Supersymmetry & Supergravity: Problem sheet 1

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Due by Friday, week 2 (January 31st), 4pm.

1. $\mathcal{N} = 2$ supersymmetric quantum mechanics

Consider a 1d field theory consisting of a free complex scalar $Z(t)$ and a free complex 1d fermion $\psi(t)$, with the Lagrangian:

$$L = \dot{\bar{Z}}\dot{Z} - m^2 \bar{Z} Z + i\bar{\psi}\dot{\psi} - m\bar{\psi}\psi \ , \tag{0.1}$$

with $m \in \mathbb{R}$ a one-dimensional mass parameter.

(1.a) By direct computation, show that this Lagrangian is invariant (up to a total derivative) under the $\mathcal{N}$=2 supersymmetry transformations:

$$\begin{aligned} \delta Z &= -\epsilon\psi \ , \\ \delta\bar{Z} &= \bar{\epsilon}\bar{\psi} \ , \\ \delta\psi &= i\bar{\epsilon}(\dot{Z} - imZ) \ , \\ \delta\bar{\psi} &= -i\epsilon(\dot{\bar{Z}} + im\bar{Z}) \ , \end{aligned} \tag{0.2}$$

with $\epsilon, \bar{\epsilon}$ some formal infinitesimal (c-number) parameters, such that:

$$\delta = \epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}} \ . \tag{0.3}$$

(1.b) Check that these transformations satisfy:

$$\delta^2 = -i\epsilon\bar{\epsilon}\left(\frac{d}{dt} - iq_f m\right) \ , \tag{0.4}$$

where $q_f = 1$ when acting on Z or ψ, and $q_f = -1$ for $\bar{Z}$ or $\bar{\psi}$. Therefore, the SUSY algebra takes the form:

$$\{\mathcal{Q}, \bar{\mathcal{Q}}\} = H - mQ_f \ . \tag{0.5}$$

Here, we have a central charge $\mathcal{Z} = -mQ_f$, with Q_f the generator of a $U(1)$ flavor symmetry. Are there any other symmetries of the Lagrangian above, appart from H and Q_f? Which of these symmetries commute with supersymmetry?

(1.c) Consider the canonical quantization of this theory, with canonical momenta $\Pi_Z = \dot{\bar{Z}}$, $\Pi_{\bar{Z}} = \dot{Z}$ and $\Pi_\psi = -i\bar{\psi}$ and the (anti)-commutators:

$$[Z, \Pi_Z] = i \ , \qquad [\bar{Z}, \Pi_{\bar{Z}}] = i \ , \qquad \{\bar{\psi}, \psi\} = 1 \ . \tag{0.6}$$

Derive the Noether charges for the two supersymmetries (0.2). Using (0.6), check that the supercharge operators indeed generate the transformation (0.2), with:

$$\delta\varphi = [\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}, \varphi] \tag{0.7}$$

in the quantum theory.

2. On the Poincaré algebra.

Consider the $so(p,q)$ "Lorentz" algebra, leaving invariant the flat metric $\eta_{\mu\nu}$ of $\mathbb{R}^{p,q}$. It is given by:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right) \ . \tag{0.8}$$

(2.a) Check explicitly that $M_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ satisfy the $so(p,q)$ algebra, provided the gamma matrices satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{0.9}$$

(1.b) The Poincaré algebra (in any signature) contains the Lorentz generators $M_{\mu\nu}$ and the translation generators P_μ, with the commutation relations:

$$[P_\mu, P_\nu] = 0 \ , \qquad [M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \ . \tag{0.10}$$

Given that:

$$P_\mu = -i\partial_\mu \ , \tag{0.11}$$

on scalar fields $\phi(x)$ (*i.e.* functions of x^μ), what is the expression for $M_{\mu\nu}$ acting on $\phi(x)$? Check that the Poincaré algebra is satisfied. (For instance, $[P_\mu, P_\nu]\phi(x) = -(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\phi(x) = 0$.)

3. Useful identities.

(2.a) Prove the following identities for the σ matrices defined in the lectures:

$$\begin{aligned} &\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu = -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \ , \\ &\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) = -2\eta^{\mu\nu} \ , \\ &(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta = -2\eta^{\mu\nu}\delta^\beta_\alpha \ . \end{aligned} \tag{0.12}$$

Using these, invert the map:

$$\widetilde{X}_{\alpha\dot{\alpha}} = X_\mu\sigma^\mu_{\alpha\dot{\alpha}} \tag{0.13}$$

which maps the vector X_μ to the bi-spinor $\widetilde{X}_{\alpha\dot{\alpha}}$.

(3.b) Check also:

$$\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \ , \tag{0.14}$$

with $\epsilon^{0123} = 1$.

3. On $SO(1,3)$.

(3.a) Show that the $so(1,3)$ algebra (0.8) can be decomposed as:

$$so(1,3) \cong su(2) \times su(2)^* \ . \tag{0.15}$$

Hint: first, write down the algebra in terms of the $SO(3)$ rotation and boost generators:

$$J^i = \frac{1}{2}\epsilon^{ijk} M_{jk} \ , \qquad K_i = M_{0i} \ . \tag{0.16}$$

Then, show that:

$$J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \tag{0.17}$$

generate the two $su(2)$ factors.

(3.b) We briefly mentioned in the lectures that there is a *group homomorphism*:

$$Sl(2,\mathbb{C}) \to SO(1,3) \tag{0.18}$$

Given a four-vector X_μ, we have a map to a 2×2 complex matrix, using the σ-matrices:

$$\widetilde{X} = X_\mu \sigma^\mu \ . \tag{0.19}$$

On X_μ, we have the action of an $SO(1,3)$ matrix Λ:

$$X_\mu \to \Lambda_\mu{}^\nu X_\nu \ , \tag{0.20}$$

while on the matrix, the corresponding action is:

$$\widetilde{X} \to N\widetilde{X}N^\dagger \ . \tag{0.21}$$

Explain why N should be an $SL(2,\mathbb{C})$ matrix. Work out the explicit map from N to Λ, and check that it is an homomorphism. Is it one-to-one?

(3.c) Write down explicitly the $SO(1,3)$ matrices acting on left- and right-handed Weyl spinors.

4. On 4d Weyl spinors and Fierzing.

(4.a) Prove that, for Grassmann-number-valued Weyl spinors:

$$\begin{aligned} &\psi\chi = \chi\psi \ , \\ &\psi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\psi \ , \end{aligned} \tag{0.22}$$

using our Wess-and-Bagger conventions. Check that $\psi\chi$ is a scalar under $SL(2,\mathbb{C})$. (Given the action, $\psi_\alpha \to N_\alpha{}^\beta\psi_\beta$ on ψ_α, one should first determine the action of $SL(2,\mathbb{C})$ on $\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta$.)

(4.b) For $\psi, \theta, \chi, \cdots$ some Grassmann-valued spinors, check the Fierz identities:

$$\theta\chi\,\psi_\alpha = -\theta\psi\,\chi_\alpha - \chi\psi\,\theta_\alpha \ , \tag{0.23}$$

and:

$$\begin{aligned} &(\theta\psi)(\bar{\chi}\bar{\eta}) = \frac{1}{2}(\theta\sigma^\mu\bar{\eta})(\bar{\chi}\bar{\sigma}_\mu\psi) \ , \\ &(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = -\frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta}) \ . \end{aligned} \tag{0.24}$$