Scientific Programming with Mathematica 2020

Problems 1

1. I have taken the following problem from the preface to Experimental Number Theory by F. Rodriguez Villegas.

   On May 30th 1799, Gauss wrote in his Scientific Diary (in Latin) :
   We have established that the Arithmetic Geometric mean between 1 and $\sqrt{2}$ *is* $\frac{\pi}{\varpi}$ *to the 11th decimal place; the demonstration of this fact will surely open an entirely new field of analysis.*

   The arithmetic-geometric mean $M(a,b)$ of two positive real numbers (a,b) is the common limit of the sequences a_n and b_n defined by

   $$a_{n+1} \;=\; \frac{1}{2}\,(a_n + b_n) \quad , \quad b_{n+1} \;=\; \sqrt{a_n b_n}$$

   Write a program to compute $M(a,b)$ to a desired accuracy. Compute also the integral

   $$\varpi \;=\; 2\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}}$$

   and check Gauss' observation to 100 figures.

2. Let F denote the function

   $$F(x) \;=\; 5^{5\mathrm{i}x}\,\frac{\Gamma(1-5\mathrm{i}x)}{\Gamma^5(1-\mathrm{i}x)}$$

   and consider the integral

   $$I(\delta) \;=\; \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{x}{1-\mathrm{e}^{-2\pi x}}\,\mathrm{re}\,(F(x))\,\cos(\delta x)\ ,$$

   with δ, real.

   We want to show that, as $\delta \to 0$,

   $$I(\delta) \;=\; \frac{\sqrt{5}}{4\pi^2}\,\log\delta + C + \mathcal{O}(\delta)\ ,$$

   with C a constant that we wish to compute.

   Use the function Series to show that, as $x \to \infty$,

   $$xF(x) \;= -\frac{\sqrt{5}}{4\pi^2 x} + \mathcal{O}(x^{-2})\ .$$

   [In order to simplify the result you should first use Normal *and then* Simplify*. You can, of course, check this using Stirling's formula.]*

Plot the integrand in the range (-3,3) and $\frac{x}{1-\mathrm{e}^{-2\pi x}}\,\mathrm{re}\,(F(x)) + \frac{\sqrt{5}}{4\pi^2 x}$ for x in the range (1,3).

By breaking up the range of integration into $(-\infty,\,1)$ and $(1,\,\infty)$ and making use of the integral

$$J(\delta) \;=\; -\frac{\sqrt{5}}{4\pi^2}\int_1^\infty \frac{\mathrm{d}x}{x}\,\cos(\delta x)\;,$$

which you should evaluate, compute C numerically.

Once you have computed C, compute it again, to higher precision, using the option WorkingPrecision $\rightarrow$ 40 for NIntegrate. Increase WorkingPrecision until you have C correct to 100 figures.

[Warning: There seems to be an unresolved bug that particularly affects the Mac front end. If you use WorkingPrecision $\rightarrow$ n for large n*, you should also set* AccuracyGoal *to about* n/2*, or Mathematica may crash, without warning.]*

3. Let f and g denote the functions

$$f(x) \;=\; \sin(x^2) + \sin^2(x)\;;\quad g(x) \;=\; \exp\left(\frac{(5-x)^2}{10}\right)\;.$$

Make a simultaneous plot of f and g for x in the range (2,8), say. How many real solutions are there to the equation $f(x) = g(x)$? Find the roots.

[If you use the function FindRoot *you may care to set* WorkingPrecision $\rightarrow$ 80*.]*

4. The function

rmat := RandomReal$[\{-1,1\},\{8,8\}]$

generates a 8×8 matrix of random numbers, in the range $(-1,1)$. Generate from this two *symmetric* random matrices, A and B with elements in the range $(-1,1)$. Let $M(t)$ be the one parameter family of matrices

$$M(t) \;=\; tA + (1-t)B\;.$$

Plot, simultaneously, the eight eigenvalues of $M(t)$ as a function of t for the range $(-1,1)$. Probably, some of the curves will appear to cross. Do they in fact do so? (If no two curves appear to cross, repeat with different values for A and B).

[To order eigenvalues, you may care to use Sort[Eigenvalues[M[t]]]*.]*

5. Write a program to compute the solutions to

$$n \;=\; x^3 + y^3\;;\quad x,\,y\;\in\;\mathbb{Z}_{>0}$$

and verify Ramanujan's famous taxicab remark, that 1729 is the smallest integer that can be written as a sum of two cubes in two different ways. What is the next smallest such integer?