi(z,\theta)=\phi(z)+\sqrt{2}\theta\psi(z)+\theta\theta F(z)\;, \tag{5.27}$$

and similarly for the anti-chiral multiplet:

$$\bar{\Phi}(\bar{z},\bar{\theta})=\bar{\phi}(\bar{z})+\sqrt{2}\bar{\theta}\bar{\psi}(\bar{z})+\bar{\theta}\bar{\theta}\bar{F}(\bar{z})\;. \tag{5.28}$$

Expressing (5.27) in terms of $(x, \theta, \bar{\theta})$, we find:

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) \\ &\quad + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) \ . \end{aligned} \tag{5.29}$$

Comparing to the general multiplet (5.18), we see that (5.29) indeed corresponds to the specialisation (5.20) of the general superfield. Similarly, for the anti-chiral superfield, we have:

$$\begin{aligned} \bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{\phi}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \bar{\theta}\bar{\theta}\bar{F}(x) \\ &\quad - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\phi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\bar{\phi}(x) \ . \end{aligned} \tag{5.30}$$

The product of two chiral superfields is again a chiral superfield, since:

$$\bar{D}_{\dot{\alpha}}(\Phi_1\Phi_2) = 0 \ . \tag{5.31}$$

We can work out the product rules in components, for instance by using the chiral coordinates z^μ, for simplicity:

$$\Phi_1\Phi_2 = (\phi_1 + \sqrt{2}\theta\psi_1 + \theta\theta F_1)(\phi_2 + \sqrt{2}\theta\psi_2 + \theta\theta F_2) \ . \tag{5.32}$$

Collecting the terms at each order in θ, one finds: [21]

$$\begin{aligned} \phi^{\Phi_1\Phi_2} &= \phi_1\phi_2 \ , \\ \psi^{\Phi_1\Phi_2} &= \psi_1\phi_2 + \psi_2\phi_1 \ , \\ F^{\Phi_1\Phi_2} &= F_1\phi_2 + F_2\phi_1 - \psi_1\psi_2 \ . \end{aligned} \tag{5.33}$$

On the other hand, the product of a chiral with an anti-chiral superfield is a general superfield; in particular, the product of Φ with its complex conjugate $\bar{\Phi}$ is a real superfield,

$$\mathcal{S}^{\bar{\Phi}\Phi} \equiv \bar{\Phi}\Phi \ , \qquad (\mathcal{S}^{\bar{\Phi}\Phi})^\dagger = \mathcal{S}^{\bar{\Phi}\Phi} \ . \tag{5.34}$$

The whole point of the superfield formalism is that it makes it easy to take "products of supermultiplets." In this example, the bottom component of $\mathcal{S}^{\bar{\Phi}\Phi}$ is the real scalar $C^{\bar{\Phi}\Phi} = \bar{\phi}\phi$, and all the other components can be found similarly to (5.33), with a little bit more algebra. Note also that the *sum* of superfields $\Phi + \bar{\Phi}$ is another real superfield.

[21] In the last line, we used the simple Fierz identity:

$$2\theta\psi_1\,\theta\psi_2 = -\theta\theta\,\psi_1\psi_2 \ .$$

This can be checked using the identity (A.4) in Appendix.

5.4 Supersymmetric Lagrangians—D-terms and F-terms

We can now answer the question of how to build supersymmetric actions—namely, an action

$$S = \int d^4x \mathscr{L}(\varphi, \partial_\mu \varphi, \cdots) \tag{5.35}$$

such that:

$$\delta S = 0 \ . \tag{5.36}$$

Note that the Lagrangian density $\mathscr{L}$ (which we call 'the Lagrangian,' for short) itself cannot be supersymmetric—if $\delta S = 0$, the commutators (4.8) imply that S is a constant. However, for the action to be invariant, it is sufficient for the supersymmetric variation of $\mathscr{L}$ to be a total derivative:

$$\boxed{\delta \mathscr{L} = \partial_\mu V^\mu_{\rm SUSY} \ .} \tag{5.37}$$

By abuse of notation, we call such a $\mathscr{L}$ 'a supersymmetric Lagrangian.'

5.4.1 D-terms

Looking at the supersymmetry transformation laws of a general multiplet in equation (5.16), we see that the only field component whose variation is a total derivative is the field $D(x)$—or, equivalently, $\widetilde{D}(x)$ in (5.14); the two definitions are related by a total derivative anyway. Indeed, since the supersymmetry parameters are constant (we are doing "rigid supersymmetry"), we have:

$$\delta D = \partial_\mu \left(-\epsilon \sigma^\mu \bar{\lambda} + \bar{\epsilon} \bar{\sigma}^\mu \lambda \right) \ . \tag{5.38}$$

Thus, there is a straighforward way to build a supersymmetric action: construct a general superfield $\mathcal{S}^L$—more precisely, $\mathcal{S}^L$ should be a real superfield, so that the action is real—, for instance by taking products and sums of elementary superfields, and consider the so-called "D-term Lagrangian:"

$$\mathscr{L}_D \equiv \frac{1}{2} D^{\mathcal{S}^L} \ . \tag{5.39}$$

This can be written more elegantly as a superspace integral over the real superfield itself:

$$\mathscr{L}_D = \int d^2\theta d^2\bar{\theta}\, \mathcal{S}^L \ . \tag{5.40}$$

The supersymmetric action is an integral over 4d $\mathcal{N}=1$ superspace, $\mathbb{R}^{1,3|4}$:

$$\boxed{S_D = \int d^4x \int d^2\theta d^2\bar{\theta}\, \mathcal{S}^L \ .} \tag{5.41}$$

5.4.2 F-terms

In the presence of chiral multiplets, there is another possibility for constructing supersymmetric Lagrangians. Indeed, from the transformations laws (4.10) of a chiral multiplet, we see that the field $F(x)$ also transforms as a total derivative:

$$\delta F = \partial_\mu \left(i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu \psi \right) \,, \tag{5.42}$$

and similarly for the anti-chiral multiplet. Thus, given any chiral multiplet Φ, we can construct the so-called *F-term and anti-F-term Lagrangians*:

$$\mathscr{L}_F = F^\Phi \,, \qquad\qquad \mathscr{L}_{\bar{F}} = \bar{F}^{\bar{\Phi}} \,, \tag{5.43}$$

which are separately supersymmetric. (We should add the two of them to obtain a real action, though.) The F-term action can be written as an integral over *half* of superspace:

$$\boxed{S_F = \int d^4x \int d^2\theta \Phi \,, \qquad\qquad S_{\bar{F}} = \int d^4x \int d^2\bar{\theta}\bar{\Phi} \,.} \tag{5.44}$$

5.5 Lagrangians of chiral multiplets

Consider now a theory of n chiral multiplet Φ^i, with $i = 1, \cdots, n$—that is, a theory of n complex bosons ϕ^i and n Weyl fermions ψ^i. Since Φ is complex, it is convenient to denote the anti-chiral multiplet by $\bar{\Phi}^{\bar{i}}$, with indices $\bar{i} = 1, \cdots, n$. A general supersymmetric Lagrangian takes the form:

$$\boxed{\mathscr{L} = \int d^2\theta d^2\bar{\theta}\, K(\bar{\Phi}, \Phi) + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}) \,.} \tag{5.45}$$

It is fully determined by two functions:

- The real superfield $K(\bar{\Phi}, \Phi)$, which is an arbitrary real function of the fundamental chiral multiples Φ and $\bar{\Phi}$. It is called the "Kähler potential" (more on this in subsection 5.8 below). It encodes the *kinetic terms* of the theory.
- The chiral superfield $W(\Phi)$, which is an arbitrary *holomorphic* function of the fundamental chiral superfields Φ^i. It is is called the *superpotential* and it encodes the *interaction terms.* The anti-holomorphic function $\bar{W}(\bar{\Phi})$ is the complex conjugate of $W(\Phi)$.

Comment on notation: The term "superpotential" can denote either the chiral superfields $W(\Phi)$, or the holomorphic function $W(\phi)$, an holomorphic function of the scalar fields ϕ^i. Of course, $W(\phi)$ is the bottom component of the superfield $W(\Phi)$.

In order to obtain a quantum field theory which is *local*, we should restrict ourselves to polynomial functions $K(\bar{\Phi},\Phi)$ and $W(\Phi)$. Even so, for general polynomials, the Lagrangian would be non-renormalisable. Recall that $\mathscr{L}$ is *renormalisable* (at tree level) if and only if every operator in the Lagrangian has engineering dimensions less or equal than 4, which ensures that the coupling constants have non-negative mass dimensions. Schematically, we denote this by:

$$[\mathscr{L}] \leq 4 \,, \qquad \text{for renormalisability.} \tag{5.46}$$

Note also that we have:

$$[d\theta] = [d\bar{\theta}] = \frac{1}{2} \,, \qquad [d^4x] = -4 \,, \tag{5.47}$$

therefore, in a supersymmetric theory, we need:

$$[K] \leq 2 \,, \qquad [W] \leq 3 \,, \qquad \text{for renormalisability.} \tag{5.48}$$

Since the engineering dimensions of a four-dimensional scalar is 1, we have $[\Phi]=1$, and therefore the only renormalisable Kähler potential is quadratic:

$$K(\bar{\Phi},\Phi) = g_{i\bar{i}}\bar{\Phi}^i\Phi^{\bar{i}} \,, \tag{5.49}$$

with $g_{i\bar{i}}$ a constant Hermitian matrix. This is often called the "canonical Kähler potential." Similarly, a renormalisable superpotential is at most cubic in the chiral superfields:

$$W = g_i\Phi^i + g_{ij}\Phi^i\Phi^j + g_{ijk}\Phi^i\Phi^j\Phi^k \,, \tag{5.50}$$

with g_i, g_{ij}, g_{ijk} some coupling constants.

5.5.1 R-symmetry and the superpotential

It is often useful to keep track of the R-symmetry, $U(1)_R$, of a given theory. Consider the $U(1)_R$ action on chiral multiplets:

$$\Phi_i \to e^{ir_i\alpha}\Phi_i \,, \qquad \bar{\Phi}_{\bar{i}} \to e^{-ir_i\alpha}\bar{\Phi}_{\bar{i}} \,, \tag{5.51}$$

which leaves the canonical Kähler potential term invariant. We say the "the chiral multiplet Φ_i has R-charge r_i, denoted as $R[\Phi_i]=r_i$ (and then $R[\bar{\Phi}_{\bar{i}}]=-r_i$). This means that the R-charges of its *component fields* are:

$$R[\phi_i] = r_i \,, \qquad R[\psi_i] = r_i - 1 \,, \qquad R[F_i] = r_i - 2 \,. \tag{5.52}$$

This $U(1)_R$ is a symmetry of the canonical kinetic terms. In fact, this is a symmetry for *any* choice of the R-charges r_i. This is best understood as follows. Consider the "reference R-symmetry" R_0 such that:

$$R_0[\Phi_i] = 0 \,. \tag{5.53}$$

We also have the ordinary *flavor symmetries* $U(1)_{\mathbf{F}_i}$ which rotate each Φ_i individually, with charge 1:

$$\mathbf{F}_i[\Phi_j] = \delta_{ij} \ . \tag{5.54}$$

A flavor symmetry, by definition, is such that all the component fields have the same flavor charge; here:

$$\mathbf{F}_i[\phi_j] = \mathbf{F}_i[\psi_j] = \mathbf{F}_i[F_j] = \delta_{ij} \ . \tag{5.55}$$

Then, a general R-symmetry as above is simply a mixing of R_0 with the flavor symmetries:

$$R = R_0 + \sum_i r_i \mathbf{F}_i \ . \tag{5.56}$$

In general, the flavor symmetry might be larger. For a free theory with diagonal kinetic term, we actually have a $U(n)$ flavor symmetry; here we just considered the Cartan subgroup $\prod_i U(1)_{\mathbf{F}_i} \subset U(n)$.

This is an important lesson, valid more generally in 4d $\mathcal{N} = 1$ supersymmetric theories: what we mean by "the R-symmetry" is often ambiguous, because one can redefine R by mixing with abelian flavor symmetries.

The interaction terms generally break the flavor symmetry of the free theory to a subgroup (possibly trivial). They also constrain the possible choices of R-symmetry. Indeed, since $R[Q_\alpha] = -1$ and $R[\bar{Q}_{\dot{\alpha}}] = 1$, the superspace coordinate themselves are *charged* under $U(1)_R$:

$$R[\theta] = 1 \ , \qquad\qquad R[\bar{\theta}] = -1 \ . \tag{5.57}$$

Since Grassman integration acts like a derivation, we also have $R[d\theta] = -1$ and therefore we see that, for the action (5.45) to be R-symmetric, the superpotential must have R-charge 2:

$$\boxed{R[W] = 2 \ ,} \tag{5.58}$$

and of course $R[\bar{W}] = -2$. A generic superpotential breaks $U(1)_R$ explicitly.

5.5.2 General superpotential

For future reference, let us compute the interaction Lagrangian for an *arbitrary* superpotential $W(\Phi)$. This is a simple exercise, generalising the product of two chiral multiplets in (5.33) to an arbitrary function $W(\Phi^i)$. It is easiest to carry out the computation in components. Given the bottom component $\Phi^W \equiv W(\phi)$, a composite scalar field, its variation is given by:

$$\delta\Phi^W = \delta\phi^i \, \partial_i W(\phi) = \sqrt{2}\epsilon\psi^i \, \partial_i W(\phi) \ , \tag{5.59}$$

where $\partial_i \equiv \frac{\partial}{\partial\phi^i}$, which allows us to read off ψ^W by comparing with (4.10), and similarly for the F-term. We then find:

$$\begin{aligned} \phi^W &= W(\phi) \ , \\ \psi^W &= \psi^i \partial_i W(\phi) \ , \\ F^W &= F^i \partial_i W(\phi) - \frac{1}{2}\psi^i\psi^j \, \partial_i\partial_j W(\phi) \ . \end{aligned} \tag{5.60}$$

Therefore, the interaction Lagrangian that follows from the superpotential is:

$$\mathscr{L}_W = \int d^2\theta \, W(\Phi) = F^i\partial_i W(\phi) - \frac{1}{2}\psi^i\psi^j \, \partial_i\partial_j W(\phi) \ . \tag{5.61}$$

Similarly, from the anti-holomorphic superpotential, we have:

$$\mathscr{L}_{\bar{W}} = \int d^2\theta \, \bar{W}(\bar{\Phi}) = \bar{F}^{\bar{i}}\bar{\partial}_{\bar{i}}\bar{W}(\bar{\phi}) - \frac{1}{2}\bar{\psi}^{\bar{i}}\bar{\psi}^{\bar{j}} \, \bar{\partial}_{\bar{i}}\bar{\partial}_{\bar{j}}\bar{W}(\bar{\phi}) \ . \tag{5.62}$$

5.6 The Wess-Zumino model

A four-dimensional supersymmetric Wess-Zumino model is simply a supersymmetric theory of chiral multiplets, with canonical kinetic term.

Kinetic term. The canonical Kähler potential gives the supersymmetric kinetic term:

$$\mathscr{L}_{\bar{\Phi}\Phi} = \int d^2\theta d^2\bar{\theta} \, \bar{\Phi}\Phi \ . \tag{5.63}$$

Expanding in component, this gives:

$$\begin{aligned} \mathscr{L}_{\bar{\Phi}\Phi} &= \frac{1}{4}\bar{\phi}\partial^2\phi + \frac{1}{4}\partial^2\bar{\phi}\phi - \frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\phi + \bar{F}F \\ &\quad - \frac{i}{2}\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \frac{i}{2}\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi \ . \end{aligned} \tag{5.64}$$

This gives:

$$\boxed{\mathscr{L}_{\bar{\Phi}\Phi} \cong -\partial_\mu\bar{\phi}\partial^\mu\phi + \bar{F}F - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi \ ,} \tag{5.65}$$

up to a total derivative. Of course, this is a free theory of a massless complex scalar and a massless Weyl fermion.

This was written for a single chiral superfield Φ, but the generalisation to n chiral superfields with canonical kinetic terms is trivial—we just have $\bar{\Phi}_i\Phi^i$, where we contracted the indices with a constant "metric" $g_{i\bar{i}}$ in the obvious way:

$$\mathscr{L}_{\bar{\Phi}\Phi} \cong g_{i\bar{j}}\left(-\partial_\mu\bar{\phi}^{\bar{j}}\partial^\mu\phi^i + \bar{F}^{\bar{j}}F^i - i\bar{\psi}^{\bar{j}}\bar{\sigma}^\mu\partial_\mu\psi^i\right) \ . \tag{5.66}$$

We can always set $g_{i\bar{j}} = \delta_{ij}$ by a $U(n)$ redefinition of the fields Φ^i.

5.6.1 Interaction terms: superpotential and scalar potential

Consider the superpotential term:

$$\mathscr{L}_{W+\bar{W}} = F^i \partial_i W - \frac{1}{2}\psi^i\psi^j\, \partial_i\partial_j W + \bar{F}^{\bar{i}}\bar{\partial}_{\bar{i}}\bar{W} - \frac{1}{2}\bar{\psi}^i\bar{\psi}^j\, \bar{\partial}_{\bar{i}}\bar{\partial}_{\bar{j}}\bar{W} \;, \tag{5.67}$$

as computed above. The equations of motion for the auxiliary fields F^i and $\bar{F}^{\bar{i}}$ are:

$$g_{i\bar{i}}F^i = -\bar{\partial}_{\bar{i}}\bar{W} \;, \qquad\qquad g_{i\bar{i}}\bar{F}^{\bar{i}} = -\partial_i W \;. \tag{5.68}$$

Imposing those relations ("integrating out" F and $\bar{F}$), we find the *scalar potential*:

$$\boxed{V_0(\phi,\bar{\phi}) = g^{\bar{i}i}\partial_i W \bar{\partial}_{\bar{i}}\bar{W} \equiv \left|\frac{\partial W(\phi)}{\partial\phi}\right|^2 .} \tag{5.69}$$

The scalar potential of a supersymmetric theory of this type is necessarily positive definite. It is given by the square of the first derivative of the superpotential, hence the name for the latter.

The other interaction terms in (5.67) are Yukawa-type couplings, involving the fermions and the bosons:

$$V = V_0(\phi,\bar{\phi}) + \frac{1}{2}\psi^i\psi^j\, \partial_i\partial_j W + \frac{1}{2}\bar{\psi}^i\bar{\psi}^j\, \bar{\partial}_{\bar{i}}\bar{\partial}_{\bar{j}}\bar{W} \;. \tag{5.70}$$

In particular, the cubic terms in W give rise to the actual Yukawa interactions, which are related by supersymmetry to the $|\phi|^4$ scalar interactions.

5.6.2 Majorana and Dirac mass terms

A quadratic superpotential is a supersymmetric *mass term.* Consider first a single chiral multiplet, Φ. This corresponds to a single Weyl fermion (the left-chiral spinor ψ and its CPT conjugate $\bar{\psi}$), making up a single Majorana fermion, and its bosonic partner ϕ. The mass term superpotential:

$$W_\mu = \frac{1}{2}\mu\Phi^2 \tag{5.71}$$

induces the masses:

$$\mathscr{L}_\mu = -|\mu|^2\bar{\phi}\phi - \frac{1}{2}(\mu\psi\psi + \bar{\mu}\bar{\psi}\bar{\psi}) \;. \tag{5.72}$$

This is a Majorana mass term for the fermion. Indeed, adding this mass term to the kinetic term (5.65), we find the Majorana equation, in Weyl spinor notation:

$$i\bar{\sigma}^\mu\partial_\mu\psi + \bar{\mu}\bar{\psi} = 0 \;, \tag{5.73}$$

The scalar mass squared is the square of the Majorana mass, $M_\phi^2 = |\mu|^2$. The equality of masses is expected from supersymmetry. If we only have this quadratic superpotential, this is still a free theory of massive bosons and fermions, and their spectrum has to be degenerate.

On the other hand, we could consider a Dirac fermion (say, the electron):

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\tilde{\psi}}^{\dot\alpha} \end{pmatrix} . \tag{5.74}$$

Those fermions sit inside two *distinct* chiral multiplets Φ and $\widetilde{\Phi}$ (together with the anti-chiral multiplets $\bar\Phi$ and $\bar{\widetilde{\Phi}}$), of charge $+1$ and -1 under a $U(1)$ symmetry, respectively (say, the electric charge, under which Ψ has charge 1). The corresponding Dirac mass term in the superpotential is simply:

$$W = m\Phi\widetilde{\Phi} , \tag{5.75}$$

corresponding to:

$$\mathscr{L}_m = -|m|^2(|\phi|^2 + |\widetilde{\phi}|^2) - m\psi\widetilde{\psi} + \bar{m}\bar{\psi}\bar{\widetilde{\psi}} . \tag{5.76}$$

Note that we can always fix the masses μ or m to be real, by a redefinition of the chiral superfields. Adding this mass term to the massless kinetic Lagrangian for Φ and $\widetilde{\Phi}$, we get the Dirac equation:

$$i\bar\sigma^\mu\partial_\mu\psi + \bar{m}\bar{\widetilde{\psi}} = 0 , \qquad i\bar\sigma^\mu\partial_\mu\widetilde{\psi} + \bar{m}\bar{\psi} = 0 . \tag{5.77}$$

The main difference between the superpotentials (5.71)ï¿½and (5.75) is what symmetries they preserve. The Majorana mass term (5.71) breaks the $U(1)$ flavor symmetry that rotates the single chiral multiplet Φ explicitly, and fixes the R-symmetry to be $R[\Phi]$=1. On the other hand, the Dirac mass term (5.75) preserves a $U(1)$ flavor symmetry (the "electric charge") out of the $U(2)$ symmetry of two massless chiral multiplets $(\Phi, \widetilde{\Phi})$; therefore, we can choose any R-symmetry with $R[\Phi] + R[\widetilde{\Phi}] = 2$.

