Supersymmetry & Supergravity: Problem sheet 2

MMathPhys, University of Oxford, HT2020, Dr Cyril Closset

Tutor: Dr Cyril Closset. TA: Horia Magureanu

Due by Friday, week 4 (February 14), 4pm.

The starred questions or subquestions, denoted by [$*$], are optional and will not be marked. They should be useful as part of your exam preparation, and can be discussed in class.

1. The 4d $\mathcal{N} = 1$ super-Poincaré algebra.

The 4d $\mathcal{N} = 1$ supersymmetry algebra reads:

$$\begin{aligned}
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \ , \\
&\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \ , \\
&[P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot\alpha}] = 0 \ , \\
&[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu} Q)_\alpha \ , \\
&[M_{\mu\nu}, \bar{Q}_{\dot\alpha}] = -i(\bar{Q}\bar{\sigma}_{\mu\nu})_{\dot\alpha} \ ,
\end{aligned} \tag{0.1}$$

together with the Poincaré algebra itself:

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}\right) \ , \\
&[P_\mu, P_\nu] = 0 \ , \qquad [M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \ .
\end{aligned} \tag{0.2}$$

(1.a) Define the supercommutator:

$$[\mathcal{O}_a, \mathcal{O}_b\} \equiv \mathcal{O}_a \mathcal{O}_b - (-1)^{\epsilon_a \epsilon_b} \mathcal{O}_b \mathcal{O}_a \ , \tag{0.3}$$

where $\epsilon_a \in \{0, 1\}$ is the $\mathbb{Z}_2$ grading of $\mathcal{O}_a$. Using the Jacobi identities of a super-algebra:

$$(-1)^{\epsilon_c \epsilon_a} [[\mathcal{O}_a, \mathcal{O}_b\}, \mathcal{O}_c\} + (-1)^{\epsilon_a \epsilon_b} [[\mathcal{O}_b, \mathcal{O}_c\}, \mathcal{O}_a\} + (-1)^{\epsilon_b \epsilon_c} [[\mathcal{O}_c, \mathcal{O}_a\}, \mathcal{O}_b\} = 0 \ ,$$

show that the 4d $\mathcal{N} = 1$ supersymmetry algebra closes.

Hint: For this computation, a useful identity is:

$$\sigma^\mu \bar{\sigma}^\nu \sigma^\rho + \sigma^\rho \bar{\sigma}^\nu \sigma^\mu = 2(\eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho) \ . \tag{0.4}$$

(1.b) [$*$] Write down the supersymmetry algebra in terms of the Majorana spinor:

$$(\mathbf{Q}_a) = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix} \ . \tag{0.5}$$

2. Supermultiplets of 4d $\mathcal{N}=2$ supersymmetry.

In this problem, we look at massless multiplets of 4d $\mathcal{N}=2$ supersymmetry.

We would like to keep track of the $U(2)_R = U(1)_r \times SU(2)_R$ symmetry, which is part of the automorphism group of the 4d $\mathcal{N}=2$ superalgebra. The supercharges:

$$(Q^I) = (Q^1, Q^2) , \qquad (\bar{Q}_I) = (\bar{Q}_1, \bar{Q}_2) , \tag{0.6}$$

transform as $SU(2)_R$ doublets.

(2.a) Write down the 4d $\mathcal{N}=2$ algebra acting on massless one-particle states with $P^\mu = (E, 0, 0, E)$. Give it in terms of creation and annihilation operators, like in the lectures.

(2.b) Describe the structure of the massless $\mathcal{N}=2$ supermultiplet with lowest helicity $\lambda = -\frac{1}{2}$ (that is, give the particle content and show how the various particles are related by supersymmetry). This supermultiplet is called the **hypermultiplet**. How does it behave under CPT? In which $SU(2)_R$ representations do the various bosons and fermions transform?

(2.c) Describe the structure of the massless $\mathcal{N}=2$ supermultiplet with lowest helicity $\lambda = -1$, adding the CPT conjugate particles if needed. This gives the so-called **4d $\mathcal{N}=2$ vector multiplet**. In which $SU(2)_R$ representations do the various bosons and fermions transform?

(2.d) [*] Decompose the 4d $\mathcal{N}=2$ vector multiplet into two 4d $\mathcal{N}=1$ supermultiplets, after choosing a particular $\mathcal{N}=1$ subalgebra of the 4d $\mathcal{N}=2$ supersymmetry algebra.