5.7 Supersymmetric vacuum equations

In our discussion of supersymmetric quantum mechanics, we saw that the energy of any state has to be *non-negative*—see equation (1.40). This is true also in 4d $\mathcal{N}=1$ supersymmetric QFT:

$$\boxed{E = -\langle\psi|P_0|\psi\rangle \geq 0 .} \tag{5.78}$$

Indeed, we have:

$$\langle\psi|Q_\alpha\bar{Q}_{\dot\beta} + \bar{Q}_{\dot\beta}Q_\alpha|\psi\rangle = 2\sigma^\mu_{\alpha\dot\beta}\langle\psi|P_\mu|\psi\rangle , \tag{5.79}$$

from the supersymmetry algebra. Taking the trace, $\mathrm{Tr}(\sigma^\mu P_\mu) = -2P_0 = 2E$ gives:

$$4E = \sum_{\alpha=1}^{2}\left(\Big|(Q_\alpha)^\dagger|\psi\rangle\Big|^2 + \Big|Q_\alpha|\psi\rangle\Big|^2\right) \geq 0 . \tag{5.80}$$

5.7.1 The supersymmetric vacuum.

Since the relation (5.80) gives the energy as a sum over perfect squares, it implies that the *vacuum* $|\text{vac}\rangle$ of the QFT is supersymmetric (*i.e.* it is preserved by the four supercharges) if and only if its energy vanishes:

$$-P_0|\text{vac}\rangle = 0 \qquad \Leftrightarrow \qquad Q_\alpha|\text{vac}\rangle = 0 \quad \text{and} \quad \bar{Q}_{\dot\alpha}|\text{vac}\rangle = 0 \; . \tag{5.81}$$

This is a very important relation, which holds in any (rigid) supersymmetric theory, in any space-time dimension. In ordinary QFT, the vacuum formally has an infinite energy, at least in a semi-classical approximation (from the zero-point energy of harmonic oscillators at every point in space), and we can always redefine (renormalise) it to be whatever we want. Supersymmetry gives a well-defined meaning to the zero of the energy, because the Hamiltonian of a supersymmetric theory is necessarily a perfect square—schematically:

$$H = |Q|^2 \; . \tag{5.82}$$

Side note: In any theory including gravity, unlike in pure QFT, there is an intrinsic meaning to the zero of the energy as well. The vacuum energy of a QFT should gravitate, like any other form of energy-impulsion. The vacuum energy gives rise to a cosmological constant (CC) term in the Einstein equations. In a non-supersymmetric theory, the natural scale of the CC is the Planck scale. In a supersymmetric theory, on the other hand, the CC would be exactly zero. Of course, our world is not supersymmetry. Then, even if supersymmetry exists a high energy, the natural scale of the CC is the supersymmetry-breaking scale, which is at least 10^3GeV. On a $\log_{10}$ scale, that is essentially half-way between the Planck scale (10^{19}GeV) and the experimentally observed value of the vacuum energy, which tiny (and positive), at about $(\rho_{\text{vac}})^{\frac{1}{4}} = 10^{-12}$GeV. We have no good idea how this tiny number comes about—that is the famous cosmological constant problem.

5.7.2 The vacuum equations in a theory of chiral multiplets

Consider a 4d $\mathcal{N} = 1$ theory of chiral multiplets only. We have seen that the scalar potential (in a theory with canonical kinetic terms) is given by:

$$V_0(\phi, \bar\phi) = \sum_i |\partial_{\phi^i} W(\phi)|^2 \; . \tag{5.83}$$

This is a sum of perfect squares. Therefore, the supersymmetric vacuum ($V_0 = 0$) exists if and only if:

$$\boxed{\frac{\partial W(\phi)}{\partial \phi^i} = 0 \; , \qquad \forall i \; .} \tag{5.84}$$

The solutions to these equations determine the possible *vacuum expectation values* (VEVs) for the scalar fields ϕ. A supersymmetric vacuum is a configuration of *constant* VEVs:

$$\phi_i = \langle \phi_i \rangle = \langle \text{vac} | \phi_i | \text{vac} \rangle \ , \tag{5.85}$$

which solve (5.84).

When the superpotential is a polynomial in the fields, the set of supersymmetric vacua is determined by a set of *algebraic* equations for ϕ^i. Moreover, the equations (5.84) are *holomorphic* equations in the complex scalars ϕ^i. Thus, determining supersymmetric vacua is really a problem in *algebraic geometry*—given a set of polynomials $p_1(z), p_2(z), \cdots$ in several complex variables $z = (z^1, z^2, \cdots, z^n) \in \mathbb{C}^n$, we want to know what is its zero set $p_1(z) = p_2(z) = \cdots = 0$.

The expression for the scalar potential V_0 is slightly modified in the case of a non-canonical kinetic term, as we will see momentarily, but the vacuum equations remain the same

5.7.3 Vacuum moduli spaces

In a theory of n chiral multiplets Φ^i, the vacuum equations (5.84) are n equations for n unknowns. Depending on the particular form of the superpotential W, there are three possibilities:

- There are no solutions. Then, supersymmetry is *spontaneously broken.* We will come back to this possibility later in the lectures.
- There are a finite number N of solutions. They correspond to N "discrete vacua," local minima of the potential with $V_0 = 0$.
- There could be a *continuum* of solutions. This is called a *vacuum moduli space.* In an ordinary QFT, any such "flat direction" in the potential would generally be lifted by quantum corrections. In a supersymmetric theory, we will see that supersymmetry actually preserves the moduli space to all orders in perturbation theory.

Let us give some simple examples:

Example 1. Consider first a theory with a single chiral multiplet Φ and a superpotential:

$$W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \ . \tag{5.86}$$

The vacuum equations reads:

$$\partial_\phi W = m\phi + \lambda\phi^2 = 0 \ , \tag{5.87}$$

so that we find two discrete vacua:

$$\phi = 0 \ , \qquad \phi = -\frac{m}{\lambda} \ . \tag{5.88}$$

Here, we take the (widespread) notational convention of denoting the VEV $\langle\phi\rangle$ of ϕ simply by ϕ. The fact that it is a VEV of a field and not the field itself should always be clear from the context.

Example 2. Consider a theory with three chiral multiplets and a cubic superpotential:

$$W = \Phi_1\Phi_2\Phi_3 \ . \tag{5.89}$$

The vacuum equations are:

$$\partial_{\phi_1} W = \phi_2\phi_3 = 0 \ , \qquad \partial_{\phi_2} W = \phi_1\phi_3 = 0 \ , \qquad \partial_{\phi_3} W = \phi_1\phi_2 = 0 \ . \tag{5.90}$$

These equations are equations for singular quadrics in $\mathbb{C}^3$. There are now *continuous solutions.* One can check that there are three branches that intersect at the origin:

$$\{\phi_1 \neq 0 \ , \phi_2 = \phi_3 = 0\} \cup \{\phi_2 \neq 0 \ , \phi_1 = \phi_3 = 0\} \cup \{\phi_3 \neq 0 \ , \phi_1 = \phi_2 = 0\} \ . \tag{5.91}$$

This is the *vacuum moduli space,* denoted by $\mathcal{M}$.

$\mathcal{M}$ as an affine variety. Note that, in the free theory of n chiral multiplets without superpotential, the vacuum moduli space is simply:

$$\mathcal{M}_{W=0} = \mathbb{C}^n \ . \tag{5.92}$$

Namely, all the scalar fields ϕ^i can take arbitrary VEVs simultaneously. A non-trivial superpotential then defines the vacuum moduli space as an *affine variety,* with coordinate ring:

$$\mathbb{C}[\phi_1, \cdots, \phi_n]/(\partial_\phi W) \ , \tag{5.93}$$

where $(\partial_\phi W)$ is the ideal generated by the n polynomials $\partial_{\phi_i} W$. (The reader not familiar with algebraic geometry can ignore this last comment.)

5.8 General Kähler potential & Kähler geometry

Finally, consider a general theory of n chiral multiplets with an general Kahler potential $K(\bar\Phi, \Phi)$ in (5.45). This is not a renormalisable theory, but it can still be considered as an effective field theory.

To understand what the general kinetic term looks like, we should build the real superfield K whose bottom component the real scalar:

$$C^K = K(\bar\phi, \phi) \ , \tag{5.94}$$

to find the D-term. This is a tedious but straightforward computation. One finds:

$$\begin{aligned} \mathscr{L}_K &= \int d^2\theta d^2\bar\theta K(\bar\Phi, \Phi) \\ &= g_{i\bar j}(\bar\phi, \phi)\left(-\partial_\mu\bar\phi^{\bar j}\partial^\mu\phi^i + \bar F^{\bar j} F^i - i\bar\psi^{\bar j}\bar\sigma^\mu\partial_\mu\psi^i\right) + \cdots \end{aligned} \tag{5.95}$$

where the ellipsis denotes additional interaction terms involving the fermions. (*Exercise: compute them.*) This is similar to (5.66), but now the coefficient of the kinetic term is a non-trivial function of the scalar fields:

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K \equiv \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} \ . \tag{5.96}$$

In a theory of scalar fields, it is always a good idea to think of the "field space" for the scalars as a *manifold*—albeit infinite-dimensional, in general. More precisely, we view the fields $\phi(x)$ as maps from space-time to a *target space*, $\mathfrak{M}$:

$$\phi \ : \mathbb{R}^{1,3} \to \mathfrak{M} \ : \ x^\mu \mapsto \phi(x) \ . \tag{5.97}$$

From this point of view, the scalar fields are viewed as *local coordinates* on the target space $\mathfrak{M}$ (of complex dimension n).

The entire field space is the infinite-dimensional space of maps (5.97). A particularly important submanifold of field space is the *vacuum moduli space,* which we discussed in the last subsection. It corresponds to the space of constant maps to target space, which is isomorphic to the target space itself. Thus, in the absence of superpotential, we have $\mathcal{M}_{W=0} = \mathfrak{M}$. In the presence of superpotential, the target-space picture still holds, except that the vacuum moduli space is a submanifold (or rather, subvariety) of $\mathfrak{M}$.

The target space can have additional structure. In a general theory of real scalars φ^i, it would simply be a real manifold. A Lagrangian of the general form:

$$\mathscr{L} = -\eta^{\mu\nu} \ g_{ij}(\varphi)\partial_\mu \varphi^i \partial_\nu \varphi^j = -\eta^{\mu\nu}(\varphi^* g)_{\mu\nu} \ , \tag{5.98}$$

defines a so-called *non-linear sigma model* (NLSM), with $g = g_{ij} d\varphi^i d\varphi^j$ a choice of Riemannian metric on target space. The kinetic term is given in term of the pullback of the metric g_{ij} through the map $\varphi : \mathbb{R}^{1,3} \to \mathfrak{M}$. [22]

Here, we are considering a theory of complex scalar fields, therefore $\mathfrak{M}$ is naturally *a complex manifold,* with complex coordinates $(z^i, \bar{z}^{\bar{i}}) = (\phi^i, \bar{\phi}^{\bar{i}})$, and a Lagrangian of the form:

$$\mathscr{L} = -g_{i\bar{j}}(\bar{\phi}, \phi)\partial_\mu \bar{\phi}^{\bar{j}} \partial^\mu \phi^i + \cdots \ . \tag{5.99}$$

Here, g is an *Hermitian metric* on $\mathfrak{M}$. A particularly nice class of Hermitian manifolds are the so-called *Kähler manifolds.* They are complex manifolds with an Hermitian metric $g_{i\bar{j}}$ such that the associated two-form ω is closed:

$$\omega \equiv 2i g_{i\bar{j}}(z, \bar{z}) dz^i \wedge d\bar{z}^{\bar{j}} \ , \qquad\qquad d\omega = 0 \ . \tag{5.100}$$

This means that:

$$\partial_i g_{j\bar{k}}(z,\bar{z}) - \partial_j g_{i\bar{k}}(z,\bar{z}) = 0 \ , \qquad \bar{\partial}_{\bar{i}} g_{k\bar{j}}(z,\bar{z}) - \bar{\partial}_{\bar{j}} g_{k\bar{i}}(z,\bar{z}) = 0 \ . \tag{5.101}$$

[22] Recall the definition of the pullback of a map $f : M \to N$ from your differential geometry or general relativity course.

These condition actually imply that the Hermitian metric of a Kahler manifold—the *Kahler metric*—can be written in terms of a real function $K(z, \bar{z})$, the *Kähler potential*, exactly as in (5.96). Thus, we conclude that:

> *the target space of a 4d $\mathcal{N} = 1$ supersymmetry field theory is a Kähler manifold.* We have just shown this for a theory consisting only of chiral multiplets, but this conclusion holds true in $\mathcal{N} = 1$ supersymmetric gauge theories as well.

The simplest example is the canonical Kähler potential:

$$K = \sum_{i=1}^{n} |\phi^i|^2 \ , \tag{5.102}$$

which obviously gives the flat metric $\delta_{i\bar{j}}$ on the target space $\mathbb{C}^n$. A more general Kähler potential simply gives a non-flat metric (on a space which is still topologically $\mathbb{C}^n$). [23]

Finally, we can also consider a non-trivial superpotential. The presence of a more general $K(\bar{\Phi}, \Phi)$ does not affect the discussion of the vacuum structure given above. It is easy to check that the scalar potential still takes the form (5.69), where now $g^{i\bar{i}}$ is the inverse of the non-trivial Kähler metric. Then, assuming that there are no metric singularities in $g_{i\bar{j}}$, the vacuum equations are still given by the critical points of the holomorphic superpotential.

6 Renormalisation of supersymmetric theories

In this section, we finally start discussing some *quantum* properties of 4d $\mathcal{N} = 1$ supersymmetric theories. Supersymmetric quantum field theories are much better behaved than non-supersymmetric ones, which is the main reason why they are so interesting and useful, from a theoretical perspective.

6.1 The Wess-Zumino model at one loop

Consider the original Wess-Zumino model, which consists of a single chiral multiplet, Φ, with canonical Kähler potential and a cubic superpotential:

$$W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \ . \tag{6.1}$$

For simplicity of notation, we take $m \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. After integrating out the auxiliary fields $F, \bar{F}$, the full Lagrangian takes the form:

$$\begin{aligned} \mathscr{L} = &-\partial_\mu \bar{\phi} \partial^\mu \phi - m^2 \bar{\phi}\phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \frac{m}{2}(\psi\psi + \bar{\psi}\bar{\psi}) \\ &- \lambda^2 \bar{\phi}^2 \phi^2 - m\lambda(\bar{\phi}^2\phi + \bar{\phi}\phi^2) - \lambda(\psi\psi\phi + \bar{\psi}\bar{\psi}\bar{\phi}) \ . \end{aligned} \tag{6.2}$$

We would like to analyse this QFT in perturbation theory, in the regime $\lambda \ll 1$.

[23] We could consider target spaces of different topologies, too.

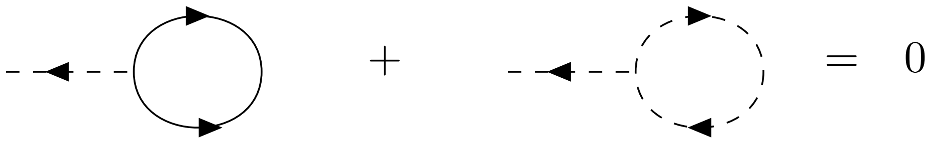


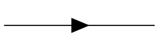
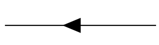
Figure 1: Tadpole Feynman diagrams for the scalar ϕ. The bosons and fermions in the loop cancel each other due to supersymmetry, so that the tadpole vanishes.

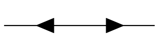
6.1.1 Feynman rules for the WZ model

The scalar propagator is given by:

$\bar{\phi}$  ϕ $= \dfrac{-i}{p^2 + m^2 - i\epsilon}\ ,$ (6.3)

as usual, and the fermions propagators are given by:

$\bar{\psi}$  ψ $= \dfrac{-i\bar{\sigma}^{\mu}p_{\mu}}{p^2 + m^2 - i\epsilon}\ ,$ ψ  $\bar{\psi}$ $= \dfrac{-i\sigma^{\mu}p_{\mu}}{p^2 + m^2 - i\epsilon}\ ,$

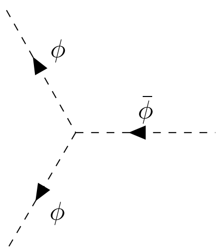
ψ  ψ $= \dfrac{-im}{p^2 + m^2 - i\epsilon}\ ,$ $\bar{\psi}$  $\bar{\psi}$ $= \dfrac{-im}{p^2 + m^2 - i\epsilon}\ .$

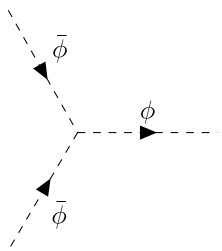
Here, the momentum always flow from left to right. Note that the $\langle\psi\psi\rangle$ and $\langle\bar{\psi}\bar{\psi}\rangle$ propagators reverse the fermion (chirality) arrow. [24]

The propagators are easily derived from the first line of (6.2), which can be written as:

$$\mathscr{L} = \bar{\phi}(\partial_\mu\partial^\mu - m^2)\phi + \frac{1}{2}(\psi^\alpha, \bar{\psi}_{\dot{\alpha}}) \begin{pmatrix} -m\delta_\alpha{}^\beta & -i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu & -m\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \begin{pmatrix} \psi_\beta \\ \bar{\psi}^{\dot{\beta}} \end{pmatrix}\ , \tag{6.4}$$

up to a total derivative. From the second line in (6.2), we also read off the interaction vertices. There are two types of cubic vertices, the scalar cubic vertex:

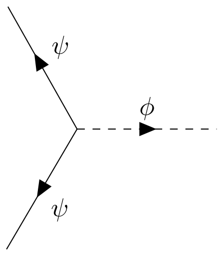


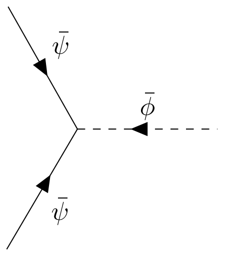
$= -im\lambda\ ,$


$= -im\lambda\ ,$ (6.5)

[24] In your QFT course, you have probably seen Feynman rules for fermions in Dirac notation, while we are using the two-component Weyl notation. It is relatively straightforward to translate expression between the two languages. For a (very) detailed discussion of this point, see [20].

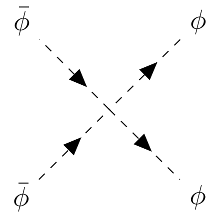
and the Yukawa interaction:



$= -i\lambda$,


$= -i\lambda$. (6.6)

There is a unique quartic vertex:



$= -i\lambda^2$ (6.7)

Using these Feynman rules, one can study the 4d $\mathscr{N} = 1$ Wess-Zumino model in perturbation theory, as we would do for any particular quantum field theory.

6.1.2 Some one-loop corrections

Recall that there are, roughly, two types of "quantum corrections" to the quantum effective action, [25] at some renormalisation group (RG) scale μ:

- **Wavefunction renormalisation.** The quantum correction to the (massless) kinetic terms take the form:

$$\mathscr{L}_{\text{eff}} = -Z_\phi\, \partial_\mu \bar\phi \partial^\mu \phi - Z_\psi\, i\bar\psi \bar\sigma^\mu \partial_\mu \psi \ , \tag{6.8}$$

  where $Z_\phi = Z_\phi(\mu)$ and $Z_\psi = Z_\psi(\mu)$ are known as the "wavefunction renormalisation" factors. One often defines the renormalised fields:

$$\phi_R = \sqrt{Z_\phi}\, \phi \ , \qquad \psi_R = \sqrt{Z_\psi}\, \psi \ , \tag{6.9}$$

  to rescale the kinetic term back to its canonical form. The so-called *anomalous dimension* of the field ϕ is defined by:

$$\gamma_\phi = -\mu \frac{\partial}{\partial \mu} \log Z_\phi \ , \tag{6.10}$$

  and it is itself a function of μ. In term of γ_ϕ, the "quantum dimension" Δ of the field ϕ is then:

$$\Delta[\phi] = 1 + \frac{1}{2}\gamma_\phi \ , \tag{6.11}$$

[25] We will be interested in the so-called Wilsonian effective action, although for the following discussion can we equivalently consider the 1PI effective action. We'll come back to this point soon.

Indeed, by constuction, $\phi_R = \sqrt{Z_\phi}\phi$ has dimension 1. Then, the *dimensionless* field:

$$\widetilde{\phi} = \mu^{-1} Z_\phi^{\frac{1}{2}} \phi \ , \tag{6.12}$$

scales as:

$$\mu \frac{\partial \widetilde{\phi}}{\partial \mu} = -\Delta[\phi]\, \widetilde{\phi} \ . \tag{6.13}$$

A similar discussion holds for the fermions, with:

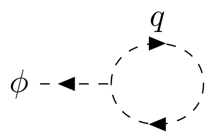
$$\Delta[\psi] = \frac{3}{2} + \frac{1}{2}\gamma_\psi \ , \qquad \gamma_\psi = -\mu \frac{\partial}{\partial \mu} \log Z_\psi \ . \tag{6.14}$$

- **Coupling constant renormalisation.** The various coupling constants g in the Lagrangian, for a term $\mathscr{L} = g\mathcal{O}(x)$, can be renormalised independently of the wavefunction renormalisation:

$$\mathscr{L}_g = g Z_g \mathcal{O}(x) = g(1 + \cdots)\mathcal{O}(x) \ , \tag{6.15}$$

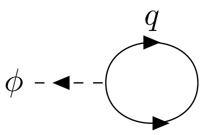
  where the ellipsis denotes the loop corrections.

Absence of tadpole. The one-loop contributions to the scalar tadpole (the one-point function of ϕ), are shown in Figure 1. The contribution from the scalar running in the loop is:



$$= (-im\lambda) \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 + m^2 - i\epsilon} \ . \tag{6.16}$$

The contribution from the fermion loop reads:



$$= (-1)(-i\lambda) \int \frac{d^4q}{(2\pi)^4} \frac{-im}{q^2 + m^2 - i\epsilon} \ , \tag{6.17}$$

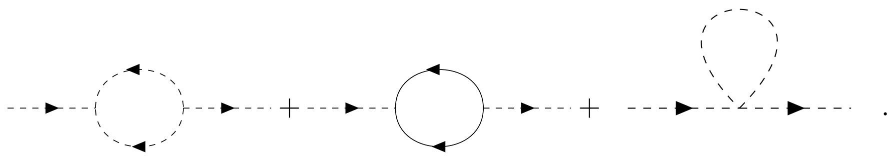
where the used the $\langle\psi\psi\rangle$ propagator in the loop. Note the overall -1 in (6.17), because of the *fermion* loop. The two (divergent) tadpole contributions cancel out precisely. The same holds, of course, for the $\bar{\phi}$ tadpole.

Let us insist on the fact that this cancellation relies on the precise values of the cubic vertices (6.5) and (6.6) ($-im\lambda$ and $-i\lambda$, respectively), which is a consequence of supersymmetry. In a non-supersymmetric theory, the two divergences would not cancel, and we would have to renormalise this term to zero "by hand."

The vanishing of the tadpole also means that the vacuum expectation value (5.88) in a supersymmetric vacuum is not renormalised at one loop (and in fact the same holds at all orders).

Scalar self-energy at zero external momentum. Consider the one-loop contribution to the self-energy of the scalar ϕ—in other words, the $\langle\bar{\phi}\phi\rangle$ two-point function. Diagrammatically, it is given by a sum of three contributions:

$\Pi_{\bar{\phi}\phi}(p^2) =$



(6.18)

This expression vanishes in the limit of zero external momentum:

$$\lim_{p^2\to 0} \Pi_{\bar{\phi}\phi}(p^2) = 0 \; . \tag{6.19}$$

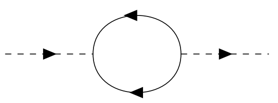
At a formal level, this is easy to check, without the need of any regularisation. From the three diagrams (6.18) with $p^\mu = 0$, we have:

$$\begin{aligned}\Pi_{\bar{\phi}\phi}(0) &= (-im\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\left(\frac{-i}{q^2+m^2}\right)^2 \\ &- (-i\lambda)^2 \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\frac{\mathrm{tr}(-\sigma^\mu\bar{\sigma}^\nu)q_\mu q_\nu}{(q^2+m^2)^2} + (-i\lambda^2)\int \frac{d^4q}{(2\pi)^4}\frac{-i}{q^2+m^2} = 0 \; .\end{aligned} \tag{6.20}$$

This "miraculous" cancelation should give us pause. What vanishes, here, is the potential divergence that would renormalise the scalar mass, m^2. For instance, if we had a fermion-fermion-scalar vertex with some random coupling $y \in \mathbb{R}$, namely:

$$\mathscr{L} = y(\psi\psi\phi + \bar{\psi}\bar{\psi}\bar{\phi}) \; , \tag{6.21}$$

the fermion in the loop would give a contribution:



$$= -(-iy)^2 \int \frac{d^4q}{(2\pi)^4}\frac{q^2}{(q^2+m^2)^2} \propto y^2\,\Lambda^2 + \cdots \; , \tag{6.22}$$

if we use a hard cut-off regulator $|q| \leq \Lambda$ (the ellipsis denotes a lower-order, logarithmic divergence). Whatever the chosen regularisation method, there is a parametrically large parameter that one must cancel "by hand" to fix the scalar mass m^2 at its physical value. In the Standard Model of particle physics, for instance, this is the one-loop contribution from fermions to the mass of the Higgs boson. That quadratic divergence cancels out in a supersymmetric theory, due to exact and opposite contributions from the other two diagrams.

Vertex corrections. One can similarly check, with some more work, that the total one-loop corrections to the vertices $\langle\bar{\phi}\phi\phi\rangle$ and $\langle\bar{\phi}\bar{\phi}\phi\phi\rangle$ are also trivial, in the limit of vanishing external momenta.

6.1.3 A simpler perturbation theory

One can better understand the nature of the cancellations due to supersymmetry by considering an (equivalent) theory where we keep the auxiliary fields $\bar{F}, F$. Then, the WZ Lagrangian:

$$\begin{aligned}\mathscr{L} = &-\partial_\mu\bar{\phi}\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \bar{F}F + m(F\phi + \bar{F}\bar{\phi} - \frac{1}{2}\psi\psi + \frac{1}{2}\bar{\psi}\bar{\psi})\\ &+ \lambda(F\phi^2 + \bar{F}\bar{\phi}^2 - \psi\psi\phi - \bar{\psi}\bar{\psi}\bar{\phi}) \ .\end{aligned} \tag{6.23}$$

In this formulation, we have the scalar propagators:

$$\langle\bar{\phi}\phi\rangle = \frac{-i}{p^2+m^2} \ , \qquad \langle\bar{F}F\rangle = \frac{ip^2}{p^2+m^2} \ , \qquad \langle\phi F\rangle = \langle\bar{\phi}\bar{F}\rangle = \frac{im}{p^2+m^2} \ , \tag{6.24}$$

while the fermion propagator are unchanged. The second line in (6.23) only gives two types of *cubic* vertices. In this language, one finds that [11]:

- The only renormalisation needed at one loop is due to the divergent contribution to the two-point functions. These give rise to a non-trivial wavefunction renormalisation (proportional to $|\lambda|^2$—namely, $Z_\phi = 1 + c_0(\mu)|\lambda|^2 + \cdots$), which is the *same* for all the chiral multiplet field components:

$$Z_\phi = Z_\psi = Z_F \equiv Z_\Phi \ . \tag{6.25}$$

- The mass m and coupling constant λ are not independently renormalised. Instead, in a convenient renormalisation scheme, the renormalised quantities can be written entirely in terms of Z_Φ:

$$m_R = Z_\Phi^{-1} m \ , \qquad \lambda_R = (Z_\Phi)^{-\frac{3}{2}}\lambda \ . \tag{6.26}$$

In summary, there is a unique divergence that appears at one-loop in the WZ model, corresponding to the *anomalous dimension* γ_ϕ of the chiral multiplet Φ. That anomalous dimension is the same for all field components—that is:

$$\Delta[\phi] = 1 + \frac{1}{2}\gamma_\phi \ , \qquad \Delta[\psi] = \frac{3}{2} + \frac{1}{2}\gamma_\phi \ , \qquad \Delta[F] = 2 + \frac{1}{2}\gamma_\phi \ . \tag{6.27}$$

All order result. These one-loop results generalise to *all order* in perturbation theory [21]. In fact, it turns out that one can always write down the effective action as:

$$\begin{aligned}\mathscr{L}_{\text{eff}} = &Z_\Phi\left(-\partial_\mu\bar{\phi}\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \bar{F}F\right) + m(F\phi + \bar{F}\bar{\phi} - \frac{1}{2}\psi\psi + \bar{\psi}\bar{\psi})\\ &+ \lambda(F\phi^2 + \bar{F}\bar{\phi}^2 - \psi\psi\phi - \bar{\psi}\bar{\psi}\bar{\phi}) \ ,\end{aligned} \tag{6.28}$$

where the only non-trivial renormalisation factor is the wavefunction renormalisation factor Z_Φ, in front of the (massless) kinetic term. Then, the coupling constants

m and λ are only renormalised in the sense that, when we write down the effective Lagrangian in term of the renormalized fields

$$\Phi_R = Z_\Phi^{\frac{1}{2}}\Phi \ , \tag{6.29}$$

we simply have:

$$\begin{aligned}\mathscr{L}_{\text{eff}} &= \left(-\partial_\mu\bar\phi_R\partial^\mu\phi_R - i\bar\psi_R\bar\sigma^\mu\partial_\mu\psi_R + \bar F_R F_R\right)\\ &\quad + m_R(F_R\phi_R + \bar F_R\bar\phi_R - \frac{1}{2}\psi_R\psi_R + \bar\psi_R\bar\psi_R)\\ &\quad + \lambda_R(F_R\phi_R^2 + \bar F_R\bar\phi_R^2 - \psi_R\psi_R\phi_R - \bar\psi_R\bar\psi_R\bar\phi_R)\ ,\end{aligned} \tag{6.30}$$

with the renormalised mass m_R and coupling constant λ_R defined in (6.26).

6.2 Wilsonian effective action and the power of holomorphy

The above discussion of the Wess-Zumino model can be generalised to any field theory of n chiral multiplets with canonical kinetic term. In the effective action, we always have:

$$\mathscr{L}_{\text{eff}} = \int d^2\theta d^2\bar\theta \sum_i Z_{\Phi^i}\bar\Phi_i\Phi^i + \int d^2\theta W(\Phi) + \int d^2\bar\theta \bar W(\Phi)\ . \tag{6.31}$$

with just a wavefunction renormalisation factor for each chiral multiplet. The simple slogan is that:

the superpotential $W(\Phi)$ is not renormalised, at all!

This is the famous *non-renormalisation theorem* of 4d $\mathcal{N}=1$ supersymmetry. In the early 1990's, Seiberg gave a very simple proof of it, based entirely on symmetry arguments [22].

6.2.1 Wilsonian effective action, in one word

At this point, we should mention, a bit more explicitly, that we are thinking of quantum corrections in the so-called Wilsonian framework. In short, the idea of the Wilsonian renormalisation group (RG) flow is to start with a field theory, with an action S, defined at a scale $\mu_0 = \Lambda_{UV}$ (which might be sent to infinity, if the theory is renormalisable), and to compute the effective action $S_{\text{eff}}(\mu)$ at a scale $\mu < \mu_0$ by "integrating out" all degrees of freedom from μ_0 down to μ. That is, in momentum space, one splits the fields into into high and low momentum modes:

$$\varphi(q) = \begin{cases}\varphi_H(q) & \text{if} \quad \mu < |q| \leq \mu_0\\ \varphi_L(q) & \text{if} \quad |q| \leq \mu\ ,\end{cases} \tag{6.32}$$

so that the path integral takes the form:

$$Z = \int [D\varphi]\ \exp\left(iS_{[}\varphi]\right) = \int [D\varphi_L][D\varphi_H]\ \exp\left(iS[\varphi_L,\varphi_H]\right)\ . \tag{6.33}$$

Then, the Wilsonian action is defined by integration over the high-momentum modes:

$$\exp\left(iS_{\text{eff},\mu}[\varphi_L]\right) = \int [D\varphi_H]\ \exp\left(iS_{\text{eff}}[\varphi_L, \varphi_H]\right)\ . \tag{6.34}$$

In general, the effective action at "low" energy μ is a very complicated (and generally unknown) sum over many operators:

$$S_{\text{eff},\mu} = \int d^4x \sum_{\mathcal{O}} g_{\mathcal{O}}(\mu)\, \mathcal{O}(x)\ . \tag{6.35}$$

Moreover, the fundamental fields of the UV description might not be the most natural *variables* to describe the low-energy effective theory. For instance, the low-energy effective action for real-world QCD is well-approximated by the so-called chiral Lagrangian describing interactions amongst *mesons.* This is a very different *low-energy effective description* from the "fundamental" quarks and gluon of QCD in the UV.

If the RG flow is reliably *perturbative,* we have more control. For a theory without massless excitations, the Wilsonian effective action is essentially the same as the 1PI effective action of standard textbook, in the limit $\mu \to 0$ (in that case, since there are no massless excitations, we can essentially "stop integrating" below the scale of the lowest excitation). One advantage of the Wilsonian framework is that we do not take the strict IR limit ($\mu \to 0$), which would correspond to "doing the full path integral"—instead, we stop "path integrating" at some intermediate "low energy" but finite energy scale μ, and obtain an "effective" QFT (*i.e.* a path integral) for the remaining low-energy modes (which may include massless particles).

For a pedagogical introduction to the Wilsonian approach to renormalisation, you are invited to read, for instance, chapter 12 of Peskin & Shroeder [23].

6.2.2 Holomorphy and non-renormalisation of the superpotential

In QFT, as in quantum mechanics, *symmetries* lead to selection rules. In the Wilsonian framework, this means that they constraint the form of the operators that can appear in the effective action.

A powerful way to make such selection rules manifest is often to treat all coupling constants g as "background fields," $g(x)$, which are frozen to some constant value $g = \langle g \rangle$—for instance, we can think that they appear as very massive fields in some larger theory; it is also perfectly fine to think of background fields only as a convenient bookkeeping device.

Then, consider a theory with global symmetry $G = U(1)$, for simplicity, and consider adding the perturbation:

$$\mathscr{L}_g = g\mathcal{O}\ , \tag{6.36}$$

which breaks that $U(1)$ *explicitly,* because the operator $\mathcal{O}$ has charge a $q[\mathcal{O}] = q_{\mathcal{O}}$. If we think of the coupling g as a *background field,* we can assign it a charge

$q[g] = -q_{\mathcal{O}}$, so that the Lagrangian term (6.36) is actually invariant. In that formulation, the $U(1)$ symmetry is now *spontaneously* broken by the "VEV" g of the background field $g(x)$. Then, the quantum corrections to any observable after the deformation must depend on g in a way that respects the symmetry.

When dealing with the superpotential of a supersymmetric theory, we apply the same logic. Consider:

$$W = \lambda \mathcal{O} \ , \tag{6.37}$$

with $\mathcal{O}$ a *chiral* superfield. Then, by supersymmetry, λ should also be thought as a chiral superfield, with lowest component a complex scalar. The coupling constant is then a VEV $\lambda \in \mathbb{C}$ of that scalar. Now, we directly have a very powerful constraint: since the superpotential is *holomorphic* in the chiral superfields, any quantum correction can only appear holomorphically in λ, as well.

Thus, a correction:

$$W = \lambda \mathcal{O} + \lambda^2 \mathcal{O}' + \cdots \ , \tag{6.38}$$

may be allowed, but any correction involving $\bar{\lambda}$ is ruled out. Moreover, an holomorphic function is entirely determined by its singularities and its asymptotics. Thus, it is not too surprising that analyticity combined with our knowledge of the weak coupling limit often completely determines the effective superpotential.

The WZ model, revisited. For illustration purpose, let us again specialise to our simple WZ model:

$$W_{\mu_0} = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 = \mu_0 \frac{\widetilde{m}}{2} + \frac{\lambda}{3}\Phi^3 \ . \tag{6.39}$$

Here, we introduced the dimensionless coupling:

$$\widetilde{m} = \frac{m}{\mu_0} \ ,$$

while λ is already dimensionless. The free theory at $W = 0$ has a $U(1) \times U(1)_R$ symmetry. Treating $\widetilde{m}$ and λ as background fields, we assign the charges:

	$U(1)$	$U(1)_R$
Φ	1	1
$\widetilde{m}$	-2	0
λ	-3	-1

(6.40)

Then, the most general form allowed for the effective superpotential as a scale $\mu < \mu_0$ is:

$$W_\mu = \mu \widetilde{m} \Phi^2 f\left(\frac{\lambda \Phi}{\mu \widetilde{m}}, \frac{\mu}{\mu_0}\right) \ , \tag{6.41}$$

with f an arbitrary function of its dimensionless, neutral parameters. The function f should be analytic in its first argument, and regular in the $\lambda \to 0$ limit. Therefore, expanding out f, we find:

$$W_\mu = \sum_{n=0}^{\infty} c_n \frac{\lambda^n}{(\mu\widetilde{m})^{n-1}} \Phi^{n+2} \ . \tag{6.42}$$

We should also have a regular $\widetilde{m} \to 0$ limit, so the terms with $n > 1$ are disallowed. Thus, we find:

$$W_\mu = c_0 \mu\widetilde{m}\Phi^2 + c_1 \lambda \Phi^3 \ , \tag{6.43}$$

for c_0 and c_1 some functions of μ/μ_0. Consider now the limit $\lambda \to 0$; then, the theory is free and the mass terms at the scales μ_0 and μ can be matched. This fixes $c_0 = \frac{1}{2}$ when $\lambda = 0$. Indeed, in a free theory we simply have a *classical* running of the mass coupling $\widetilde{m}$, which is just dimensional analysis:

$$W_\mu = \frac{1}{2}\mu\widetilde{m}(\mu)\Phi^2 \ , \qquad \widetilde{m}(\mu) = \frac{m}{\mu} = \frac{\mu_0}{\mu}\widetilde{m}(\mu_0) \ . \tag{6.44}$$

At $\lambda \neq 0$, we can match the term c_1 in (6.43) by comparing to perturbation theory. We can just consider the tree-level approximation, which fixes $c_1 = \frac{1}{3}$.

In conclusion, we find:

$$W_\mu = \mu \frac{\widetilde{m}(\mu)}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \ , \tag{6.45}$$

where the only "renormalisation" of the couplings between the UV scale μ_0 and the IR scale μ is through the *classical* scaling of the coupling constants. This sort of analysis can be generalised to any $W(\Phi)$ in a theory of only chiral multiplet. We can then conclude that the superpotential is not renormalised at any order in perturbation theory (and even, in fact, non-perturbatively).

6.3 "Exact" β-functions for the physical couplings

The above discussion was valid in a choice of renormalisation scheme as in (6.31), where the wavefunction renormalisation factor appears explicitly in front of the Kähler potential. Thus, the precise statement of the non-renormalisation theorem is that *there exists* a (supersymmetry-preserving) renormalisation scheme, which we might call the *"holomorphic scheme,"* in which the superpotential is not renormalised, and the only renormalisation is through wave-function renormalisation that only affects the Kähler potential.

Classical β-functions. Recall that the β-function of any coupling g is defined as:

$$\boxed{\beta(g) = \mu \frac{\partial g(\mu)}{\partial \mu} \ .} \tag{6.46}$$

These first order equations determine how the coupling constants "flow" as we change the scale μ.

In the holomorphic scheme, the running of the superpotential coupling constants is purely classical. Consider a coupling:

$$W = \lambda \mathcal{O} = \lambda \prod_i (\Phi^i)^{d_i} \,, \tag{6.47}$$

with $\mathcal{O}$ an operator of classical dimension $\Delta \equiv \sum_i d_i$, so that λ has classical dimension $3 - \Delta$. We define the dimensionless coupling, at any scale μ, to be:

$$\widetilde{\lambda} = \mu^{\Delta - 3} \lambda \,. \tag{6.48}$$

This obviously gives the "classical" β-function:

$$\beta(\widetilde{\lambda}) = (\Delta - 3)\widetilde{\lambda} \,. \tag{6.49}$$

Recall that, from this (classical) β function, we classify the possible operators $\mathcal{O}$ into:

- **Relevant operators**, if the β function is negative. That is, if $\Delta < 3$. These couplings—which, for $\Delta = 2$, are just mass terms—go to zero in the UV, but dominate in the IR.
- **Irrelevant operators,** if the β function is positive. That is, if $\Delta > 3$. Such couplings blow up in the UV and they make the theory power-counting non-renormalisable.
- **Marginal operators, if $\beta = 0$.** These are the *classically* marginal operators of dimension $\Delta = 3$.

This is just a fancy way to do dimensional analysis.

Physical coupling constants. While the so-called *holomorphic superpotential couplings*, that appear in W, are not renormalised in the holomorphic scheme, does that mean that a physical observer in, say, the WZ model, would conclude that the quartic vertex (6.7) does not change as we vary the energy of incoming particles in some scattering experiment? Indeed, no. The point is that, in computing such *physical observables,* we would consider the canonically normalised fields:

$$\Phi^i_R = Z^{\frac{1}{2}}_{\Phi^i} \Phi^i \,. \tag{6.50}$$

In term of these, the superpotential coupling (6.47) takes the form:

$$W = \lambda_R \mathcal{O}_R = \lambda_R \prod_i (\Phi^i_R)^{d_i} \,, \qquad \lambda_R \equiv \left(\prod_i (Z_{\Phi^i})^{-\frac{d_i}{2}} \right) \lambda \,. \tag{6.51}$$

with λ_R the "physical" coupling constant. We again define the dimensionless coupling:

$$\widetilde{\lambda}_R = \mu^{\Delta - 3} \left(\prod_i (Z_{\Phi^i})^{-\frac{d_i}{2}} \right) \lambda \,. \tag{6.52}$$

Its β-function is then given by:

$$\boxed{\beta(\widetilde{\lambda}_R) = \left(-3 + \sum_i \left(1 + \frac{1}{2}\gamma_{\phi^i}\right) d_i\right) \widetilde{\lambda}_R \ ,} \tag{6.53}$$

with γ_{ϕ^i} the anomalous dimension of Φ^i, as defined in (6.10) The equation (6.53) is an "exact" expression for the β-function of λ_R, in the sense that the running of the physical coupling λ_R is entirely determined if we know the exact quantum dimensions (6.11) of the chiral fields Φ^i. Of course, unlike the superpotential, the anomalous dimensions γ_ϕ (or, equivalently, the wavefunction renormalisation factors Z_Φ) receive corrections at every order in perturbation theory.

6.4 General comment on non-renormalisation theorems

This concludes this discussion of supersymmetric theories of chiral multiplets. It is worth pausing to absorb the main lesson. We have seen that superpotential terms, often reffered to as "F-terms," are not renormalised. This is a very general lesson that apply to other supersymmetric theories, and even to string theory: supersymmetry often allows us to consider some *"holomorphic sector"* of the larger theory, which contains observables that are not renormalised at all, or only renormalised at one-loop order. Such non-renormalisation theorems are very powerful, since they often allow us to reach interesting conclusions about supersymmetric QFTs which are otherwise in a strong coupling regime.

However, we should always keep in mind that the "holomorphic sector" is not the full theory. To answer many finer questions—or, indeed, many basic questions such as "what are the physical Yukawa coupling constants?"—, we will inevitably need some knowledge of the quantum corrections to the Kähler potential—often referred to as "D-terms." That remains a very hard problem beyond perturbation theory, just like in any other non-supersymmetric QFT.

7 4d $\mathcal{N} = 1$ supersymmetry, part II: gauge theories

In this section, we discuss how to write down 4d $\mathcal{N} = 1$ supersymmetric theories that contain *gauge fields.*

7.1 Classical and quantum gauge theory: executive summary

Let us first briefly review (non-supersymmetric) gauge theories, mostly to set up our notation.

7.1.1 Classical gauge theory

We denote by G a Lie group, with $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Let T^a denote the Hermitian generators, which satisfy: [26]

$$[T_a, T_b] = i f_{ab}{}^c T_c \ , \tag{7.1}$$

with $f_{ab}{}^c$ the structure constants. We normalise the generators such that, in the adjoint representation, we have:

$$\mathrm{Tr}(T^a T^b) = k\delta^{ab} \ , \qquad k > 0 \ . \tag{7.2}$$

Let A_μ denote a *gauge field* for some "gauge group" G. This is a covector field valued in the adjoint representation of $\mathfrak{g}$—that is, $\mathfrak{g}$ itself (or rather, $i\mathfrak{g}$):

$$A_\mu(x) = A_\mu^a(x)\, T_a \ . \tag{7.3}$$

For many purposes, it is more convenient to view A_μ as a Lie-algebra-valued one-form:

$$A \equiv A_\mu(x) dx^\mu \ , \tag{7.4}$$

although we will not emphasise that geometric viewpoint in the following.

Non-abelian gauge field. Mathematically, a gauge field A is a connection on a principal G-bundle over space-time, $P \to \mathbb{R}^{1,3}$. For physicists, that essentially means that we *declare* that two gauge fields A_μ and A'_μ are physically equivalent if they are related as:

$$\boxed{A'_\mu = g(x)\,(A_\mu + i\partial_\mu)\, g^{-1}(x) \ ,} \tag{7.5}$$

with $g(x)$ some group-valued function:

$$g(x) : \mathbb{R}^{1,3} \to G \ . \tag{7.6}$$

This is called a *gauge transformation.* We mostly care about infinitesimal gauge transformations connected to the identity. Consider:

$$g(x) = e^{i\alpha(x)} \ , \qquad \alpha(x) \equiv \alpha^a(x)\, T_a \in i\mathfrak{g} \ . \tag{7.7}$$

Then, we have:

$$\delta_\alpha A_\mu = \partial_\mu \alpha + i[\alpha, A_\mu] \ , \tag{7.8}$$

with $\delta_\alpha A_\mu \equiv A'_\mu - A_\mu$ at first order in α.

We define the field-strength of the gauge field (a.k.a. the curvature of the connection A) by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \ . \tag{7.9}$$

[26] Note that the actual generator of the Lie algebra $\mathfrak{g}$ is iT_a, which we denote by $T_a \in i\mathfrak{g}$. We follow the usual physics notation where T_a are Hermitian generators. Mathematicians would consider the anti-Hermitian generator $T_a^{\mathrm{math}} = iT_a \in \mathfrak{g}$, so that the commutator of two generators is a generators, instead of i times a generators in the "physics" convention.

By construction, we have:

$$F'_{\mu\nu} = gF_{\mu\nu}g^{-1} \;, \qquad \leftrightarrow \qquad \delta_\alpha F_{\mu\nu} = i[\alpha, F_{\mu\nu}] \;, \tag{7.10}$$

under gauge transformation.

Abelian gauge field. An abelian gauge field is a special case of the above, when $G = U(1)^n$. For each $U(1)$ factor, we have an abelian gauge field A_μ which transforms simply as:

$$\delta_\alpha A_\mu = \partial_\mu \alpha \;. \tag{7.11}$$

In that case, the field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \;, \tag{7.12}$$

is gauge-invariant.

Yang-Mills action. Consider G a simple gauge group. The Yang-Mills (YM) action reads:

$$S_{YM} = \int d^4x \, \mathrm{tr}\left(-\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}\right) \;. \tag{7.13}$$

This is obviously gauge invariant—that is, invariant under any gauge transformation (7.5) of the gauge field A_μ. Here, g^2 is the Yang-Mills *gauge coupling.* The YM action is the canonical kinetic term for a gauge field.