3. Superspace differential operators.

Consider 4d $\mathcal{N}=1$ superspace:

$$\mathbb{R}^{3,1|4} = ISO(1,3|4)/SO(1,3) . \tag{0.7}$$

The differential operators representing the supersymmetry algebra on superspace are given by:

$$\begin{aligned}
\mathbf{Q}_\alpha &= -i(\partial_\alpha - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu) , \\
\bar{\mathbf{Q}}_{\dot\alpha} &= i(\bar{\partial}_{\dot\alpha} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu) , \\
\mathbf{P}_\mu &= -i\partial_\mu , \\
\mathbf{M}_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu - (\theta\sigma_{\mu\nu})^\alpha\partial_\alpha + (\bar{\sigma}_{\mu\nu}\bar{\theta})^{\dot\alpha}\partial_{\dot\alpha}) .
\end{aligned} \tag{0.8}$$

(3.a) Derive the expression for $\mathbf{M}_{\mu\nu}$ using the coset manifold construction of superspace.

(3.b) [*] By explicit computation, check that the operators (0.8) satisfy the supersymmetry algebra. For simplicity, check only the two (anti)commutators $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}$ and $[M_{\mu\nu}, Q_\alpha]$. The following identity may be useful:

$$\sigma^{\mu\nu}\sigma^\rho - \sigma^\rho\bar{\sigma}^{\mu\nu} = \eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu .$$

4. Supersymmetry variations of a chiral multiplet.

Recall the supersymmetry variations for the chiral and anti-chiral multiplets:

$$\begin{aligned} &\delta\phi = \sqrt{2}\epsilon\psi \ , && \delta\bar{\phi} = \sqrt{2}\bar{\epsilon}\bar{\psi} \ , \\ &\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu\phi + \sqrt{2}\epsilon_\alpha F \ , && \delta\bar{\psi}^{\dot{\alpha}} = i\sqrt{2}(\bar{\sigma}^\mu\epsilon)^{\dot{\alpha}}\partial_\mu\bar{\phi} + \sqrt{2}\bar{\epsilon}^{\dot{\alpha}}\bar{F} \ , \\ &\delta F = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \ , && \delta\bar{F} = i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\bar{\psi} \ . \end{aligned} \tag{0.9}$$

Let us also write down the Lagrangian:

$$\mathscr{L}_{\text{kin}} = -\partial_\mu\bar{\phi}\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \bar{F}F \ , \tag{0.10}$$

and:

$$\mathscr{L}_W = F^i\partial_i W - \frac{1}{2}\psi^i\psi^j\,\partial_i\partial_j W \ , \tag{0.11}$$

for $W = W(\phi)$ an arbitrary superpotential.

(4.a) By explicit computation using the SUSY variations (0.9), show that the Lagrangian $\mathscr{L}_{\text{kin}}$ is supersymmetric:

$$\delta\mathscr{L}_{\text{kin}} = \partial_\mu(\cdots) \ . \tag{0.12}$$

(4.b) [$*$] Similarly, show by explicit computation that the interaction Lagrangian $\mathscr{L}_W$ is supersymmetric.

5. Chiral superfields.

Consider the explicit form:

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) \ &= \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) \\ &\quad + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi(x) \ . \end{aligned} \tag{0.13}$$

for the chiral superfield, and the differential operators:

$$\begin{aligned} &\mathbf{Q}_\alpha = -i\left(\partial_\alpha - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\right) \ , && \bar{\mathbf{Q}}_{\dot{\alpha}} = i\left(\bar{\partial}_{\dot{\alpha}} - i(\theta^\alpha\sigma^\mu)_{\dot{\alpha}}\partial_\mu\right) \ , \\ &\mathrm{D}_\alpha = \partial_\alpha + i(\sigma^\mu\bar{\theta})\partial_\mu \ , && \bar{\mathrm{D}}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \ . \end{aligned} \tag{0.14}$$

(5.a) By explicit computation, check that:

$$\bar{\mathrm{D}}_{\dot{\alpha}}\Phi = 0 \ .$$

(5.b) Consider the general non-linear kinetic term for a single chiral multiplet:

$$\mathscr{L}_K = \int d^2\theta d^2\bar{\theta} K(\bar{\Phi}, \Phi) \ , \tag{0.15}$$

with $K(\bar{z}, z)$ a real function of a single complex variable z, with complex conjugate $\bar{z}$. Expanding the superfield Φ into component fields and performing the superspace integral, write down the Lagrangian $\mathscr{L}_K$ in components. Check that you recover (0.10) in the special case $K(\bar{z}, z) = |z|^2$.

(5.c) Rederive the chiral multiplet supersymmetry transformation laws (0.9) using the superspace definition:

$$\delta\Phi = i(\epsilon\mathbf{Q} + \bar{\epsilon}\mathbf{Q})\Phi \ .$$

6. Supersymmetric gauge transformations.

Given a real superfield $\mathcal{S}$ ($(\mathcal{S})^\dagger = \mathcal{S}$), we may define the transformation:

$$\mathcal{S} \to \mathcal{S} + \Phi + \bar{\Phi} \ , \tag{0.16}$$

where Φ and $\bar{\Phi}$ are a chiral multiplet and its Hermitian conjugate.