Matter fields. Given a gauge field A_μ, we can introduced *charged matter fields* φ (which might be scalars or spinors) in some representation $\mathfrak{R}$ of G. By definition, the matter field φ transforms in the representation $\mathfrak{R}$ if we have: [27]

$$\varphi' = R(g)\varphi \;, \qquad\qquad R(g) \in \mathfrak{R} \;, \tag{7.14}$$

under the gauge transformation (7.5). Note that $R(g)$, like g itself, depends on the space-time coordinates. At the level of the Lie algebra, we have:

$$\delta_\alpha \varphi = i\alpha^{(\mathfrak{R})}\varphi = i\alpha^a T_a^{(\mathfrak{R})}\varphi \;, \tag{7.15}$$

with $T_a^{(\mathfrak{R})}$ the the Lie algebra generators in the representation $\mathfrak{R}$. For instance, if φ is in the adjoint representation, we simply have $\delta_\alpha\varphi = i[\alpha, \varphi]$.

The gauge-covariant derivative (or *covariant derivative*, for short) is defined as:

$$D_\mu \varphi = (\partial_\mu - iA_\mu)\varphi \;, \tag{7.16}$$

[27] Mathematically, a charged field is a section of a vector bundle $V \to E \to \mathbb{R}^{1,3}$ associated to the principal G-bundle $G \to P \to \mathbb{R}^{1,3}$, with V the representation vector space.

with the gauge field acting in the appropriate representation, $A_\mu = A_\mu^a T_a^{(\mathfrak{R})}$. By construction, $D_\mu \varphi$ transforms covariantly, in the same representation as φ: $\delta_\alpha D_\mu \varphi = i\alpha D_\mu \varphi$. Note also that we have:

$$[D_\mu, D_\nu]\varphi = -iF_{\mu\nu}\varphi \ . \tag{7.17}$$

In term of the covariant derivative, the gauge transformation of the gauge field itself can be written as:

$$\delta_\alpha A_\mu = \mathrm{D}_\mu \alpha \ . \tag{7.18}$$

We can easily write down gauge-invariant kinetic terms, by replacing the derivatives with covariant derivatives—for instance:

$$\mathscr{L} = -D_\mu \bar{\phi} D^\mu \phi \ , \tag{7.19}$$

for a scalar field. Here, we assume that ϕ and $\bar{\phi}$ transform in conjugate representations.

7.1.2 Quantum gauge theory: running of the gauge coupling

To properly quantise a gauge theory, recall that one has to carefully "fix a gauge." There are various methods to do that, at various levels of mathematical sophistication. Then, the gauge-fixed quantum theory makes perfect sense, at least in perturbation theory.

In our discussion of supersymmetric gauge theories, we will need to keep in mind two important aspects of the quantum theory: [28]

- The YM gauge coupling constants undergo RG flow—they vary as we vary the RG scale μ. Depending on the sign of the β-function, the theory is either asymptotically free (free in the UV, and strongly-coupled in the IR), or IR free. Only asymptotically free theories are believed to be "fully consistent QFTs," but any gauge theory can be considered as an effective QFT, in the Wilsonian framework.

- Gauge theories can be *anomalous.* This means that the gauge invariance of the classical action is violated by the quantum effective action (due to a non-trivial transformation of the path-integral measure). When that is the case, the theory is inconsistent.

In the following, we review some key results, without derivation.

[28] This will be useful for the 'Advanced Supersymmetry' lectures. The discussion of gauge theories in these lectures will be mostly classical.

One-loop β function. Consider a four-dimensional YM gauge theory coupled to complex scalars ϕ and Weyl fermions ψ in representation $\mathfrak{R}_\phi$ and $\mathfrak{R}_\psi$ of the gauge group, respectively. (The representations can be reducible.) Schematically, the "minimal-coupling" Lagrangian reads:

$$\mathscr{L} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - D_\mu \bar{\phi} D^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi + \cdots \ , \tag{7.20}$$

where the trace over the gauge indices is left implicit. Here, the YM coupling only appears in the front of the first term. The physical (or "canonical") normalisation, however, corresponds to rescaling the gauge field as $A_\mu = g A_\mu^{(c)}$, so that positive powers g appears in interactions vertices, including through the covariant derivative $D_\mu = \partial_\mu - i g A_\mu^{(c)}$.

The Yang-Mills coupling g^2 runs with energy. At one-loop, the β-function is given by:

$$\beta\left(\frac{1}{g^2}\right) = \frac{b_0}{8\pi^2} \ , \qquad b_0 = \frac{11}{6} T(\text{adj}) - \frac{1}{6} T(\mathfrak{R}_\phi) - \frac{1}{3} T(\mathfrak{R}_\psi) \ , \tag{7.21}$$

thus $1/g^2$ runs logarithmically in μ:

$$\frac{1}{g(\mu)^2} = \frac{1}{g(\mu_0)^2} + \frac{b_0}{8\pi^2} \log \frac{\mu}{\mu_0} \ . \tag{7.22}$$

In (7.21), $T(\mathfrak{R})$ denotes the quadratic index of the representation $\mathfrak{R}$, defined by:

$$\operatorname{tr}\left(T_a^{(\mathfrak{R})} T_b^{(\mathfrak{R})}\right) = T(\mathfrak{R}) \frac{\delta_{ab}}{2} \ . \tag{7.23}$$

(If $\mathfrak{R}$ is reducible, $T(\mathfrak{R})$ is the sum of the indices of its irreducible representations.) In the following, we will only deal with $G = U(1)$ or $G = SU(N)$. We normalise the generators so that the index is equal to 1 for the fundamental and antifundamental representations of $SU(N)$:

$$T(\mathbf{N}) = T(\overline{\mathbf{N}}) = 1 \ , \qquad \text{for} \quad \mathfrak{g} = su(N) \ . \tag{7.24}$$

Then, we have:

$$T(\text{adj}) = 2N \ , \qquad \text{for} \quad \mathfrak{g} = su(N) \ , \tag{7.25}$$

for the adjoint representation. Note that, for $G = U(1)$, we have $T(\text{adj}) = 0$ and thus (7.21) specialises to:

$$b_0^{G=U(1)} = -\frac{1}{6} \sum_\phi (q_\phi)^2 - \frac{1}{3} \sum_\psi (q_\psi)^2 \ , \tag{7.26}$$

a sum over the $U(1)$ charges, q, of the matter fields. Note that $b_0^{G=U(1)} < 0$, so that the $U(1)$ gauge coupling (*i.e.* the effective "electric charge") goes to zero at low energy, [29] and blows up at high energy (that is called a Landau pole).

[29] If all the matter fields are massless. Matter fields with a non-zero mass m decouple from the RG flow at scales $\mu < m$. That is what happens in the real-world QED, since the electron has a mass.

For G a simple Lie group without too many matter fields, we can have $b_0 > 0$. In that case, the theory is asymptotically free, meaning that the YM coupling g^2 goes to zero in the UV limit. Conversely, it becomes large at low energy. One defines the *dynamically-generated* scale by:

$$\Lambda = \mu e^{-\frac{8\pi^2}{b_0 g^2(\mu)}} . \tag{7.27}$$

This scale is independent of μ; more precisely, if we fix $g^2(\mu_0)$ in the UV, then $\Lambda(\mu) = \Lambda(\mu_0)$ at any scale, in the one-loop approximation, due to (7.21). This is the (infrared) scale at which the gauge coupling blows up, and perturbation theory become unreliable.

Anomaly-free conditions. Consistency of the quantum theory requires that the gauge symmetry G be *non-anomalous.* Four-dimensional anomalies are only due to *chiral (Weyl) fermions.* The potential gauge anomaly is proportional to the following numerical coefficients:

$$\mathcal{A}_{abc} = \text{tr}\left(T_a^{(\mathfrak{R}_\psi)}\{T_b^{(\mathfrak{R}_\psi)}, T_c^{(\mathfrak{R}_\psi)}\}\right) = \frac{1}{2} A\big(\mathfrak{R}^{(\mathfrak{R}_\psi}\big)\, d_{abc} , \tag{7.28}$$

where d_{abc} is a symmetric invariant tensor. Amongst simple Lie groups, it is non-trivial only for $\mathfrak{g} = SU(N)$ with $N \geq 3$. In that case, we have the cubic index coefficients:

$$A(\mathbf{N}) = 1 , \quad A(\overline{\mathbf{N}}) = -1 , \quad A(\text{adj}) = 0 , \tag{7.29}$$

for the (anti)fundamental, anti-fundamental and adjoint representations, respectively. In particular, a QCD-like gauge theory with $G = SU(N)$ with N_f^+ fermions in the fundamental representation and N_f^- fermions in the anti-fundamental representations (and possibly a number of fermions in the adjoint) will be anomaly-free if and only if:

$$A\big(\mathfrak{R}^{(\mathfrak{R}_\psi}\big) = N_f^+ - N_f^- = 0 . \tag{7.30}$$

Thus, we must have as many fundamental as anti-fundamental fermions, so that the total gauge anomaly vanishes. The number $N_f = N_f^+ = N_f^-$ is called the "number of flavors." In QCD, for instance, we have $G = SU(3)$ and $N_f = 6$ (called up, down, strange, charm, top, bottom).

Similarly, in an abelian gauge theory with $G = U(1)$, we have the anomaly-free conditions:

$$\sum_\psi (q_\psi)^3 = 0 , \qquad \sum_\psi q_\psi = 0 . \tag{7.31}$$

The first condition is the vanishing of the cubic anomaly (7.28), while the second one is the vanishing of the so-called gravitational-gauge mixed anomaly.

7.2 Abelian vector multiplet

Let us now consider the 4d $\mathcal{N} = 1$ supersymmetric version of a gauge theory. The first ingredient is the *vector multiplet*, which combines the gauge fields A_μ with a fermionic superpartner $\lambda, \bar{\lambda}$ called the *gaugino.*

We first consider the supersymmetric version of an abelian theory, $G = U(1)$. That is, we want to build a supersymmetric version of Maxwell theory, with a gauge field (*i.e.* an 'electromagnetic potential') A_μ subject to the gauge invariance:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \ . \tag{7.32}$$

Looking at the general multiplet (5.17), we see that it contains a four-vector v_μ, which we would like to identify with A_μ. In fact, the gauge field is real, so a natural guess is that it will sit in a *real multiplet,* which satisfies $\mathcal{S}^\dagger = \mathcal{S}$.

A massless gauge field A_μ has 2 degrees of freedom off-shell—recall the helicity states $\lambda = \pm 1$ from the discussion around equation (3.45). On the other hand, an off-shell gauge field should have $4 - 1 = 3$ degrees of freedom—that is, 4 degrees of freedom modulo one degree of freedom which is pure gauge, due to (7.32). The fermion superpartner consists of one Weyl fermion, which we denote by $\lambda, \bar{\lambda}$, for a total of 4 off-shell degrees of freedom. Thus, we would like to introduce one real auxiliary field, which we denote D, to preserve the fermion-boson degeneracy.

The real superfield seems to contain too many fields, but there is a simply way out. To combine gauge invariance with supersymmetry, one should really find a superfield generalisation of the gauge transformation (7.32).

Definition: An abelian [30] *vector superfield*, V, is a real superfield,

$$V^\dagger = V \ , \tag{7.33}$$

subject to the gauge equivalence relation:

$$\boxed{V \to V + \frac{i}{2}\left(\Omega - \bar{\Omega}\right) \ ,} \tag{7.34}$$

where Ω, $\bar{\Omega}$ are chiral and anti-chiral multiplets, respectively, conjugate to each other.

This is obviously compatible with the reality condition (7.33). We write the corresponding infinitesimal gauge transformation as:

$$\delta_\Omega V = \frac{i}{2}\left(\Omega - \bar{\Omega}\right) \ . \tag{7.35}$$

The supersymmetric gauge transformation (7.34) gives us the expected counting of off-shell degrees of freedom. The real multiplets has $8 + 8$ real degrees of freedom, but the Ω-valued gauge equivalence removes $4 + 4$ degrees of freedom.

[30] Here, for $G = U(1)$. Of course, for any abelian gauge group $G = U(1)^n$, we would just have n distinct vector superfields.

The expansion of the superfield V in components is given by:

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \frac{i}{2}\theta\theta M(x) - \frac{i}{2}\bar\theta\bar\theta\bar M(x) \\ &\quad - \theta\sigma^\mu\bar\theta A_\mu(x) + i\theta\theta\bar\theta\left(\bar\lambda(x) + \frac{i}{2}(\bar\sigma^\mu\partial_\mu\chi(x))\right) \\ &\quad - i\bar\theta\bar\theta\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(D(x) + \frac{1}{2}\partial^2 C(x)\right) , \end{aligned} \tag{7.36}$$

just like in (5.18), with $v_\mu = A_\mu$. It is easy to work out the form of the gauge transformation (7.34) in component. Let us denote by:

$$\Omega = (\omega, \psi^\Omega, F^\Omega) , \qquad \bar\Omega = (\bar\omega, \bar\psi^{\bar\Omega}, \bar F^{\bar\Omega}) , \tag{7.37}$$

the chiral multiplet component fields. Then, we have

$$\begin{aligned} &C \to C + \frac{i}{2}(\omega - \bar\omega) , && A_\mu \to A_\mu + \partial_\mu\left(\frac{\omega+\bar\omega}{2}\right) , \\ &\chi \to \chi + \frac{1}{\sqrt{2}}\psi^\Omega , && \lambda \to \lambda \\ &\bar\chi \to \bar\chi + \frac{1}{\sqrt{2}}\bar\psi^{\bar\Omega} , && \bar\lambda \to \bar\lambda \\ &M \to M + F^\Omega , && D \to D . \\ &\bar M \to \bar M + \bar F^{\bar\Omega} , && \end{aligned} \tag{7.38}$$

Thus, the field A_μ transforms like an abelian gauge field (7.32), as needed, with the real gauge function:

$$\alpha(x) = \frac{\omega(x) + \bar\omega(x)}{2} . \tag{7.39}$$

We also see that the field components $C, \chi, \bar\chi, M$ and $\bar M$ are *pure gauge*—that is, they are gauge-equivalent to zero.

7.2.1 Supersymmetry in the Wess-Zumino gauge

For many purposes, it will be useful to fix the so-called Wess-Zumino (WZ) gauge, defined by:

$$C = \chi = \bar\chi = M = \bar M = 0 . \tag{7.40}$$

Note that the WZ gauge is not compatible with supersymmetry. Indeed, even if we start from the WZ gauge (7.40), we see from (5.16) that a supersymmetry transformation will generate the new components:

$$\begin{aligned} &\delta C = 0 , \quad && \delta\chi_\alpha = i(\sigma^\mu\bar\epsilon)_\alpha A_\mu , && \delta M = 2\bar\epsilon\bar\lambda , \\ & && \delta\bar\chi_{\dot\alpha} = -i(\epsilon\sigma^\mu)_{\dot\alpha} A_\mu , && \delta\bar M = 2\epsilon\lambda . \end{aligned} \tag{7.41}$$

However, one can compensate (7.41) with another gauge transformation, to *restore* the WZ gauge. Namely, let us define the gauge transformation:

$$\delta_{\Omega_{\rm WZ}} V = \frac{i}{2}(\Omega_{\rm WZ} - \bar\Omega_{\rm WZ}) \ , \tag{7.42}$$

with the gauge parameters:

$$\begin{aligned} &\omega^{\Omega_{\rm WZ}} = 0 \ , && \bar\omega^{\bar\Omega_{\rm WZ}} = 0 \ , \\ &\psi_\alpha^{\Omega_{\rm WZ}} = -i\sqrt{2}(\sigma^\mu\bar\epsilon)_\alpha A_\mu \ , && \bar\psi_{\dot\alpha}^{\bar\Omega_{\rm WZ}} = i\sqrt{2}(\epsilon\sigma^\mu)_{\dot\alpha} A_\mu \ , \\ &F^{\Omega_{\rm WZ}} = -2\bar\epsilon\bar\lambda \ , && \bar F^{\bar\Omega_{\rm WZ}} = -2\epsilon\lambda \ . \end{aligned} \tag{7.43}$$

Then, by construction, the *modified* supersymmetry transformation:

$$\hat\delta \equiv \delta + \delta_{\Omega_{\rm WZ}} \tag{7.44}$$

preserves the WZ gauge.

Supersymmetry transformations. A vector multiplet *in WZ gauge* only contains the physical fields:

$$V_{\rm WZ} = (A_\mu, \lambda, \bar\lambda, D) \ . \tag{7.45}$$

The corresponding superfield reads:

$$V = -\theta\sigma^\mu\bar\theta A_\mu + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta\, D \ , \tag{7.46}$$

After fixing the WZ gauge, we still have the residual gauge invariance with parameters:

$$\Omega = \bar\Omega = (\alpha, 0, 0) \ , \tag{7.47}$$

which just gives the $U(1)$ gauge transformation (7.32). Note that $\lambda, \bar\lambda$ and D are gauge-invariant. The supersymmetry transformations of $V_{\rm WZ}$ are:

$$\begin{aligned} &\hat\delta A_\mu = i\epsilon\sigma_\mu\bar\lambda + i\bar\epsilon\bar\sigma_\mu\lambda \ , \\ &\hat\delta\lambda_\alpha = i\epsilon_\alpha D + (\sigma^{\mu\nu}\epsilon)_\alpha F_{\mu\nu} \ , \\ &\hat\delta\bar\lambda_{\dot\alpha} = -i\bar\epsilon_{\dot\alpha} D - (\bar\epsilon\bar\sigma^{\mu\nu})_{\dot\alpha} F_{\mu\nu} \ , \\ &\hat\delta D = -\epsilon\sigma^\mu\partial_\mu\bar\lambda + \bar\epsilon\bar\sigma^\mu\partial_\mu\lambda \ , \end{aligned} \tag{7.48}$$

with $F_{\mu\nu}$ the (gauge-invariant) field strength defined in (7.12). Note that we have:

$$\hat\delta F_{\mu\nu} = i\epsilon(\sigma_\nu\partial_\mu\bar\lambda - \sigma_\mu\partial_\nu\bar\lambda) + i\bar\epsilon(\bar\sigma_\nu\partial_\mu\lambda - \bar\sigma_\mu\partial_\nu\lambda) \ . \tag{7.49}$$

7.2.2 The abelian field-strength multiplet

The fields:

$$\lambda_\alpha \ , \qquad \bar{\lambda}_{\dot{\alpha}} \ , \qquad F_{\mu\nu} \ , \qquad D \ , \tag{7.50}$$

form a *gauge-invariant* supersymmetry multiplet on their own. In fact, looking at the supersymmetry variation of λ in (7.48), we see that it is proportional to a ϵ only, without $\bar{\epsilon}$ contribution. Thus, we may suspect that we can organise the fields (7.50) into a pair of chiral and anti-chiral multiplets, which we will call $\mathcal{W}_\beta$ and $\bar{\mathcal{W}}_{\dot{\beta}}$—they are just like any chiral multiplet, except that they have an overall spinor index:

$$\begin{aligned} \phi_\beta^{\mathcal{W}} &= \lambda_\beta \ , \\ \psi_{\alpha\beta}^{\mathcal{W}} &= -\frac{i}{\sqrt{2}}\varepsilon_{\alpha\beta} D - \frac{1}{\sqrt{2}}(\sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu} \ , \\ F_\beta^{\mathcal{W}} &= i(\sigma^\mu \partial_\mu \bar{\lambda})_\alpha \ . \end{aligned} \tag{7.51}$$

Note that the bispinor $\psi_{\alpha\beta}^{\mathcal{W}}$ has a natural decomposition into a scalar (the field D) and an self-dual two-form (namely, the self-dual part of $F_{\mu\nu}$), in agreement with (2.25).

Definition: Given an abelian vector superfield V, the *field strength chiral and anti-chiral superfields* are defined as:

$$\boxed{\mathcal{W}_\alpha = -\frac{i}{4}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{D}_\alpha V \ , \qquad \bar{\mathcal{W}}_{\dot{\alpha}} = -\frac{i}{4}\mathrm{D}\mathrm{D}\bar{\mathrm{D}}_{\dot{\alpha}} V \ .} \tag{7.52}$$

The superfields $\mathcal{W}$ and $\bar{\mathcal{W}}$, defined in this way, are fully gauge invariant under the gauge transformation (7.34). Indeed, a gauge transformation gives:

$$\delta_\Omega \mathcal{W}_\alpha = \frac{1}{8}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{D}_\alpha(\Omega - \bar{\Omega}) = -\frac{1}{8}\bar{\mathrm{D}}^{\dot{\alpha}}\{\bar{\mathrm{D}}_{\dot{\alpha}}, \mathrm{D}_\alpha\}\Omega = 0 \ , \tag{7.53}$$

where we used $\bar{D}_{\dot{\alpha}}\Omega = 0$, $\mathrm{D}_\alpha\bar{\Omega} = 0$ and (5.7). From the definition (7.52), it also follows that:

$$\bar{\mathrm{D}}_{\dot{\beta}}\mathcal{W}_\alpha = 0 \ , \qquad \mathrm{D}_\beta\bar{\mathcal{W}}_{\dot{\alpha}} = 0 \ . \tag{7.54}$$

We also have the non-trivial identity:

$$\boxed{\mathrm{D}^\alpha\mathcal{W}_\alpha - \bar{\mathrm{D}}_{\dot{\alpha}}\mathcal{W}^{\dot{\alpha}} = 0 \ .} \tag{7.55}$$

This is the superspace generalisation of the Bianchi identity:

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ . \tag{7.56}$$

In components, we have:

$$\mathcal{W}_\beta(z,\theta) = \lambda_\beta - \theta^\alpha\left((\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu} + i\varepsilon_{\alpha\beta}D\right) + i\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\beta \ , \tag{7.57}$$

and similarly for $\bar{\mathcal{W}}_{\dot{\beta}}$. Here, we used the chiral coordinate z^μ for simplicity.

7.3 Non-abelian vector multiplet

Consider now a non-abelian vector superfield:

$$V \; : \; \mathbb{R}^{1,3|4} \to i\mathfrak{g} \; , \tag{7.58}$$

with $\mathfrak{g}$ the Lie algebra of some compact gauge group G.

Definition: For any $\mathfrak{g}$, a vector superfield V is a real superfield subject to the gauge equivalence:

$$\boxed{e^{-2V} \to e^{i\bar{\Omega}} e^{-2V} e^{-i\Omega} \; ,} \tag{7.59}$$

with Ω, $\bar{\Omega}$ some chiral and anti-chiral multiplet valued in the adjoint representation of $\mathfrak{g}$, and conjugate to each other.

The corresponding infinitesimal gauge transformation is given by:

$$\delta_\Omega V = \frac{i}{2}\left(\Omega - \bar{\Omega}\right) + \frac{i}{2}[\Omega + \bar{\Omega}, V] \; , \tag{7.60}$$

generalising (7.35) to the non-abelian case.

7.3.1 Supersymmetry in the Wess-Zumino gauge

We can again make use of the large gauge invariance and go to a Wess-Zumino gauge, exactly as in (7.40). In WZ gauge, the vector superfield reads:

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\, D \; , \tag{7.61}$$

where all the fields are valued in the adjoint representation of $\mathfrak{g}$. We can again define modified supersymmetry transformations compatible with the WZ gauge,

$$\hat{\delta} \equiv \delta + \delta_{\Omega_{\rm WZ}} \tag{7.62}$$

with the compensating gauge parameters given in (7.43). We have:

$$\frac{i}{2}(\Omega_{\rm WZ} + \bar{\Omega}_{\rm WZ}) = \theta\sigma^\mu\bar{\epsilon}A_\mu + \bar{\theta}\bar{\sigma}^\mu\epsilon A_\mu - i\theta\theta\bar{\epsilon}\bar{\lambda} - i\bar{\theta}\bar{\theta}\epsilon\lambda + \cdots \; , \tag{7.63}$$

where the ellipsis denotes higher-order terms in $\theta, \bar{\theta}$. Then, a direct computation shows that, in the WZ gauge, the supersymmetry transformations of the vector multiplet take the form:

$$\begin{aligned}
\hat{\delta}A_\mu &= i\epsilon\sigma_\mu\bar{\lambda} + i\bar{\epsilon}\bar{\sigma}_\mu\lambda \; , \\
\hat{\delta}\lambda_\alpha &= i\epsilon_\alpha D + (\sigma^{\mu\nu}\epsilon)_\alpha F_{\mu\nu} \; , \\
\hat{\delta}\bar{\lambda}_{\dot{\alpha}} &= -i\bar{\epsilon}_{\dot{\alpha}} D - (\bar{\epsilon}\bar{\sigma}^{\mu\nu})_{\dot{\alpha}} F_{\mu\nu} \; , \\
\hat{\delta}D &= -\epsilon\sigma^\mu D_\mu\bar{\lambda} + \bar{\epsilon}\bar{\sigma}^\mu D_\mu\lambda \; ,
\end{aligned} \tag{7.64}$$

where $F_{\mu\nu}$ is the non-abelian field strength, and D_μ is the covariant derivative—for instance, $D_\mu\lambda = \partial_\mu\lambda - i[A_\mu, \lambda]$. The new Lie-algebra commutators terms in (7.64) arise from the commutator in:

$$\delta_{\Omega_{\rm WZ}} V = \frac{i}{2}\left(\Omega_{\rm WZ} - \bar{\Omega}_{\rm WZ}\right) + \frac{i}{2}[\Omega_{\rm WZ} + \bar{\Omega}_{\rm WZ}, V] \; .$$

7.3.2 Non-abelian field-strength superfield

The non-abelian field-strength superfield should transform covariantly under gauge transformations, namely:

$$\mathcal{W}_\alpha \to e^{i\Omega}\mathcal{W}_\alpha e^{-i\Omega} \;, \qquad \bar{\mathcal{W}}_{\dot\alpha} \to e^{i\bar\Omega}\bar{\mathcal{W}}_{\dot\alpha} e^{-i\bar\Omega} \;, \tag{7.65}$$

Note that these gauge transformations are compatible with the chirality conditions:

$$\bar{\mathrm{D}}_{\dot\alpha}\mathcal{W}_\beta = 0 \;, \qquad \mathrm{D}_\alpha \bar{\mathcal{W}}_{\dot\beta} = 0 \;. \tag{7.66}$$

This can be achieved with the following definition:

$$\boxed{\mathcal{W}_\alpha = \frac{i}{8}\bar{\mathrm{D}}\bar{\mathrm{D}}e^{2V}\mathrm{D}_\alpha e^{-2V} \;, \qquad \bar{\mathcal{W}}_{\dot\alpha} = -\frac{i}{8}\mathrm{D}\mathrm{D}e^{-2V}\bar{\mathrm{D}}_{\dot\alpha}e^{2V} \;.} \tag{7.67}$$

which reduces to (7.52) in the abelian case. Expanding out the superfield, we find:

$$\begin{aligned}
\mathcal{W}_\beta(z,\theta) &= \lambda_\beta - \theta^\alpha \left((\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu} + i\varepsilon_{\alpha\beta}D\right) + i\theta\theta(\sigma^\mu D_\mu\bar\lambda)_\beta \;, \\
\bar{\mathcal{W}}_{\dot\beta}(\bar z,\bar\theta) &= \bar\lambda_{\dot\beta} - \bar\theta^{\dot\alpha}\left((\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}F_{\mu\nu} - i\varepsilon_{\dot\alpha\dot\beta}D\right) - i\bar\theta\bar\theta(D_\mu\lambda\,\sigma^\mu)_{\dot\beta} \;,
\end{aligned} \tag{7.68}$$

in the chiral and anti-chiral coordinates, respectively.

7.4 The super-Yang-Mills Lagrangian

Given the field-strength superfield, it is very easy to build a supersymmetric Lagrangian. A very important property of the vector multiplet of 4d $\mathcal{N}=1$ supersymmetry is that the field-strength sits inside a *chiral superfield*. Indeed, one can easily check that the Yang-Mills term can be build from a *F-term:*

$$-\frac{1}{2g^2}\int d^2\theta\,\mathrm{tr}\,(\mathcal{W}^\alpha\mathcal{W}_\alpha) = -\frac{1}{4g^2}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \cdots \;, \tag{7.69}$$

where the ellipsis denotes the supersymmetric completion. Here, we introduced the real YM coupling constant, g^2, as in (7.13). However, recall that the coupling constants appearing in F-terms are naturally seen as complex couplings, which enter holomorphically. In super-Yang-Mills (SYM) theory, it is customary to define the *holomorphic gauge coupling:*

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \;. \tag{7.70}$$

We will discuss the meaning of θ, the so-called θ-angle, momentarily. Then, the full Lagrangian of SYM can be written in superspace simply as:

$$\boxed{\mathscr{L}_{\mathrm{SYM}} = -\frac{\tau}{16\pi i}\int d^2\theta\,\mathrm{tr}\;\mathcal{W}^\alpha\mathcal{W}_\alpha + \frac{\bar\tau}{16\pi i}\int d^2\bar\theta\,\mathrm{tr}\;\bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} \;.} \tag{7.71}$$

This gives:

$$\mathscr{L}_{\rm SYM} = \frac{1}{g^2}\,{\rm tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar\lambda\bar\sigma^\mu D_\mu\lambda + \frac{1}{2}D^2\right) - \frac{\theta}{64\pi^2}\,{\rm tr}\left(\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\ . \quad (7.72)$$

The term proportional to $1/g^2$ is the expected supersymmetric completion of the YM kinetic, with a standard kinetic term for the gaugino λ, a Weyl fermion (that is, a Majorana fermion) in the adjoint representation, and a quadratic term for the auxiliary field D.

Note that, in the above, we implicitly assumed that the gauge group G was simple (or $U(1)$). In the case when G is the product of several simple factors and $U(1)$'s—for instance, $G = SU(3)\times SU(2)\times U(1)$ as in the Standard Model—, there is an independent gauge coupling constant τ for each gauge group.

Gaugino R-charge. As we can see from (7.61), the gaugino λ has R-charge 1:

$$R[\lambda] = 1\ , \qquad R[\bar\lambda] = -1\ . \quad (7.73)$$

Therefore, the chiral superfield $\mathcal{W}_\alpha$ also has R-charge 1, so that the SYM Lagrangian preserves $U(1)_R$.

7.5 Charged matter fields and supersymmetric Lagrangians

Matter fields in 4d $\mathcal{N}=1$ supersymmetric gauge theories sit in chiral multiplets, Φ. We take Φ to transform in some (generally reducible) representation $\mathfrak{R}$ of G. Therefore, $\bar\Phi$ transforms in the conjugate representation $\bar{\mathfrak{R}}$.

A gauge transformation acts on the matter superfields as:

$$\Phi \to e^{i\Omega}\Phi\ , \qquad \bar\Phi \to \bar\Phi e^{-i\bar\Omega}\ . \quad (7.74)$$

The minimal coupling to the vector multiplet takes the form:

$$\boxed{\mathscr{L}_{\bar\Phi\Phi} = \int d^2\theta d^2\bar\theta\,\bar\Phi e^{-2V}\Phi\ .} \quad (7.75)$$

This Lagrangian is obviously gauge invariant under supersymmetric gauge transformations, (7.59) together with (7.74). In the Wess-Zumino gauge, using the explicit expression (7.61) for V and the fact that:

$$V^3 = 0\ , \quad (7.76)$$

it is easy to check that:

$$\boxed{\mathscr{L}_{\bar\Phi\Phi} = -D_\mu\bar\phi D^\mu\phi - i\bar\psi\bar\sigma^\mu D_\mu\psi + \bar F F - \bar\phi D\phi - i\sqrt2\bar\phi\lambda\psi + i\sqrt2\bar\lambda\bar\psi\phi\ .} \quad (7.77)$$

Note the coupling to the D term, as well as the Yukawa coupling involving the chiral fermion and the gaugino.

7.6 Scalar potential and classical vacuum equations

Using the above, we can write down the supersymmetric Lagrangian for a completely general renormalizable supersymmetric gauge theory.

For simplicity, let us choose G to be a simple compact Lie group, and consider some matter field in chiral multiplet, Φ, in some (generally reducible) representation $\mathfrak{R}$. The full Lagrangian for the vector and chiral multiplets can be written compactly, in superspace, as: [31]

$$\begin{aligned} \mathscr{L} &= \int d^2\theta d^2\bar\theta\, \bar\Phi e^{-2V}\Phi - \frac{\tau}{16\pi i}\int d^2\theta\, \mathrm{tr}(\mathcal{W}\mathcal{W}) + \frac{\bar\tau}{16\pi i}\int d^2\bar\theta\, \mathrm{tr}(\bar{\mathcal{W}}\bar{\mathcal{W}}) \\ &\quad + \int d^2\theta W(\Phi) + \int d^2\bar\theta \bar W(\bar\Phi)\ . \end{aligned} \tag{7.78}$$

Here, $W(\Phi)$ is some *gauge invariant* holomorphic polynomial in Φ (which we take to be at most cubic in the renormalisable theory).

Note that the Lagrangian of any renormalisable 4d $\mathcal{N}=1$ supersymmetric gauge theory is fully determined by the data of:

- The gauge group G with gauge coupling(s) $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.
- The representation $\mathfrak{R}$ for the chiral multiplets.
- The superpotential $W(\Phi)$.

All the various interactions terms are then determined by the combination of gauge invariance and supersymmetry, as well as by W.

7.6.1 Classical scalar potential and vacuum manifold

By looking at the classical Lagrangian in components, it is easy to study the classical scalar potential of the gauge theory. The adjoint-valued auxiliary field D enters as:

$$\mathscr{L} \supset \frac{1}{2g^2}D^2 - \bar\phi D\phi\ , \tag{7.79}$$

in the WZ gauge. The equations of motions for the auxiliary fields $D = D^a T_a$ give:

$$D_a = g^2\, \bar\phi T_a^{(\mathfrak{R})}\phi\ , \qquad a = 1, \cdots, \dim(G)\ . \tag{7.80}$$

Integrating out D, we then find the scalar potential:

$$\boxed{V_0 = \sum_\phi \left|\frac{\partial W}{\partial\phi}\right|^2 + \frac{g^2}{2}\sum_{a=1}^{\dim(G)}\left(\bar\phi T_a^{(\mathfrak{R})}\phi\right)^2\ .} \tag{7.81}$$

[31] The generalization to a general gauge group $G = \prod_s G_s \times \prod_I U(1)_I$, which is a product of simple and abelian factors, is straightforward. We just introduce a gauge coupling g_s^2 for each simple group G_s and e_I^2 for each $U(1)_I$ factor.

The first term is the contribution from the superpotential, which we discussed in previous sections, while the second term can be viewed as a contribution from the gauge interactions themselves. The real operators:

$$\mu_a(\phi, \bar{\phi}) \equiv \bar{\phi} T_a^{(\mathfrak{R})} \phi \tag{7.82}$$

are often called the "moment map operators."

Since the scalar potential is again a sum of perfect squares, the classical vacuum equations of a supersymmetric gauge theory are:

$$\boxed{\partial_\phi W = 0 \;, \quad \forall \phi \;, \qquad\qquad \mu_a(\phi, \bar{\phi}) = 0 \;, \quad \forall a \;.} \tag{7.83}$$

Any two solutions to (7.83) related by a (constant) gauge transformations are physically equivalent. So, we introduce the equivalence relation on the space of constant field values:

$$\phi' \sim \phi \qquad \text{if} \qquad \exists \;(\alpha^a) \in \mathbb{R}^{\dim(G)} \;\text{ such that }\; \phi' = e^{i\alpha^a T_a^{(\mathfrak{R})}} \phi \;. \tag{7.84}$$

The constant values of the scalar field $\phi \in \Phi$ span the vector space:

$$V_{\mathfrak{R}} \cong \mathbb{C}^n \;, \qquad n \equiv \dim(\mathfrak{R}) \;, \tag{7.85}$$

on which the representation $\mathfrak{R}$ acts. Then, the *vacuum manifold* of the gauge theory takes the general form:

$$\boxed{\mathcal{M} = \{\phi \in V_{\mathfrak{R}} \mid \partial_\phi W = 0 \;,\; \mu_a = 0\}/G \;,} \tag{7.86}$$

where the quotient by the gauge group corresponds to the equivalence relation (7.84). In our discussion of theories with only chiral multiplets, we saw that the vacuum moduli space was a purely algebraic object—in particular, everything was holomorphic in ϕ. This is apparently not the case in a gauge theory, since the formula (7.86) is non-holomorphic in two ways: the moment maps μ_a are real, and the gauge equivalence (7.84) is in terms of real gauge parameters α^a.

Nonetheless, there is a simple-looking (although by no mean obvious) way to rewrite (7.86) more algebraically. It turns out that imposing the vanishing of the moment maps, $\mu_a = 0$, and then dividing by G, is *equivalent* to dividing by the *complexified gauge group:*

$$\boxed{\mathcal{M} = \{\phi \in V_{\mathfrak{R}} \mid \partial_\phi W = 0\}/G_{\mathbb{C}} \;.} \tag{7.87}$$

In this approach, we are considering the space of complexified gauge orbits (or, more precisely, their closure), under the $G_{\mathbb{C}}$ action:

$$\phi' \sim \phi \qquad \text{if} \qquad \exists \;(\omega^a) \in \mathbb{C}^{\dim(G)} \;\text{ such that }\; \phi' = e^{i\omega^a T_a^{(\mathfrak{R})}} \phi \;. \tag{7.88}$$

The fact that the two approaches (7.86) and (7.87) reproduce the same moduli space was shown explicitly in [24]. [32]

Conceptually, this was to be expected: the fact that we only divide by real gauge transformations in (7.86) is an artefact of the WZ gauge. The supersymmetric gauge transformations on chiral superfields,

$$\delta_\Omega \Phi = e^{i\Omega}\Phi \ , \tag{7.89}$$

are really $G_{\mathbb{C}}$-valued gauge transformations. More generally, the F-term contributions to the Lagrangian of any supersymmetric gauge theory are invariant under the complexified gauge group $G_{\mathbb{C}}$, while the total Lagrangian (in particular, the D-term kinetic term for matter fields) is only G-invariant.

Finally, it is non-obvious but nonetheless true that the vacuum moduli space $\mathcal{M}$ of a gauge-theory is also a Kähler manifold, just like in the case without gauge fields.

8 Spontaneous supersymmetry breaking

The "real world" is not supersymmetric—we clearly do not observe a massless "photino," the fermionic superpartner of a photon, nor an electrically charged scalar, the "selectron" at 0.5 MeV, which would be the scalar superpartner of the electron.

Thus, if supersymmetry is part of a more fundamental theory of Particle Physics, it should be *spontaneously broken.* There should be a supersymmetry-breaking mass scale, $M_{\rm SUSY}$, which is likely larger than the TeV scale. In any such theory, this scale gives the approximate mass-splitting between supersymmetric partners:

$$|m_{\rm boson} - m_{\rm fermion}| = M_{\rm SUSY} \ . \tag{8.1}$$

At very high energy, $\mu \gg M_{\rm SUSY}$, the mass splitting can be neglected and the theory looks supersymmetric, while the vacuum of the theory (our world, presumably) is not supersymmetric.

In this section, we give a brief theoretical discussion of supersymmetry breaking in general. In the next section, we will discuss attempts to apply supersymmetry to Particle Physics.

8.1 The supercurrent multiplet

In any local 4d $\mathcal{N} = 1$ supersymmetric QFT, the supercharges are the integral of conserved *supersymmetry currents* over a space-like slice:

$$\boxed{Q_\alpha = \int_\Sigma d^3x\, S^0_\alpha \ , \qquad \bar Q_{\dot\alpha} = \int_\Sigma d^3x\, \bar S^0_{\dot\alpha} \ .} \tag{8.2}$$

[32] Mathematically, it is a non-trivial equivalence between Kähler quotients (corresponding to (7.86)) and Geometric Invariant Theory (GIT) quotients (corresponding to (7.87)).

The supersymmetric current—or *supercurrent*—is of a Majorana-spinor worth of conserved currents:

$$S^\mu_\alpha(x) \ , \qquad S^\mu_{\dot\alpha}(x) \ , \qquad\qquad \partial_\mu S^\mu_\alpha = 0 \ , \qquad \partial_\mu S^\mu_{\dot\alpha} = 0 \ , \tag{8.3}$$

for a total of 12 independent real *fermionic* local operators. In any given theory, defined by a Lagrangian, the supercurrent can computed explicitly by the usual Noether procedure. For instance, consider the theory of a free massless chiral multiplet:

$$\mathscr{L} = -\partial_\mu\bar\phi\partial^\mu\phi - i\bar\psi\bar\sigma^\mu\partial_\mu\psi + \bar F F \ . \tag{8.4}$$

One easily finds:

$$S^\mu_\alpha = \sqrt{2}\partial_\nu\bar\phi\,(\sigma^\nu\bar\sigma^\mu\psi)_\alpha \ , \qquad \bar S^\mu_{\dot\alpha} = \sqrt{2}\left(\bar\psi\bar\sigma^\mu\sigma^\nu\right)_{\dot\alpha}\partial_\nu\phi \ , \tag{8.5}$$

which is conserved upon using the equations of motion, $\partial_\mu\partial^\mu\phi = 0$ and $\bar\sigma^\mu\partial_\mu\psi = 0$.

Note that the supercurrent is not fully determined by the Noether procedure—it can be "improved" by adding a term which is automatically conserved:

$$S^\mu_\alpha \to S^\mu_\alpha + (\sigma^{\mu\nu})_\alpha{}^\beta\partial_\nu\eta_\beta \ , \tag{8.6}$$

for some spinor η.

Like any local operator in a supersymmetry theory, S^μ_α must be part of a supersymmetric multiplet, which is called the *supercurrent multiplet*. From the supersymmetry algebra, it is clear that the supercurrent is in the same multiplet as the *energy-momentum tensor*, $T_{\mu\nu}$. Indeed, $T_{\mu\nu}$ is itself the local current for the conserved momentum:

$$P_\mu = \int_\Sigma d^3x\; T^0{}_\mu \ , \qquad\qquad \partial_\mu T^{\mu\nu} = 0 \ . \tag{8.7}$$

and we must therefore have:

$$\{Q_\alpha, \bar S_{\mu\dot\beta}\} = 2\sigma^\nu_{\alpha\dot\beta}T_{\mu\nu} \ , \tag{8.8}$$

in order to reproduce the 4d $\mathcal{N}=1$ supersymmetry algebra. (The relation (8.8) holds modulo terms that do not contribute upon integration over space.) The energy-momentum tensor itself contains only 6 independent real bosonic operator. Thus, we need at least 6 more real bosonic operators to match the fermionic degrees of freedom of the supercurrent.

The most general supercurrent multiplet actually contains 16+16 real operators, and is called the $\mathcal{S}$-multiplet [25, 26]. Here, we will focus on a special case, that contains $12+12$ operators and was first derived by Ferrara and Zumino (FZ) [27].

In superspace, the FZ supercurrent multiplet is a superfield $\mathcal{J}_\mu$ that satisfies:

$$\boxed{\bar D^{\dot\alpha}\mathcal{J}_{\alpha\dot\alpha} = D_\alpha X \ , \qquad\qquad \bar D_{\dot\alpha}X = 0 \ .} \tag{8.9}$$

In components, we have [26]:

$$\begin{aligned}\mathcal{J}_\mu &= j_\mu - i\theta\left(S_\mu + \frac{1}{3}\sigma_\mu\bar\sigma_\nu S^\nu\right) + i\bar\theta\left(\bar S_\mu + \frac{1}{3}\bar\sigma_\mu\sigma_\nu\bar S^\nu\right) + \frac{i}{2}\theta\theta\partial_\mu\bar X \\ &\quad - \frac{i}{2}\bar\theta\bar\theta\partial_\mu X + 2\theta\sigma^\nu\bar\theta\left(T_{\nu\mu} - \frac{1}{3}\eta_{\mu\nu}T^\rho{}_\rho + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\partial^\rho j^\sigma\right) + \cdots\ ,\end{aligned} \tag{8.10}$$

where the ellipsis denotes higher-order terms in $\theta, \bar\theta$. Here, the operators $j^\mu, X, \bar X$ provides 6 real operators (here j^μ is not a conserved current, in general, despite the notation).

8.2 Spontaneous supersymmetry breaking and goldstino

Classical argument. Consider now the situation when the vacuum is not invariant under supersymmetry. The scalar potential of a supersymmetric gauge theory reads:

$$V_0 = \sum_\phi \left|\frac{\partial W}{\partial\phi}\right|^2 + \frac{g^2}{2}\sum_{a=1}^{\dim(G)}\left(\bar\phi T_{\mathbf{a}}^{(\mathfrak{R})}\phi\right)^2\ . \tag{8.11}$$

Supersymmetry is spontaneously broken, at the classical level, if the vacuum has non-zero energy. Let us define by:

$$\bar f_i \equiv \frac{\partial W}{\partial\phi^i}\ , \qquad d_{\mathbf{a}} \equiv \bar\phi T_{\mathbf{a}}^{(\mathfrak{R})}\phi\ , \tag{8.12}$$

the values of the "F-terms" and "D-terms" in the vacuum, with VEVs $\phi = \langle\phi\rangle$ for the fundamental scalars. Here, the index $\mathbf{a}$ runs over the generators of G, and i is an index for the (generally reducible) representation $\mathfrak{R}$. Supersymmetry is broken is some f_i or d_a are non-vanishing. Consider, then, any (classical) vacuum:

$$\frac{\partial V}{\partial\phi^i} = 0\ , \qquad \frac{\partial V}{\partial\bar\phi_i} = 0\ , \tag{8.13}$$

which gives:

$$\frac{\partial W}{\partial\phi^i\partial\phi^j}f^j + g^2\bar\phi_j(T_{\mathbf{a}})^j{}_i d^{\mathbf{a}} = 0\ , \qquad \forall i\ . \tag{8.14}$$

and its Hermitian conjugate. Moreover, the gauge-invariance of the superpotential gives:

$$\begin{aligned}\delta_\Omega W = 0 \qquad &\Rightarrow \qquad \bar f_j(T_{\mathbf{a}})^j{}_i\phi^i = 0\ , \\ \delta_{\bar\Omega}\bar W = 0 \qquad &\Rightarrow \qquad \bar\phi_j(\bar{T_{\mathbf{a}}})^j{}_i f^i = 0\ .\end{aligned} \tag{8.15}$$

We can write the conditions (8.14) and (8.15) as:

$$\mathcal{M}_{\frac{1}{2}}\begin{pmatrix} f^i \\ \widetilde{d}^{\mathbf{a}}\end{pmatrix} = 0\ , \qquad \mathcal{M}_{\frac{1}{2}} \equiv \begin{pmatrix} \frac{\partial W}{\partial\phi^i\partial\phi^j} & i\sqrt{2}\bar\phi_k(T_{\mathbf{a}})^k{}_j \\ i\sqrt{2}\bar\phi_k(T_{\mathbf{b}})^k{}_i & 0\end{pmatrix}\ , \tag{8.16}$$

where we also defined:

$$\widetilde{d}^{\mathbf{a}} \equiv -\frac{ig^2}{\sqrt{2}} d^{\mathbf{a}} \ , \tag{8.17}$$

for convenience of notation. By inspection of the fermion mass terms:

$$\mathscr{L}_m = -\frac{1}{2}\frac{\partial W}{\partial\phi^i\partial\phi^j}\psi^i\psi^j - i\sqrt{2}\bar{\phi}_k\lambda^{\mathbf{a}}(T_{\mathbf{a}})^k{}_i\psi^i + \text{h.c.} \ , \tag{8.18}$$

we see that $\mathcal{M}_{\frac{1}{2}}$ is just the *fermion mass matrix*:

$$\mathscr{L}_m = -\frac{1}{2}\begin{pmatrix}\psi & \lambda\end{pmatrix}\mathcal{M}_{\frac{1}{2}}\begin{pmatrix}\psi \\ \lambda\end{pmatrix} + \text{h.c.} \ . \tag{8.19}$$

Incidentally, note that the gaugino mass terms are consistent with the Higgs mechanism—they vanish for $\langle\phi\rangle = 0$, and otherwise are equal to the mass of the massive W-bosons.