(6.a) Write down this transformation in terms of the components fields of the real superfields,

$$\mathcal{S} = \left(C, \chi \ , \bar{\chi} \ , M \ , \bar{M} \ , v_\mu \ , \lambda \ , \bar{\lambda} \ , D\right) \ ,$$

and of the component fields of Φ and $\bar{\Phi}$. (For instance, for the bottom component, we obviously have: $C \to C + \phi + \bar{\phi}$.) Why is this called a "gauge transformation"?

(6.b) Show that (0.16) leaves the D-term action:

$$S = \int d^4x \int d^2\theta d^2\bar{\theta}\, \mathcal{S} \tag{0.17}$$

invariant.

Additional problems (optional, not marked).

7. [$*$] Manipulating Grassmann numbers.

Let η^i denote a set of n Grassmann coordinates ($i = 1, \cdots, n$), which satisfy the Grassmann algebra:

$$\{\eta^i, \eta^j\} = 0 \ . \tag{0.18}$$

Let us define the integration over the η coordinates as:

$$\int d^n\eta = \int d\eta^n \cdots d\eta^2 d\eta^1 \ . \tag{0.19}$$

(7.a) Prove that:

$$\int d^n\eta \, e^{\frac{1}{2}A_{ij}\eta^i\eta^j} = \mathrm{Pf}(A) \ ,$$

where $A_{ij} = A_{ji}$ is an antisymmetric matrix.

(7.b) For a single variable η, let us define the Dirac delta function $\delta(\eta - \theta)$ by:

$$\int d\eta \, \delta(\eta - \theta) f(\eta) = f(\theta) \ , \qquad \forall f \ . \tag{0.20}$$

Show that $\delta(\eta - \theta) = \eta - \theta$.

8. [*] The $\mathcal{N} = 4$ vector multiplet.

Similarly to problem 2 above, let us study the massless representations of the 4d $\mathcal{N} = 4$ supersymmetry algebra. In this case, we keep track of the $SU(4)_R$ R-symmetry; the supercharges (Q^I) and $(\bar{Q}_I)$ transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$, respectively.

(8.a) Describe the unique massless 4d $\mathcal{N} = 4$ supermultiplet compatible with *rigid* supersymmetry. How do its components transform under $SU(4)_R$? This multiplet is called the **4d $\mathcal{N} = 4$ vector multiplet.** (Why?)

(8.b) Decompose the $\mathcal{N} = 4$ vector multiplet into 4d $\mathcal{N} = 1$ supermultiplets.

9. [*] $\mathcal{N} = 3$ one-particle multiplet.

Find the most general massless one-particle multiplet of $\mathcal{N} = 3$ rigid supersymmetry in a QFT (without gravity), and organise the states in representations of the $U(3)_R \cong SU(3)_R \times U(1)_R$ R-symmetry. Work out how the $\mathcal{N} = 3$ multiplet is embedded into the $\mathcal{N} = 4$ multiplet. What can you conclude?
[*The following mathematical fact may be useful: the branching rules for the Lie algebra decomposition $SU(4) \to SU(3) \times U(1)$ are:*

$$\mathbf{4} \to \mathbf{1}_{-3} \oplus \mathbf{3}_1 \ , \qquad \mathbf{6} \to \mathbf{3}_{-2} \oplus \bar{\mathbf{3}}_2 \ , \qquad \mathbf{10} \to \mathbf{1}_{-6} \oplus \mathbf{3}_{-2} \oplus \mathbf{6}_2 \ ,$$

for the first few irreducible representations of $SU(4)$.]

10. [*] Coset manifold: a classic example.

Consider the group $G = SU(2)$, with group elements given by:

$$g = e^{i\epsilon_i T^i} \in G \ , \qquad [T^i, T^j] = i\epsilon^{ij}{}_k T^k \ . \tag{0.21}$$

(Of course $i = 1, 2, 3$.) We consider the subgroup $H = U(1)$ with a single generator:

$$h = e^{i\varepsilon T^3} \in H \ . \tag{0.22}$$

Show explicitly that the coset manifold G/H is the two-sphere:

$$\mathcal{M} = G/H \cong S^2 \ . \tag{0.23}$$

Hint: One can consider an explicit realisation of $SU(2)$ in terms of $T^i = \frac{1}{2}\sigma^i$, with σ^i the Pauli matrices. Then, the general element g takes the form:

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \ , \qquad a, b \in \mathbb{C} \ , \quad \text{such that } |a|^2 + |b|^2 = 1 \ . \tag{0.24}$$

(Check this.) Show that, in this parameterisation, the action $g \to gh$ is given by:

$$a \to a e^{i\frac{\varepsilon}{2}} \ , \qquad b \to b e^{-i\frac{\varepsilon}{2}} \ . \tag{0.25}$$

Use this to argue that the coset manifold is indeed S^2. Can you find a convenient set of coordinates on the coset? How does $G = SU(2)$ act on those coordinates?