Any non-supersymmetric vacuum satisfy:

$$\mathcal{M}_{\frac{1}{2}}\begin{pmatrix}f^i \\ \widetilde{d}^{\mathbf{a}}\end{pmatrix} = 0 \ , \qquad \begin{pmatrix}f^i \\ \widetilde{d}^{\mathbf{a}}\end{pmatrix} \neq 0 \ . \tag{8.20}$$

Thus, the fermion mass matrix has *at least one vanishing eigenvalue.* In other words, there is necessarily a *massless fermion* in the spectrum of low-energy excitations around the vacuum. This is the analogue of the Goldstone theorem for bosonic symmetries, here in the case of supersymmetry, and the massless fermion is called the *goldstino.*

Non-perturbative argument. While we the above argument for the existence of a Goldstino was only in the tree-level approximation, the existence of a massless Golstino is actually true in the full QFT, similarly to the case of a Goldstone boson. It follows from the supersymmetric Ward identity:

$$\Big\langle \partial^\mu S_{\mu\alpha}(x)\bar{S}_{\nu\dot{\beta}}(0)\Big\rangle = -\delta^4(x)\Big\langle i\{Q_\alpha,\, S_{\nu\dot{\beta}}\}\Big\rangle \ . \tag{8.21}$$

Using (8.8), we have:

$$p^\mu \Big\langle S_{\mu\alpha}(p)\bar{S}_{\nu\dot{\beta}}(-p)\Big\rangle = -2\sigma^\mu_{\alpha\dot{\beta}}\langle T_{\nu\mu}\rangle = -2\sigma^\mu_{\alpha\dot{\beta}}\eta_{\mu\nu}\, E_0 \tag{8.22}$$

in momentum space. In the last line, we used that:

$$\langle T_{\mu\nu}\rangle = \eta_{\mu\nu}E_0 \ , \tag{8.23}$$

in a Poincaŕe-invariant vacuum (with $E_0 \geq 0$, with $E_0 = 0$ if the vacuum is supersymmetric). [33] This implies that: [34]

$$\Big\langle S_{\mu\alpha}(p)\bar{S}_{\nu\dot{\beta}}(-p)\Big\rangle \supset (\sigma_\mu\bar{\sigma}^\rho\sigma_\nu)_{\alpha\dot{\beta}}\frac{p_\rho}{p^2} \ . \tag{8.24}$$

[33] We assumed that $T_{\mu\nu}$ is symmetric, which can always be achieved by an improvement transformation.

[34] Indeed, we have: $p^\mu(\sigma_\mu\bar{\sigma}^\rho\sigma_\nu)p_\rho = -p^2\sigma_\nu$, as one can easily check using (A.1).

We therefore see the necessary appearance of a *pole* in the two-point function of the supercurrent, whenever the vacuum energy E_0 is non-zero. This corresponds to the presence of a massless fermion ψ^G_α, which is created out of the vacuum by Q_α.

At this point, we might worry that the existence of a Goldstino precludes any realistic model of supersymmetry in the real world, since we do not observe such a massless particle. The way out is that supersymmetry, ultimately, must be also coupled to gravity; this makes supersymmetry "gauged" (with space-time dependent supersymmetry parameters $\epsilon, \bar{\epsilon}$), in which case the Goldstino is "eaten" by the gravitino, in a supersymmetric version of the Higgs mechanism, and is then safely of the order of the supersymmetry-breaking scale M_{SUSY}.

8.3 Supersymmetric mass sum rule

In the tree-level approximation—that is, by looking at the classical Lagrangian—, one can derive additional constraints on the spectrum in a vacuum of a supersymmetric theory with spontaneously broken supersymmetry. Let m_s denote the mass of (mass eigenstates) particles of spin s. We must always have that the *supertrace* over the full mass matrix vanishes:

$$\text{STr}(M^2) \equiv \sum_{\text{scalars}} (m_0)^2 - 2 \sum_{\text{Weyl fermions}} (m_{\frac{1}{2}})^2 + 3 \sum_{\text{vectors}} (m_1)^2 = 0 \; . \tag{8.25}$$

The factors 2 and 3 accounts for the helicities of the fermions and vectors, respectively.

(The case with only chiral multiplet is discussed on a problem sheet. The general case with gauge field included is similar; see chapter 27 of [2] for details.)

In a supersymmetric vacuum, the spectrum is perfectly degenerate between bosons and fermions. The supersymmetric mass sum rule implies that, when supersymmetry is spontaneously broken, the fermion and bosons masses can differ but still organise themselves around some "average,"

$$\sum_{\text{bosons}} m^2 = \sum_{\text{fermions}} m^2 \; . \tag{8.26}$$

Furthermore, by symmmetry or gauge invariance, this same sum rule holds independently in each sector with a given set of charges under global and gauge symmetries.

8.4 Mechanisms of supersymmetry breaking

Writing down a theory that breaks supersymmetry spontaneously is somewhat of an art, although there are many examples on the market.

Supersymmetry-breaking models are usually separated into "F-term or D-term SUSY-breaking," depending on whether f_i or $d_{\mathbf{a}}$ in (8.12) is non-zero. (Of course, both could also be non-zero.) Let us briefly discuss some examples.

8.5 F-term supersymmetry breaking

For some superpotential $W(\Phi)$, whether in a theory of chiral multiplet or in a larger gauge theory, it might happen that the F-terms are non-zero at the minimim of V_0, namely:

$$\bar{f}_i = \frac{\partial W}{\partial \phi^i} \neq 0 \ , \tag{8.27}$$

as we discussed earlier. Then, the vacuum is non-supersymmetric with energy $V_0 = |f|^2$.

O'Raifeartaigh model. One of the first such models, historically speaking, was the O'Raifeartaigh model, which has a three chiral fields and a superpotential:

$$W_{\mathrm{O'R}} = \alpha Y + \beta Y X^2 + \gamma X Z \ , \tag{8.28}$$

which has no supersymmetric vacua. At the minimum of the superpotential, we have:

$$\bar{f}_X = 0 \ , \qquad \bar{f}_Y = \alpha \neq 0 \ , \qquad \bar{f}_Z = 0 \ . \tag{8.29}$$

We studied this one on a problem sheet. Classically, any VEV for Y is allowed in this vacuum, while $X = Z = 0$. Quantum mechanically, that "pseudo-modulus" Y is lifted at one-loop. (In the absence of supersymmetry, the moduli space is no longer protected; the one-loop corrected potential is known as the Coleman-Weinberg effective potential.)

One can cook up many models of F-term supersymmetry breaking, although it is a surprisingly hard thing to do. The essential mechanism is always similar.

One general comment one can make about these models is that they often look *fined-tuned.* For instance, in the O'Raifeartaigh model, there are many small deformations of the superpotential that would lead to *supersymmetry restoration.* In particular, adding a mass term for Z:

$$W = W_{\mathrm{O'R}} + m Z^2 \ , \tag{8.30}$$

we can now find supersymmetric vacua (two of them). However, this particular deformation breaks the R-symmetry of the model (which was $r = 0, 2, 2$ for X, Y, Z). It is a folk-theorem that you need an R-symmetry in order to have F-term supersymmetry-breaking.

8.6 D-term supersymmetry breaking and Fayet-Iliopoulos model

In general, the F-terms are the only genuine source of supersymmetry breaking. By gauge invariance, setting $\bar{f}_i = 0$ implies that one can always set:

$$d_a = \bar{\phi} T_a \phi = 0 \ , \tag{8.31}$$

too. Therefore it seems that we cannot have "genuine" D-term breaking with $\bar{f}_i = 0$ and $d_a \neq 0$.

8.6.1 The Fayet-Iliopoulos term

There is an interesting exception, however, when (part of) the gauge group is abelian. For each $U(1) \subset G$, we can add another supersymmetric term in the Lagrangian, which is simply:

$$\mathscr{L}_{\rm FI} = 2 \int d^2\theta d^2\bar{\theta}\, \xi\, V_{U(1)} = \xi D \;, \tag{8.32}$$

with $\xi \in \mathbb{R}$ a real coupling of mass dimension 2, called an *Fayet-Iliopoulos (FI) parameters*. This FI term is obviously gauge and supersymmetry invariant, and can only be written for an abelian vector multiplet.

The FI term corrects the D-term of an abelian theory is an important way. Consider a $G = U(1)$ theory with n chiral multiplets of charges q_i, such that:

$$\sum_i q_i^3 = 0 \;, \qquad \sum_i q_i = 0 \;, \tag{8.33}$$

to cancel the gauge anomaly. The D-term equations are now:

$$\frac{1}{g^2} D_a = \sum_i q_i |\phi_i|^2 - \xi = 0 \;. \tag{8.34}$$

(In term of the "moment map" operator defined in (7.82), we have $\mu = \xi$, and the FI parameter ξ is known mathematically as the "level" of the moment map.)

8.6.2 FI-term-induced supersymmetry breaking

The simplest "D-term supersymmetry breaking model" is a $U(1)$ theory with a non-zero FI term and two chiral multiplets $\Phi_\pm$ of charge ± 1 and a Dirac mass term:

$$W = m \Phi_+ \Phi_- \;. \tag{8.35}$$

In other words, this is a supersymmetric version of QED with a massive electron. We have the F-term and D-term conditions:

$$m\phi_+ = m\phi_- = 0 \;, \qquad |\phi_+|^2 - |\phi_-|^2 - \xi = 0 \;. \tag{8.36}$$

Obviously this system has not solution. The scalar potential has a minimum at $\phi_\pm = 0$, with:

$$\langle V_0 \rangle = g^2 \xi^2 > 0 \;. \tag{8.37}$$

8.6.3 Further comments

This conclude our brief introduction to spontaneous supersymmetry breaking (SSB). Note that the above models of SSB are essentially semi-classical: one finds semi-classical vacua, and then one can study the perturbative corrections around them.

It is also possible to have supersymmetry breaking in a non-perturbative regime. This is particularly true in asymptotically-free gauge theories, which are strongly coupled in the infrared (IR). Then, supersymmetry can sometimes be broken spontaneously due to non-trivial superpotential terms that are generated, by strong quantum effect, in the IR. Supersymmetric quantum field theories that realise this scenario are known as 'dynamical supersymmetry breaking' (DSB) models [18].

9 Supersymmetry and the Standard Model

Supersymmetry has long been the leading contender for Beyond the Standard Model (BSM) physics, although so far there has been no hint of it from the LHC.

In some narrow sense, BSM supersymmetry refers to any theory of Particle Physics generalising the standard model at high-enough energy, whose particle spectrum would include the superpartners of the known fundamental particles:

quarks	$\to$	sqarks (scalars)
leptons $(e^-, \mu^-, \tau^-, \nu)$	$\to$	sleptons (scalars)
$SU(3)$ gauge bosons (gluons)	$\to$	gluinos (fermions)
$SU(2) \times U(1)$ gauge bosons $W^3, W^{\pm}, B_\mu$	$\to$	winos and bino (fermions)
Higgs boson(s)	$\to$	Higgsinos (fermions)

If the superpartners are heavy enough, they might have avoided detection to this day. On the other hand, the general expectation from "naturalness" was that the superpartners should appear at the electroweak (EW) scale, to stabilise the Higgs mass.

In the following, we discuss some elementary aspects of the supersymmetric "completion" of the Standard Model. One important point is that the corresponding "minimally supersymmetric Standard Model" (MSSM) is not a supersymmetric theory with spontaneous supersymmetry breaking. Indeed, that possibility is ruled out by the mass sum rule (8.25), which would imply that at least some of the superpartners are rather light, and should have been detected.

Instead, the MSSM is a supersymmetric Lagrangian complemented by terms that break supersymmetry *explicitly*. These terms, called "soft terms," are chosen so that the theory still protects the mass of the scalars from quadratic divergences. (We say that the UV behaviour remains "soft.")

9.1 The Standard Model (lightning review)

Let us first review the Standard Model itself. It is a gauge theory based on the gauge group:

$$G = SU(3) \times SU(2) \times U(1)_Y \ . \tag{9.1}$$

The $SU(3)$ gives the "colored" interactions of the strong force (mediated by 8 gauge bosons, the gluons). The $SU(2) \times U(1)_Y$ gauge group governs the electroweak force

	$SU(3)$	$SU(2)$	$U(1)_Y$	$U(1)_{\text{EM}}$	$U(1)_L$	$U(1)_B$
$q_i = (u_i, d_i)$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$	$(\frac{2}{3}, -\frac{1}{3})$	0	$\frac{1}{3}$
$\widetilde{u}^i$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-\frac{2}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$
$\widetilde{d}^i$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
$\hat{\ell}_i = (\nu_i, \ell_i)$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$(0, -1)$	1	0
$\widetilde{\nu}^i$	$\mathbf{1}$	$\mathbf{1}$	0	0	-1	0
$\widetilde{\ell}^i$	$\mathbf{1}$	$\mathbf{1}$	1	1	-1	0
$\Phi = (\Phi^+, \Phi^0)$	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$	$(1, 0)$	0	0

Table 3: The matter content of the SM, including the hypothetical gauge-singlet "right-handed" neutrinos $\widetilde{\nu}^i$, and the Higgs scalar Φ. Here, all the fermions are given as left-chiral Weyl spinors ψ_α; their anti-particles are the right-chiral spinors $\bar{\psi}_{\dot{\alpha}}$ of opposite charges. The "flavor" index i runs over the SM "generations," $i = 1, 2, 3$. The last two columns give the lepton and baryon numbers, respectively.

(mediated by 3+1 gauge bosons, $W_\mu^\pm, W_\mu^3$ for $SU(2)$ and B_μ for $U(1)_Y$). The charge Y of the abelian factor $U(1)_Y$ is called the weak hypercharge. [35] The $U(1)$ of electromagnetism is obtained after electroweak symmetry breaking:

$$SU(2) \times U(1)_Y \to U(1)_{\text{EM}} \; , \tag{9.2}$$

with the electric charge given by:

$$Q_{\text{EM}} = T^3_{SU(2)} + Y \; . \tag{9.3}$$

The matter content of the standard model is summarised in Table 3, in two-component Weyl spinor notation. Note that we give the fermionic content in Weyl notation. The corresponding Dirac fermions, in the more usual four-component spinor notation, are:

$$U_i = \begin{pmatrix} u_i \\ \bar{\widetilde{u}}^i \end{pmatrix} \; , \qquad D_i = \begin{pmatrix} d_i \\ \bar{\widetilde{d}}^i \end{pmatrix} \; , \qquad L_i = \begin{pmatrix} \ell_i \\ \bar{\widetilde{\ell}}^i \end{pmatrix} \; , \tag{9.4}$$

for the up and down quarks (that is the name for the first generation, $i = 1$; for $i = 2, 3$, they are called the charm and strange quarks, and top and bottom quarks, respectively), and for the charged leptons (electron, muon and tau). Note that the Dirac fermions (9.4) transform covariantly under $SU(3) \times U(1)_{\text{EM}}$ but not under the full SM gauge group—in other words, the SM is a *chiral* theory.

In Table 3, we also indicated the lepton and baryon numbers, which are $U(1)$ symmetries of the SM Lagrangian.

[35] Conventionally, it is given in units of $\frac{1}{6}$, namely we have $Y[\psi] \in \frac{1}{6}\mathbb{Z}$ for every fermion.

Note: The following discussion assumes some working knowledge of anomalies in QFT, but you can ignore that part of the discussion on a first reading. Further background on anomalies will be provided in the 'Advanced supersymmetry' lectures.

Anomaly cancellation. The general anomaly-free condition has to be satisfied, namely:

$$\mathcal{A}_{abc} = \mathrm{tr}\left(T_a^{(\mathfrak{R}_\psi)}\{T_b^{(\mathfrak{R}_\psi)}, T_c^{(\mathfrak{R}_\psi)}\}\right) = 0 \ , \tag{9.5}$$

for $G = SU(3) \times SU(2) \times U(1)_Y$, where the trace is over all the fermions in the (reducible) representation $\mathfrak{R}_\psi$ of G. It is easy to check that all the gauge anomalies cancel, as needed for consistency. For each generation, we have:

$$\begin{aligned}
&\mathrm{tr}(SU(3)^3) = 2 - 2 = 0 \ , \\
&\mathrm{tr}(SU(3)^2 U(1)_Y) = 2\frac{1}{6} - \frac{2}{3} + \frac{1}{3} = 0 \ , \\
&\mathrm{tr}(SU(2)^2 U(1)_Y) = 3\frac{1}{6} - \frac{1}{2} = 0 \ , \\
&\mathrm{tr}(U(1)_Y^3) = 6(\frac{1}{6})^3 + 3(-\frac{2}{3})^3 + 3(\frac{1}{3})^3 + 2(-\frac{1}{2})^3 + 1^3 = 0 \ ,
\end{aligned} \tag{9.6}$$

and similarly $\mathrm{tr}(U(1)_Y) = 0$ for the mixed $U(1)_Y$-gravitational anomaly.

Chiral anomalies. The baryons and lepton number symmetries are separately anomalous, with the chiral anomalies:

$$\begin{aligned}
&\mathrm{tr}(SU(2)^2 U(1)_L) = \mathrm{tr}(SU(2)^2 U(1)_B) = 1 \ , \\
&\mathrm{tr}(U(1)_Y^2 U(1)_L) = \mathrm{tr}(U(1)_Y^2 U(1)_B) = -\frac{1}{2} \ ,
\end{aligned} \tag{9.7}$$

but the difference:

$$L - B \tag{9.8}$$

is non-anomalous, and therefore an exact symmetry of the Standard Model.

The Higgs sector. The Higgs field Φ is a scalar doublet of the $SU(2)$ gauge group:

$$\Phi \equiv \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} \ , \tag{9.9}$$

which acquires a VEV:

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \ , \qquad v \in \mathbb{R} \ , \tag{9.10}$$

breaking the electroweak gauge group as in (9.2). The SM vacuum therefore sees one real scalar excitation, the Higgs boson $h_{\rm SM}$, which appears as:

$$\Phi^+ = G^+ \ , \qquad \bar{\Phi}^+ = G^- \ , \qquad \Phi^0 = v + \frac{1}{\sqrt{2}}(h_{\rm SM} + iG^0) \ . \tag{9.11}$$

The other three real scalars $G^0, G^{\pm}$, corresponding to the three broken $SU(2)\times U(1)$ generators, become part of the massive vector bosons $W^{\pm}$ and Z^0.

Most of the Standard Model Lagrangian is fully determined by gauge invariance. The crucial exception is the Higgs sector. Firstly, we do not know for sure what is the self-coupling of the Higgs; one usually assume a simple potential that leads to the Higgs VEV (9.10).

More importantly for our purpose, the coupling of the Higgs field to fermions arise from Yukawa interactions, which are not fully determined by gauge invariance:

$$\mathscr{L}_{\text{Yukawa}} = (Y_u)^i{}_j \varepsilon_{rs} \Phi^r q_i^s \widetilde{u}^j - (Y_d)^i{}_j \bar{\Phi}_r q_i^r \widetilde{d}^j - (Y_\ell)^i{}_j \bar{\Phi}_r \hat{\ell}_i^r \widetilde{\ell}^j + \text{h.c.} \; . \tag{9.12}$$

Here, the indices $r = 1, 2$ and $s = 1, 2$ are $SU(2)$ gauge indices, and Y_u, Y_d, Y_ℓ are the Yukawa coupling constants, which must be determined experimentally. After electroweak symmetry breaking, these terms provide the Dirac masses $M = vY$ to the quarks and leptons (9.4).

9.2 The supersymmetric SM

Let us now consider the *supersymmetric* version of the Standard Model. None of the fermions (quarks and leptons) and bosons (gauge fields and Higgs) of the SM can be paired by supersymmetry, due to the gauge representations. Instead, we need to introduce superpartners for every known particles—whose conventional names were given at the beginning of this section.

The supersymmetric SM consists of $SU(3)\times SU(2)\times U(1)$ multiplets multiplets, which contains fermions (the *gauginos*) in the corresponding adjoint representations. All the left-chiral fermions of Table 3 also become part of chiral multiplets, denoted by:

$$\begin{aligned} &\mathcal{Q}_i = (Q_i, q_i) \;, \qquad && \widetilde{\mathcal{U}}^i = (\widetilde{U}^i, \widetilde{u}^i) \;, \qquad && \widetilde{\mathcal{U}}^i = (U^i, \widetilde{u}^i) \;, \\ &\hat{\mathcal{L}}_i = (\hat{L}_i, \hat{\ell}_i) \;, && \widetilde{\mathcal{N}}^i = (\widetilde{N}^i, \widetilde{\nu}^i) \;, && \widetilde{\mathcal{L}}^i = (\widetilde{L}^i, \widetilde{\ell}^i) \;, \end{aligned} \tag{9.13}$$

with the curly letter denoting the chiral multiplets (or superfields), the capital letters denoting the complex scalar superpartners, and the lowercase letters denoting the left-chiral fermions.

Finally, any supersymmetric version of the SM must have *two* Higgs fields distinct Higgs fields. This is apparent, firstly, from the SM Yukawa Lagrangian (9.12), which cannot arise from a superpotential term, since it contains both the scalar ϕ and its complex conjugate $\bar{\phi}$ coupling to bilinears in the left-chiral fermions. Instead, in a supersymmetric version of (9.12), we need at least two Higgs doublets to give their masses to the up and down quarks separately.

Another reason we need at least two Higgs doublets (and, in fact, an even number of doublets) is because of anomaly cancellations. The superpartner of the Higgs field Φ is a fermion in the $\mathbf{2}_{\frac{1}{2}}$ of $SU(2)\times U(1)_Y$, which has a gauge anomaly. We need a second Higgs doublet with charges $\mathbf{2}_{-\frac{1}{2}}$ in order to have an anomaly-free supersymmetric version of the SM. (The gauginos do not introduce additional

	$SU(3)$	$SU(2)$	$U(1)_Y$	$U(1)_{\text{EM}}$	$U(1)_L$	$U(1)_B$	Π_R
$\mathcal{Q}_i = (\mathcal{U}_i, \mathcal{D}_i)$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$	$(\frac{2}{3}, -\frac{1}{3})$	0	$\frac{1}{3}$	-1
$\widetilde{\mathcal{U}}^i$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-\frac{2}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	-1
$\widetilde{\mathcal{D}}^i$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	-1
$\hat{\mathcal{L}}_i = (\mathcal{N}_i, \mathcal{L}_i)$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$(0, -1)$	1	0	-1
$\widetilde{\mathcal{N}}^i$	$\mathbf{1}$	$\mathbf{1}$	0	0	-1	0	-1
$\widetilde{\mathcal{L}}^i$	$\mathbf{1}$	$\mathbf{1}$	1	1	-1	0	-1
$\Phi_1 = (\Phi_1^0, \Phi_1^-)$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$(0, -1)$	0	0	1
$\Phi_2 = (\Phi_2^+, \Phi_2^0)$	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$	$(1, 0)$	0	0	1

Table 4: Chiral superfields in the supersymmetric SM. We denote by Φ both the Higgs superfields and the Higgs scalar components. The last column denotes the R-parity of the chiral multiplet.

gauge anomalies because they are in adjoint representations—in particular, they have zero weak hypercharge.)

The chiral multiplet content of the supersymmetric SM is summarised in Table 4. The Higgs sector interactions follow from the following gauge-invariant superpotential:

$$\boxed{W = -(y_u)^i{}_j \Phi_2 \mathcal{Q}_i \widetilde{\mathcal{U}}^j + (y_d)^i{}_j \Phi_1 \mathcal{Q}_i \widetilde{\mathcal{D}}^j + (y_\ell)^i{}_j \Phi_1 \hat{\mathcal{L}}_i \widetilde{\mathcal{L}}^j \ ,} \tag{9.14}$$

with the Yukawa coupling constants y corresponding to Y in (9.12). Here, we used the shorthand notation:

$$\Phi X = \varepsilon_{rs} \Phi^r X^s = \Phi^2 X^1 - \Phi^1 X^2 \ , \tag{9.15}$$

for the contraction of two $SU(2)$ doublets. (The contraction of color indices is left implicit.) We thus have:

$$\begin{aligned} W = \quad & - (y_u)^i{}_j (\Phi_2^0 \mathcal{U}_i - \Phi_2^+ \mathcal{D}_i) \widetilde{\mathcal{U}}^j - (y_d)^i{}_j (\Phi_1^0 \mathcal{U}_i - \Phi_1^- \mathcal{U}_i) \widetilde{\mathcal{D}}^j \\ & - (y_\ell)^i{}_j (\Phi_1^0 \mathcal{L}_i - \Phi_1^- \mathcal{N}_i) \widetilde{\mathcal{L}}^j \ . \end{aligned} \tag{9.16}$$

We can obtain Dirac masses for the quarks and leptons from the VEVs of Φ_1^0 and Φ_2^0, namely:

$$\langle \Phi_1 \rangle = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \ , \qquad \langle \Phi_2 \rangle = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \ . \tag{9.17}$$

In phenomenological studies of the MSSM, one often encounters the ratio:

$$\tan \beta \equiv \frac{v_2}{v_1} \ . \tag{9.18}$$

Another F-term that one can write down, which preserves both baryons and fermion number, is a mass term for the Higgs field:

$$W_\mu = \mu\Phi_1\Phi_2 = \mu(\Phi_1^-\Phi_2^+ - \Phi_1^0\Phi_2^0) \ , \tag{9.19}$$

which is known as the "μ-term." By itself, this term would forbid the VEV (9.17); however, it turns out to be necessary for phenomenological reason. In the actual MSSM, including supersymmetry-breaking soft-terms, the electroweak symmetry is broken well below $M_{\rm SUSY}$, and the SUSY-breaking terms are essential in obtaining the EW-breaking vacuum observed in the real world. (We refer to chapter 28 of Weinberg [2] for the gory details.)

R-parity. One can also write various renormalisable superpotential terms that break the lepton and baryon numbers (and, in particular, the anomaly-free $L-B$):

$$W_{\alpha,\beta,\gamma} = \alpha^{ij}{}_k \mathcal{Q}_i \hat{\mathcal{L}}_j \widetilde{\mathcal{D}}^k + \beta^{ij}{}_k \hat{\mathcal{L}}_i \hat{\mathcal{L}}_j \widetilde{\mathcal{L}}^k + \gamma_{ijk} \widetilde{\mathcal{D}}^i \widetilde{\mathcal{D}}^k \widetilde{\mathcal{U}}^k \ . \tag{9.20}$$

The last term is a "supersymmetric baryon superfield," with the $SU(3)$ gauge indices contracted with ϵ_{abc}. These terms cannot appear in any realistic theory, since they would lead to experimentally excluded processes, such as proton decay ($p \to \pi^0 + e^+$ through exchange of squarks and sleptons), at a dramatic rate. Of course, these terms are in fact excluded if $L - B$ is an exact $U(1)$ symmetry of physics beyond the SM—the superpotential (9.20) has charge $L - B = 1$.

There are strong theoretical reasons to believe, however, that the "ultimate theory," including quantum gravity, does not have any exact continuous symmetry. In the study of supersymmetric BSM physics, a weaker assumption is usual made, that we need to preserve a $\mathbb{Z}_2$ discrete symmetry, called *R-parity* defined as:

$$\Pi_R = (-1)^F(-1)^{3(L-B)} \ , \tag{9.21}$$

with $(-1)^F$ the fermion number. This is a $\mathbb{Z}_2 \subset U(1)_R$ *discrete R symmetry* in the supersymmetric SM. The R-parity of the chiral multiplets is shown in Table 4. Note that if a scalar has R-parity ± 1, then its fermion partner has parity $\mp$. Thus, R-parity assign even parity $(+1)$ to the non-supersymmetric SM fields (including the two Higgs scalars Φ_1 and Φ_2), and odd parity (-1) to the superpartners. The superpotential terms (9.14) and (9.19) are consistent with R-parity, while (9.20) is not. [36]

An interesting consequence of R-parity, if it is indeed a good symmetry of supersymmetric BSM physics, is that the lightest supersymmetric particle (often called the "LSP") has to be stable (since it could only decay into another R-parity-odd particle). This provides a natural candidate for dark matter.

[36] As we understand it today, quantum gravity would probably not allow even exact discrete symmetry such as R-parity. Then, for our purposes, we may just declare that R-parity is conserved in the "low-energy" theory well below the quantum-gravity scale.

9.3 The MSSM: supersymmetry-breaking soft terms

What is called the *minimal supersymmetric Standard Model* (MSSM) is not actually a supersymmetric theory. It is the supersymmetric SM (SSM) corrected with "soft terms" that break supersymmetry explicitly:

$$\mathscr{L}_{\text{MSSM}} = \mathscr{L}_{\text{SSM}} + \mathscr{L}_{\text{soft}} \ . \tag{9.22}$$

The soft terms denote all the possible R-parity invariant terms that are *super-normalisable* and break supersymmetry explicitly—that is, of engineering dimension < 4. They are given by:

$$\begin{aligned}\mathscr{L}_{\text{soft}} = \quad & -(M^2_Q)^i{}_j \bar{Q}^j Q_i - (M^2_{\widetilde{U}})^j{}_i \bar{\widetilde{U}}_j \widetilde{U}^i - (M^2_{\widetilde{D}})^j{}_i \bar{\widetilde{D}}_j \widetilde{D}^i - (M^2_{\hat{L}})^i{}_j \bar{\hat{L}}^j \hat{L}_i \\ & -(M^2_{\widetilde{L}})^j{}_i \bar{\widetilde{L}}_j \widetilde{L}^i - (\lambda m_{\text{gaugino}} \lambda) + \text{h.c.} \\ & -(A_u)^i{}_j \Phi_2 Q_i \widetilde{U}^j + (A_d)^i{}_j \Phi_1 Q_i \widetilde{D}^j + (A_\ell)^i{}_j \Phi_1 \hat{L}_i \widetilde{L}^j \\ & -(C_u)^i{}_j \bar{\Phi}_1 Q_i \widetilde{U}^j + (C_d)^i{}_j \bar{\Phi}_2 Q_i \widetilde{D}^j + (C_\ell)^i{}_j \bar{\Phi}_2 \hat{L}_i \widetilde{L}^j \\ & -B\mu\, \Phi_1 \Phi_2 + \text{h.c.}\end{aligned} \tag{9.23}$$

The first two lines are explicit mass terms for the scalar superpartners and for the $SU(3) \times SU(2) \times U(1)_Y$ gauginos. The A-terms, C-terms and $B\mu$-terms introduce further interactions amongst scalars. In phenomenological studies of the MSSM, the C-terms are usually set to zero, although not for especially good reasons [2].

The MSSM soft terms (9.23) introduce over *100 new parameters*, which is not a particularly economical extension of the Standard Model. It should be thought as the minimal *low energy effective field theory* below the supersymmetry-breaking scale. The fundamental theory would hopefully look elegant; the detailed mechanism of supersymmetry breaking then determines the MSSM parameters at low energy.

In any case, the MSSM is usually the starting point for phenomenological studies—that is, to extract concrete predictions for collider experiments. Due to the large number of free parameters, one often focusses on special parameter subspace (for instance setting all the scalar masses M^2 in (9.23) equal and diagonal), for simplicity. We will not discuss the MSSM phenomenology further in these lectures.

9.4 Hidden sector and supersymmetry-breaking mediation

To conclude this brief introduction to "supersymmetry and the real world," we should give a rough idea of how the MSSM can be embedded in a more fundamental theory.

Since supersymmetry cannot be broken spontaneously in the SM itself, we need some "auxiliary" mechanism. The idea is that there exists an "hidden sector" which breaks supersymmetry spontaneously. A popular assumption is that supersymmetry is broken dynamically by strong-coupling effects, in some strongly-coupled supersymmetric gauge theory. The point is that this supersymmetry-breaking dynamics, whatever it is, must happen amongst quantum fields that are completely decoupled from the ordinary matter of the Standard Model. (This is not completely crazy. We know from astronomical observation that we do not understand most of the matter content of the Universe, anyway.)

At first approximation, we would have a "tensor product" fundamental theory at high energy:

$$\mathcal{T} = (\text{SUSY SM}) \otimes \widetilde{\mathcal{T}}_{\text{hidden}} \ , \tag{9.24}$$

with $\widetilde{\mathcal{T}}_{\text{hidden}}$ the hidden-sector QFT, which is supersymmetric but breaks supersymmetry spontaneously at a scale $\widetilde{M}_{\text{SUSY}}$.

Then, one assumes a *mediation mechanism* that "transmits" the supersymmetry-breaking effects from $\widetilde{\mathcal{T}}_{\text{hidden}}$ to the supersymmetric SM, thereby generating soft terms as in (9.23) (as well as many other non-renormalisable terms supressed by $1/\widetilde{M}_{\text{SUSY}}$ to some power).

There are two well-studied mediation mechanisms, which can be summarised as follows:

9.4.1 Gauge mediation

One possibility is that there exists (very massive) messenger superfields X which are charged under the SM gauge group and also couple to the hidden sector:

$$(\text{SUSY SM}) \quad \overset{X}{\longleftrightarrow} \quad \widetilde{\mathcal{T}}_{\text{hidden}} \ . \tag{9.25}$$

The messenger fields must be in a pseudo-real representation of the SM gauge group, so that they can obtain a very large mass M_X. (This also ensures that they do not introduce any gauge anomalies). Then, perturbative processes involving the messenger fields induces the MSSM soft terms. For instance, the gaugino masses are proportional to the SM gauge couplings, $g^2 = g_s^2, g_{\text{SU}(2)}^2, g_Y^2$:

$$m_{\text{gaugino}} \propto g^2 M_X \ , \tag{9.26}$$

while the squarks and slepton masses are proportional to:

$$M^2 \propto g^4 M_X^2 \ , \tag{9.27}$$

very schematically. This is an attractive scenario since it is very predictive, giving us many precise relations amongst the soft terms of the MSSM, which must all be proportional the SM gauge couplings.

9.4.2 Supergravity mediation

Another popular mediation mechanism is supergravity mediation, in which the only "messenger particles" are the gravitational interactions:

$$(\text{SUSY SM}) \quad \overset{\text{gravity}}{\longleftrightarrow} \quad \widetilde{\mathcal{T}}_{\text{hidden}} \ . \tag{9.28}$$

More precisely, the dominant supersymmetry-breaking "messengers" would be supersymmetric partners of the graviton, in a *supergravity* theory. Since gravity couples universally to every type of energy-momentum, supergravity mediation is always present. "Supergravity mediation" denotes the situation where this is the dominant contribution, in the absence of any gauge-mediation mechanim.

Suppose, for instance, that supersymmetry is broken by a non-zero F-term, $\langle F_X \rangle \neq 0$, in the hidden sector, so that the SUSY-breaking scale is $M_{\text{SUSY}} = \sqrt{\langle F_X \rangle}$—that is, we have some hidden-sector field X such that $X = 0$ and $F_X \neq 0$ in the supersymmetry-breaking vacuum.. Then, all the soft masses of the MSSM are proportional to:

$$m_{\text{gaugino}} \sim M \sim \frac{\langle F_X \rangle}{M_P} = \frac{M_{\text{SUSY}}^2}{M_P} \ , \tag{9.29}$$

with M_P the Planck mass. These soft terms simply arise from power-supressed higer-dimensional operators such as:

$$\int d^2\theta d^2\bar{\theta} \frac{1}{M_P^2} \bar{X} X \bar{\mathcal{Q}} \mathcal{Q} + \cdots \ , \tag{9.30}$$

which would arise at scales $\mu < M_P$, in an effective field theory approach, from "integrating out" the gravitational interactions.

9.5 Concluding comments

There has been a huge amount of literature on supersymmetry-breaking mediation mechanisms. The general motivation was to provide theoretically well-motivated predictions for the physics 'beyond the Standard Model' (BSM), if BSM physics is supersymmetric. In the absence of any hint of TeV-scale BSM physics from the LHC at CERN, however, this kind of theoretical work has slowed down significantly in recent years.

Note also that, from a phenomenological (*i.e.* experiment-based) perspective, one can remain agnostic about any 'mediation mechanism' and just deal with the MSSM, a non-supersymmetry theory, as a possible scenario for physics at the TeV scale—a scenario with many free parameters, which must be fitted experimentally. This is what the physicists working on the LHC data are doing, among other things: they are slowly putting (more and more) stringent constraints on the parameters of the MSSM.

To conclude with a note of cautious optimism, we should also say that the fact that supersymmetry is not apparent in accelerator experiments *today* does not mean it is not a symmetry of Nature at even higher energies. Of course that is one possibility: supersymmetry is a beautiful theoretical construct without relevance to the real world. The second possibility is simply that humans in the XXIst century do not have the tools to probe experimentally the relevant (astonishingly small) length scales. Certainly, String Theory, as understood today, suggests that supersymmetry is needed to formulate quantum gravity; yet, the scale of supersymmetry breaking could then be more naturally tied to the string length, which may be closer to the Planck length (10^{19} GeV) than to the TeV (10^3 GeV) scale.

10 A brief introduction to supergravity

Supergravity refers to any classical theory that combines General Relativity (GR) and supersymmetry. We insist on the fact that supergravity theories are *classical.* The difficulties in "quantising" supergravity theories are as severe as in GR.

Nonetheless, if supersymmetry is part of the physical world, it has to be combined with gravity, and supergravity then provides something like the "tree level" approximation for the coupling of supersymmetric QFTs to gravity.

The other main motivation to study supergravity is that it appears as the low-energy effective field-theory description of String Theory, which is believed to be a consistent theory of Quantum Gravity.

10.1 Gauging the supercurrent: 4d $\mathcal{N} = 1$ linearised supergravity

It is sometimes useful to think of General Relativity (GR) as a "gauge theory" for Poincaré invariance. In this approach, we start with a Poincaré-invariant field theory on $\mathbb{R}^{1,d-1}$, which then admits an energy-momentum tensor, which we can choose to be symmetric:

$$T_{\mu\nu} = T_{\nu\mu} \ . \tag{10.1}$$

This $T_{\mu\nu}$ encodes all the conserved currents for Poincaré invariance, $ISO(1, d-1)$, in flat space. Then, naively, we could attempt to "gauge" $ISO(1, d-1)$ by allowing the Poincaré transformations to depend on the space-time point. This correspond to adding a "gauge field" for $ISO(1, d-1)$, denoted by $\Delta g_{\mu\nu}$, which couples to the energy-momentum tensor as:

$$\mathscr{L} = \Delta g_{\mu\nu} T^{\mu\nu} \ , \tag{10.2}$$

at first order in $\Delta g_{\mu\nu}$. This is really equivalent to considering a non-trivial *metric* on space-time:

$$g_{\mu\nu} = \eta_{\mu\nu} + \Delta g_{\mu\nu} \ . \tag{10.3}$$

Of course, in GR, the energy-momentum tensor can be *defined* as the reaction of the system to a variation of the metric. The "gauge group" of GR correspond to the diffeomorphisms of the space-time manifold. At the linearised level, differomorphisms act on the metric as:

$$\delta_\xi g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ , \tag{10.4}$$

for some arbitrary covector ξ_μ.

Then, as in the notion of "gauging" in QFT, we should also introduce a kinetic term for the metric. This is provided by the Einstein-Hilbert action, which is essentially fixed by the requirement of diff invariance.

10.1.1 Linearised old-minimal supergravity

Consider now 4d $\mathcal{N} = 1$ supersymmetry. We have seen in section 8.1 that $T_{\mu\nu}$ sits together with the supercurrent in a supermultiplet (known as the FZ multiplet):

$$\mathcal{J}_\mu = \left(j_\mu \ , \ S^\mu_\alpha \ , \ \bar{S}^\mu_{\dot{\alpha}} \ , T_{\mu\nu}, X \ , \ \bar{X}\right) \ . \tag{10.5}$$

Then, as in GR, we would like to understand *supergravity* as a gauging of super-Poincaré invariance. The fields that couple to the supercurrent operators (10.5) are:

$$\mathcal{H}_\mu = \left(b_\mu \ , \ \Psi^\mu_\alpha \ , \ \bar{\Psi}^\mu_{\dot{\alpha}} \ , g_{\mu\nu}, M \ , \ \bar{M}\right) \ . \tag{10.6}$$

This is known as the "old minimal supergravity multiplet" [28, 29, 30], for historical reasons. (There exists another $\mathcal{N} = 1$ supregravity theory called "new minimal," which couples to a slightly different current multiplet.)

At first order in the supergravity fields, we can understand the supersymmetric coupling between (10.5) and (10.6) in ordinary superspace. Consider the supercurrent superfield $\mathcal{J}_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}\mathcal{J}_\mu$. The first-order coupling to "sources" must be of the form:

$$\mathscr{L}_{\mathcal{H}\mathcal{J}} = \int d^2\theta d^2\bar{\theta}\, \mathcal{H}_{\alpha\dot{\alpha}}\mathcal{J}^{\alpha\dot{\alpha}} \ . \tag{10.7}$$

Due to the superspace definition (8.9) of the FZ multiplet, we have a *gauge invariance*:

$$\mathcal{H}_{\alpha\dot{\alpha}} \to \mathcal{H}_{\alpha\dot{\alpha}} + \mathrm{D}_\alpha \bar{L}_{\dot{\alpha}} + \bar{\mathrm{D}}_{\dot{\alpha}} L_\alpha \ . \tag{10.8}$$

with the superfield L_α satisfying the constraint:

$$\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{D}^\alpha L_\alpha - \mathrm{D}\mathrm{D}\bar{\mathrm{D}}_{\dot{\alpha}}\bar{L}^{\dot{\alpha}} = 0 \ . \tag{10.9}$$

Using the gauge freedom (10.8), one can fix a WZ-type gauge for the (linearised) supergravity multiplet $\mathcal{H}_\mu$, so that only the components (10.6) survive. After fixing the WZ gauge, there are still *residual gauge transformations*, which include *linearized diffeomorphism* and a spinor-valued gauge transformation for the gravitino:

$$\delta_L g_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \ , \qquad \delta_L\Psi_{\mu\alpha} = \partial_\mu\eta_\alpha \ , \cdots \tag{10.10}$$

In the WZ gauge, the Lagrangian (10.8) is simply:

$$\mathscr{L}_{\mathcal{H}\mathcal{J}} = \Delta g_{\mu\nu}T^{\mu\nu} + \Psi_\mu S^\mu + \bar{\Psi}_\mu\bar{S}^\mu + b_\mu j^\mu + MX + \bar{M}\bar{X} \ . \tag{10.11}$$

We see that the *gravitino* $\Psi_{\mu\alpha}$ couples to the supercurrent, as expected. Note that the gravitino variation in (10.10) is dual to the conservation of the supercurrent:

$$\delta_L\left(\int d^4x \ \Psi_\mu S^\mu\right) = \int d^4x \ \partial_\mu\eta\, S^\mu = -\int d^4x \ \epsilon\, \partial_\mu S^\mu = 0 \ . \tag{10.12}$$

10.2 Full $\mathcal{N} = 1$ supergravity: general strategy

The supergravity fields b_μ and X in (10.6) turn out to be *auxiliary* in old-minimal supergravity (similarly to the auxiliary field D in the vector multiplet of an $\mathcal{N} = 1$ supersymmetric gauge theory). Their presence allows us to write down *off-shell* supergravity transformations. This is very useful, since it allows to consider $\mathcal{N} = 1$ supergravity theories in two-steps, similarly to what we did for rigid supersymmetry:

1. Write down off-shell supergravity transformations, which generalise the supersymmetry transformation to include diff invariance and local supersymmetry transformations, wherein the supersymmetry-transformation parameters ϵ become functions of space-time, $\epsilon(x)$. In particular, we have the gravitino variation:
$$\delta_\epsilon \Psi_\mu = \mathscr{D}_\mu \epsilon \ , \tag{10.13}$$
with $\mathscr{D}_\mu$ a supergravity-covariant derivative, which itself depends on Ψ. [37] Note that, in the linearized limit around flat space, the gravitino gauge-transformation parameter η in (10.10) is independent of the constant spinor ϵ of flat-space supersymmetry. Once we go to the non-linear theory and make supersymmetry into a *local* (gauge) transformation, however, one has to identify $\epsilon(x) = \eta(x)$.

2. Write down supergravity-invariant actions. In particular, the kinetic term for the supergravity fields is often called calls 'the' supergravity action, S_{SUGRA}. This action should generalise the Einstein-Hilbert action of GR to a diff-invariant and locally-supersymmetric action:
$$S_{\text{SUGRA}} = \frac{1}{8\pi G} \int d^4x \sqrt{g} \left(R + \bar{\Psi} \not{D} \Psi + \cdots \right) \ , \tag{10.14}$$
very schematically, with G denoting Newton's constant. Thus, we also need to understand better the gravitino kinetic term, $\bar{\Psi} \not{D} \Psi$.

There are no real conceptual problems in carrying out these two steps in detail for old-minimal 4d $\mathcal{N} = 1$ supergravity (or any other supergravity theory with an off-shell formulation). It just gets rather technical.

In the rest of this section, we give some more details about the minimal supersymmetry action and its off-shell presentation. We will focus, in particular, on the truly new ingredient in supergravity, compared to either GR or flat-space QFT: this is the presence of the gravitino Ψ_μ, which should be understood as the "gauge field" for local supersymmetry transformations.

[37] This is similar to the gauge transformation $\delta_\alpha A_\mu = D_\mu \alpha$ for a non-abelian gauge field.

10.3 Veirbein, spin connection, and curved-space spinors

To discuss supergravity, we need to first set our notation for pseudo-Riemannian geometry. Let us consider the pseudo-Riemannian manifold $\mathcal{M}$ with local coordinates x^μ, and the metric $g_{\mu\nu}$ with signature $(-1,1,1,1)$:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \ . \tag{10.15}$$

In curved space, classical scalar fields ϕ are functions $\mathcal{M} \to \mathbb{C}$ (or $\mathcal{M} \to \mathbb{R}$, for a real scalar). We can also consider various tensors, such as, for instance, the abelian field-strength $F_{\mu\nu}$, which transforms covariantly under diffeomorphisms (in other words, it is a 2-form: $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$).

Recall that a vector $X^\mu(x)$ is, by definition, a vector in the tangent space $T_x\mathcal{M}$ at x. The vector field $X = X^\mu\partial_\mu$ is then a section of the tangent bundle $T\mathcal{M}$, which we denote by: [38]

$$X \in \Gamma(T\mathcal{M}) \ . \tag{10.16}$$

Similarly, a one-form $\omega = \omega_\mu dx^\mu$ is a section of the cotangent bundle, $T^*\mathcal{M}$. In general, we may consider any tensor $T^{\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_m}$, namely:

$$T \in \Gamma(\underbrace{T\mathcal{M} \otimes \cdots \otimes T\mathcal{M}}_{n \text{ times}} \otimes \underbrace{T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M}}_{m \text{ times}}) \ . \tag{10.17}$$

Therefore, given any classical field theory in $\mathbb{R}^{1,3}$ that contains only *bosons*—more precisely, fields of integer spin or helicity. [39]—, we can easily define the theory coupled to gravity by writing all the fields as appropriate tensors, as we learned in elementary GR.

Fermions—or rather, any fields of *half-integer spin*—are a bit more subtle in curved space-time. This is because spinors are *not* ordinary tensors. This is related to the fact they they transform in representations of the spin group $Spin(1,3) \cong Sl(2,\mathbb{C})$ that are not representations of $SO(1,3)$—the representations 'of half-integer spin'. Mathematically, we say that a Dirac spinor $\Psi = (\psi, \bar\chi)$ is a section of a spin bundle:

$$\Psi \in \Gamma(S_+ \oplus S_-) \ . \tag{10.18}$$

Here, $S_\pm$ are two-dimensional complex vector bundles corresponding to the left-chiral and right-chiral Weyl spinors, respectively—that is, $S_\pm$ is the data of a vector space $V = \mathbb{C}^2$ to every point $x \in \mathcal{M}$, with the fields $(\psi_\alpha) \in \Gamma(S_+)$ and $(\bar\chi^{\dot\alpha}) \in \Gamma(S_+)$, and transition functions between coordinate patches that are precisely $Sl(2,\mathbb{C})$ transformations, acting just like the flat-space spin-$\frac{1}{2}$ Lorentz transformations on spinors.

[38] A 'section' of a bundle over $\mathcal{M}$ is essentially a 'nice' continuous choice of vector in $T_x\mathcal{M}$ over every point x. We will not review differential geometry here. Let us just say that the tangent bundle is a special case of the concept of 'vector bundle', which assigns a vector space to every point on x.

[39] In a unitary relativistic QFT, they are then boson, by the spin-statistic theorem.

10.3.1 Veirbein

To deal with spinors in GR, it is most convenient to introduce an auxiliary construction, called a 'vierbein' (in four dimensions). This is a choice of an orthonormal frame at every point in $\mathcal{M}$—that is, we pick four distinct co-vectors $e^a \in \Gamma(T^*\mathcal{M})$, with $a = 0, \cdots, 3$, such that:

$$e^a e^b \eta_{ab} = g_{\mu\nu} dx^\mu dx^\nu \qquad \leftrightarrow \qquad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab}\, e^a e^b \;, \tag{10.19}$$

where we sum over all repeated indices. Here, $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Note that the veirbein e^a_μ is a sort of 'square root' of the metric tensor:

$$g_{\mu\nu} = \eta_{ab}\, e^a_\mu e^b_\nu \;. \tag{10.20}$$

Since the metric is invertible, we also have the inverse veirbein, denoted by e^a_μ. All frame indices are raised and lowered with the flat metric η_{ab}. Here and in the following, $\mu, \nu, \cdots$ denote the 'curved' local coordinate indices, and $a, b, \cdots$ denote the 'flat' frame indices. We can write any tensor on $\mathcal{M}$ in terms of coordinate indices—and vice-versa—, using the veirbein and its inverse:

$$X^a = e^a_\mu X^\mu \;, \qquad \omega_a = e^\mu_a \omega_\mu \;, \qquad \text{etc.} \tag{10.21}$$

Given a choice of frame, we can now write spinors as if we were in flat space—indeed, we are always *locally* in flat space, by the equivalence principle. In particular, we define the γ^a matrices as before, and the corresponding σ^a, $\bar\sigma^a$ matrices as in section 2.2. The only difference is that, in curved space, we must distinguish between flat and curved indices. For instance, while the matrices $(\sigma^a) = (\sigma^0, \sigma^i)$ are defined as before (they are the constant matrices (2.26)), the matrices σ^μ would denote:

$$\sigma^\mu \equiv e^\mu_a \sigma^a \;, \tag{10.22}$$

which do depend non-trivially on the coordinates x^μ, through the veirbein. In particular, for the 4d γ-matrices, we have:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \qquad \leftrightarrow \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \;. \tag{10.23}$$

Now, the spinors ψ_α and $\bar\eta^{\dot\alpha}$ are defined as in flat space, using the orthonormal frame basis. They transforms in the appropriate $Sl(2, \mathbb{C})$ 'Lorentz' representations according to:

$$\psi_\alpha \to (\mathbf{M}_{ab}\psi)_\alpha = -i(\sigma_{ab})_\alpha{}^\beta \psi_\beta \;, \qquad \bar\eta^{\dot\alpha} \to (\mathbf{M}_{ab}\bar\psi)^{\dot\alpha} = -i(\bar\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\bar\eta^{\dot\beta} \;. \tag{10.24}$$

Beware of the notation: In this section, the indices $a, b, \cdots$ are frame indices, not spinor indices. We will only deal with 2-component spinors in the following, with Weyl-spinor indices α, $\dot\alpha$.

10.3.2 Covariant derivatives, spin connection and curvature

Consider any field φ, which may carry any combination of frame, coordinate and spinor indices. To compare the field at two different points, $\varphi(x)$ and $\varphi(x+\Delta x)$, we need a notion of parallel transport, just like for the vectors.

Recall that, for vectors and covectors, we have a particularly 'nice' connection which is torsion-free, the Levi-Civita connection, denoted by $\Gamma^\rho_{\mu\nu}$, which can be written directly in terms of the first derivatives of the metric tensor:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{10.25}$$

Then, the covariant derivatives are given by:

$$\nabla_\mu X^\nu = \partial_\mu X^\nu - \Gamma^\nu_{\mu\rho} X^\rho \ , \qquad \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \Gamma^\rho_{\mu\nu}\omega_\rho \ , \tag{10.26}$$

and similarly for any tensor. Let us assume for a moment that our general field φ only has frame and spinor indices—this can always be done by absorbing all coordinate indices with the veirbein, as in (10.21). This means that φ transforms in some (generally reducible) representation $\mathfrak{R}$ of the Lorentz group. Then, the appropriate covariant derivative takes the form:

$$\nabla_\mu \varphi = \partial_\mu \varphi - \frac{i}{2}\omega_\mu{}^{ab}\mathbf{M}^{(\mathfrak{R})}_{ab}\varphi \ , \tag{10.27}$$

where $\omega_{\mu ab}$ is the so-called *spin connection*, and $\mathbf{M}^{(\mathfrak{R})}_{ab}$ is the Lorentz matrix in the representation $\mathfrak{R}$. In particular, the covariant derivatives on Weyl spinors are given by:

$$\begin{aligned} \nabla_\mu \psi &= (\partial_\mu - \frac{1}{2}\omega_{\mu ab}\sigma^{ab})\psi \ , \\ \nabla_\mu \bar{\eta} &= (\partial_\mu - \frac{1}{2}\omega_{\mu ab}\bar{\sigma}^{ab})\bar{\eta} \ . \end{aligned} \tag{10.28}$$

By comparing (10.27) with $\varphi = \omega$ with (10.26), one can show that the spin connection is related to the Levi-Civita connection by:

$$\omega_{\mu a}{}^b = e^\nu_a e^b_\rho \Gamma^\rho_{\mu\nu} - e^b_\nu \partial_\mu e^\nu_a \ . \tag{10.29}$$

This is equivalent to the statement that the covariant derivative of the veirbein vanishes—that is:

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \Gamma^\rho_{\mu\nu} e^a_\rho - \omega_{\mu b}{}^a e^a_\nu = 0 \ . \tag{10.30}$$

Finally, the curvature tensor of the spin connection is given by:

$$R_{\mu\nu a}{}^b(\omega) = \partial_\mu \omega_{\nu a}{}^b - \partial_\nu \omega_{\mu a}{}^b + \omega_{\nu a}{}^c \omega_{\mu c}{}^b - \omega_{\mu a}{}^c \omega_{\nu c}{}^b \ . \tag{10.31}$$

This is simply the ordinary Riemann tensor with two frame indices:

$$R_{\mu\nu a}{}^b = e^\rho_a e^b_\lambda R_{\mu\nu\rho\lambda} \ . \tag{10.32}$$

Finally, the Ricci scalar is defined as:

$$R = e^a_\mu e^\nu_b R^\mu{}_{\nu a}{}^b(\omega) \ . \tag{10.33}$$

10.4 Gravitino and the Rarita-Schwinger equation

The gavitino field, $\Psi_{\alpha\mu}, \bar{\Psi}^{\dot{\alpha}}_{\mu}$, is a massless spin-$\frac{3}{2}$ field, and we need to understand better how it propagates—that is, what is the analogue of the Dirac equation for spin-$\frac{1}{2}$ fermions?

First of all, we note that the 'spinor-tensor' field $\Psi_{\alpha a} = e^{\mu}_{a}\Psi_{\alpha\mu}$, by itself, does transforms in an irreducible Lorentz representation. Instead, we have:

$$(\frac{1}{2}, 0) \otimes (\frac{1}{2}, \frac{1}{2}) = (1, \frac{1}{2}) \oplus (0, \frac{1}{2}) \tag{10.34}$$

The irreducible representation corresponding to the first term on the RHS, $(1, \frac{1}{2})$, plus its CPT conjugate $(\frac{1}{2}, 1)$, will give us the gravitino. The second term, $(0, \frac{1}{2})$, should be eliminated from the physical description, somehow; this corresponds to a projection:

$$(\bar{\sigma}^{\mu}\Psi_{\mu})^{\dot{\alpha}} = 0 \ . \tag{10.35}$$

Secondly, as already mentioned, the fact that the gravitino couples to the conserved current:

$$\Delta\mathscr{L} = \Psi_{\mu}S^{\mu} \ , \tag{10.36}$$

implies the presence of a gauge invariance:

$$\Psi_{\mu\alpha} \to \Psi_{\mu\alpha} + \partial_{\mu}\eta_{\alpha} \ . \tag{10.37}$$

These two points are related: one can view the constraint (10.35) as a *gauge-fixing* condition, which then imposes $\bar{\sigma}^{\mu}\partial_{\mu}\eta = 0$ on the gauge parameter (that is, η then solves the massless Dirac equation). Whatever gauge one picks, the point is that the $(0, \frac{1}{2})$ components in (10.34) correspond to some unphysical 'pure gauge' degrees of freedom.

Any kinetic Lagrangian for the gravitino should be gauge-invariant under (10.37). For instance, the following "field strength" is obviously gauge invariant:

$$\Psi_{\mu\nu} \equiv \partial_{\mu}\Psi_{\nu} - \partial_{\nu}\Psi_{\mu} \ . \tag{10.38}$$

In the linearized theory around flat space, we can take:

$$\mathscr{L}_{RS} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\Psi}_{\mu}\bar{\sigma}_{\nu}\partial_{\rho}\Psi_{\lambda} + h.c. \ . \tag{10.39}$$

This is gauge invariant up to a total derivative. The corresponding equation of motion is known as the massless Rarita-Schwinger equation:

$$\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\nu}\partial_{\rho}\Psi_{\lambda} = 0 \ , \qquad \epsilon^{\mu\nu\rho\sigma}\sigma_{\nu}\partial_{\rho}\bar{\Psi}_{\lambda} = 0 \ , \tag{10.40}$$

It describes the propagation of a massless particle of helicity $|\lambda| = \frac{3}{2}$ in flat space.

10.5 The old-minimal 4d $\mathcal{N}=1$ supergravity action

Finally, we now briefly describe the structure of 4d $\mathcal{N}=1$ so-called 'old minimal' supergravity. We will state the key results without proof, since the complete derivation is rather tedious and very technical. For further references, see the original literature [28, 30, 29], as well as at the books *e.g.* [2, 1].

10.5.1 Gravitino and local supersymmetry

Before proceeding to the supergravity action, let us emphasize the key role that the gravitino plays in supergravity theories. It acts as a *gauge-field for local supersymmetry.* A gauge-field for any 'gauge symmetry' $\mathcal{G}$ is a one-form valued in the Lie algebra of $\mathcal{G}$. Here, the supersymmetry transformation parameters ϵ_α are spinor-valued, and correspondingly $\Psi_{\mu\alpha}$ can be viewed as a spinor-valued one-form.

Any gauge field allows us to define '$\mathcal{G}$-covariant derivatives.' In the case of local supersymmetry, this should go as follows. Consider, for simplicity, a multiplet (φ, χ) of bosons and fermions that would be of the form:

$$\delta\varphi = \epsilon\chi \ , \qquad \delta\chi = 0 \ , \tag{10.41}$$

in flat space. In this toy model, if we wanted to make the supersymmetry transformation local, $\epsilon = \epsilon(x)$, then derivatives of the field φ would transform non-covariantly as:

$$\partial_\mu\varphi \to \delta(\partial_\mu\varphi) = \epsilon\partial_\mu\chi + (\partial_\mu\epsilon)\chi \ . \tag{10.42}$$

Instead, if we define a 'supersymmetry-covariant' derivative of the form:

$$\mathcal{D}_\mu\varphi = \partial_\mu\varphi - \Psi_\mu\chi \ , \tag{10.43}$$

thne we have:

$$\delta(\mathcal{D}_\mu\varphi) = \epsilon\partial_\mu\chi + (\partial_\mu\epsilon)\chi - (\delta\Psi_\mu)\chi \ , \tag{10.44}$$

and one can remove the dependence on the derivatives of ϵ if we declare that:

$$\delta\Psi_\mu = \partial_\mu\epsilon \ , \tag{10.45}$$

under local supersymmetry. All the art of supergravity is to make this kind of reasoning work at the full non-linear level in the gravitino and in the metric, in curved space-time.

10.5.2 Gravitino-twisted spin connection

In the non-linear supergravity theory, covariant derivatives should be taken using a modified spin-connection, which includes pieces quadratic in the gravitino:

$$\hat\omega_{\mu ab} = \omega_{\mu ab} + \frac{i}{4}e^c_\mu\left(\Psi_a\sigma_b\bar\Psi_c - \Psi_c\sigma_b\bar\Psi_a + \Psi_c\sigma_a\bar\Psi_b - \Psi_b\sigma_a\bar\Psi_c - \Psi_b\sigma_c\bar\Psi_a + \Psi_a\sigma_c\bar\Psi_b\right) \ . \tag{10.46}$$

Then, we define the modified Ricci scalar as in (10.33), namely:

$$\mathcal{R} = e^a_\mu e^\nu_b \, R^\mu{}_{\nu a}{}^b(\hat{\omega}) \; . \tag{10.47}$$

This of course reduces to the Ricci scalar R for $\Psi_\mu = \bar{\Psi}_\mu = 0$. We also define a covariant derivative for the gravitino:

$$\begin{aligned} \widetilde{\mathscr{D}}_\mu \Psi_{\nu\alpha} &= \partial_\mu \Psi_{\nu\alpha} - \frac{1}{2}\hat{\omega}_{\mu ab}(\sigma^{ab}\Psi_\nu)_\alpha \; , \\ \widetilde{\mathscr{D}}_\mu \bar{\Psi}^{\dot{\alpha}}_\nu &= \partial_\mu \bar{\Psi}^{\dot{\alpha}}_\nu - \frac{1}{2}\hat{\omega}_{\mu ab}(\bar{\sigma}^{ab}\bar{\Psi}_\nu)^{\dot{\alpha}} \; . \end{aligned} \tag{10.48}$$

10.5.3 The 4d $\mathcal{N}=1$ supergravity action

In terms of these quantities, the full supergravity action reads:

$$\begin{aligned} S_{\text{SUGRA}} \;&= \frac{1}{4\pi G}\int d^4x\sqrt{-g}\Bigg[\frac{1}{2}\mathcal{R} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\bar{\Psi}_\mu\bar{\sigma}_\nu\widetilde{\mathscr{D}}_\rho\Psi_\lambda - \Psi_\mu\sigma_\nu\widetilde{\mathscr{D}}_\rho\bar{\Psi}_\lambda\right) \\ &\qquad\qquad + \frac{1}{3}\bar{M}M - \frac{1}{3}b_\mu b^\mu\Bigg] \; . \end{aligned} \tag{10.49}$$

The auxiliary fields b_μ and $M, \bar{M}$ were introduced in subsection 10.1.1. This action is invariant under the full non-linear off-shell local $\mathcal{N}=1$ supersymmetry transformations:

$$\begin{aligned} \delta e^a_\mu &= i\left(\Psi_\mu\sigma^a\bar{\epsilon} - \epsilon\sigma^a\bar{\Psi}_\mu\right) \; , \\ \delta\Psi_{\mu\alpha} &= -2\mathscr{D}_\mu\epsilon_\alpha + ie^c_\mu\left(\frac{1}{3}M(\sigma_c\bar{\epsilon})_\alpha + b_c\epsilon_\alpha + \frac{1}{3}b^d(\epsilon\sigma_d\bar{\sigma}_c)\right) \; , \\ \delta\bar{\Psi}^{\dot{\alpha}}_\mu &= \cdots \; , \\ \delta M &= \cdots \; , \\ \delta\bar{M} &= \cdots \; , \\ \delta b_\mu &= \cdots \; . \end{aligned} \tag{10.50}$$

The full transformatiosn can be found in [1]. They are not particularly illuminating by themselves. Just note that we indeed have a derivative of the supersymmetry parameter in the gravitino variation, which now involves the modified covariant derivative on spinors:

$$\mathscr{D}_\mu\epsilon = (\partial_\mu - \frac{1}{2}\hat{\omega}_{\mu ab}\sigma^{ab})\epsilon \; . \tag{10.51}$$

One can similarly consider coupling the supergravity sector to matter, which generalizes the minimal coupling (10.11) to the full non-linear theory.

There are also a superspace approach to $\mathcal{N}=1$ supergravity, to which the Wess and Bagger book provides a good introduction [1]. The basic idea is to develop a theory of general covariance in superspace, generalizing the logic of General Relativity. This allows, in particular, to derive the connection (10.46) from first principle (or 'almost') as a component of a superspace connection.

A Useful identities

In this appendix, we collect some useful identities. The proofs are left as an exercise for the reader.

A.1 Useful identities

Recall:

$$
\begin{aligned}
(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta &= -2\eta^{\mu\nu}\delta_\alpha{}^\beta \ , \\
(\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)_\alpha{}^\beta &= -2\eta^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta} \ .
\end{aligned}
\tag{A.1}
$$

We also have:

$$
\begin{aligned}
\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\dot\beta}_{\mu}\beta &= -2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} \ , \\
\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) &= -2\eta^{\mu\nu} \ ,
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\sigma^\mu\bar\sigma^\nu\sigma^\rho + \sigma^\rho\bar\sigma^\nu\sigma^\mu &= 2(\eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho) \ , \\
\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho + \bar\sigma^\rho\sigma^\nu\bar\sigma^\mu &= 2(\eta^{\mu\rho}\bar\sigma^\nu - \eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\nu}\bar\sigma^\rho) \ ,
\end{aligned}
\tag{A.3}
$$

Involving $\theta, \bar\theta$ (or any two spinors):

$$
\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta \ , \qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \ , \tag{A.4}
$$

$$
\bar\theta^{\dot\alpha}\theta^\alpha = \frac{1}{2}\theta\sigma^\mu\bar\theta\,\bar\sigma^{\dot\alpha\alpha}_\mu \ . \tag{A.5}
$$

$$
\theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta = -\frac{1}{2}\theta\theta\,\bar\theta\bar\theta\,\eta^{\mu\nu} \ . \tag{A.6}
$$

The matrices $(\sigma^{\mu\nu})_\alpha{}^\beta$ defined in (2.37) are also (imaginary)-self-dual (SD) two-forms:

$$
\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = \sigma^{\mu\nu} \ , \tag{A.7}
$$

while $\bar\sigma^{\mu\nu}$ is (imaginary)-anti-self-dual (ADS):

$$
\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma} = -\bar\sigma^{\mu\nu} \ . \tag{A.8}
$$

Here, we have $\epsilon^{0123} = +1$. Note also that those matrices are traceless with the natural position of indices, and therefore they are symmetric in the spinor indices when bringing both indices down: $(\sigma^{\mu\nu})_{\alpha\beta} = (\sigma^{\mu\nu})_{\beta\alpha}$, and similarly, $(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta} = (\bar\sigma^{\mu\nu})_{\dot\beta\dot\alpha}$.

We also have the useful identity:

$$
\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \ . \tag{A.9}
$$

A.2 Fierz identities for 4d Weyl spinors

A basic identity involving three left-chiral Weyl spinors is:

$$\theta\chi\,\psi_\alpha = -\theta\psi\,\chi_\alpha - \chi\psi\,\theta_\alpha \,, \tag{A.10}$$

and similarly for right-chiral Weyl spinors:

$$\bar\theta\bar\chi\,\bar\psi_{\dot\alpha} = -\bar\theta\bar\psi\,\bar\chi_{\dot\alpha} - \bar\chi\bar\psi\,\bar\theta_{\dot\alpha} \,. \tag{A.11}$$

From these follow various other useful identities, such as, for instance:

$$(\sigma^\mu\bar\epsilon)_\alpha\,\epsilon\psi = -\epsilon\sigma^\mu\bar\epsilon\,\psi_\alpha + \bar\epsilon\bar\sigma^\mu\psi\,\epsilon_\alpha \,, \tag{A.12}$$

We also have:

$$\begin{aligned} \theta\psi\,\bar\chi_{\dot\alpha} &= -\frac{1}{2}\psi\sigma^\mu\bar\chi\,(\theta\sigma_\mu)_{\dot\alpha} \,, \\ \bar\chi\bar\theta\,\psi_\alpha &= -\frac{1}{2}\psi\sigma^\mu\bar\chi\,(\sigma_\mu\bar\theta)_\alpha \,. \end{aligned} \tag{A.13}$$

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