Scientific Programming with Mathematica 2020 Problems 2

1. Prove that, with the natural choice of square roots

$$\sqrt{5} + \sqrt{22 + 2\sqrt{5}} = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}$$

You may find the function MinimalPolynomial useful.

2. A common method of ensuring the integrity of documents and files is to compute a number that is very sensitive to changes in the underlying document. One such is Adler-32, this proceeds as follows. Given a string $c_1c_2 \ldots c_n$, we compute two integers m an n from the character codes C_j of the c_j .

$$m = 1 + C_1 + C_2 + \dots + C_n \mod 65521$$

$$n = (1 + C_1) + (1 + C_1 + C_2) + \dots + (1 + C_1 + C_2 \dots + C_n) \mod 65521.$$

we then form the checksum m + 65536n. The integer 65536 is 2^{16} and 65521 is the largest prime less than this. Write a script that computes the checksum. You may find the function Accumulate useful.

Compute the checksum for the following string (due to John Donne):

No man is an Iland, intire of itselfe; every man is a peece of the Continent, a part of the maine; if a Clod bee washed away by the Sea, Europe is the lesse, as well as if a Promontorie were, as well as if a Manor of thy friends or of thine owne were; any mans death diminishes me, because I am involved in Mankinde; And therefore never send to know for whom the bell tolls; It tolls for thee.

Check your answer by evaluating Hash[str, "Adler32"], for this string.

3. Conventions differ between references so we understand a modular form of weight k to satisfy

$$f(\gamma \tau) = (cz+d)^k f(\tau)$$
 for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

There is another relation, that is not a special case of the above if $N \neq 1$:

$$f\left(-\frac{1}{N\tau}\right) = \epsilon N^{k/2} \tau^k f(\tau) , \qquad (*)$$

where $\epsilon = \pm 1$ is a sign that depends on the particular form f.

The particular form that we need is a weight 4 form for the group $\Gamma_0(14)$. I do not know of a closed expression, but the LMFDB, http://www.lmfdb.org, provides, for form 14.4.1.a, the expansion

$$f_4(\tau) = q - 2q^2 + 8q^3 + 4q^4 - 14q^5 - 16q^6 - 7q^7 - 8q^8 + 37q^9 + \dots ; \quad q = e^{2\pi i\tau}$$

You can download the first 200 terms, say, which suffice for most reasonable purposes. The *L*-function associated with f is given in terms of the Mellin transform g by

$$L(s) = \frac{(2\pi)^s}{\Gamma(s)} g(s) \quad \text{with} \quad g(s) = \int_0^\infty \mathrm{d}y \, f(\mathrm{i}y) y^{s-1}$$

A standard manoeuvre with regard to the Mellin transform g consists of breaking the range of integration at $y = N^{-1/2}$ and using (*) to rewrite the integral for the lower part of the range. In this way we have

$$g(s) = \int_{N^{-1/2}}^{\infty} dy f(iy) \left\{ y^{s-1} + (-1)^{k/2} \epsilon N^{\frac{k}{2}-s} y^{k-1-s} \right\} ,$$

for the particular form we are concerned with $\epsilon = +1$. The advantage of this representation is that, owing to the nonzero lower limit of integration, one can integrate the *q*-series and this yields good convergence.

Compute L(2) to at least 50 figures.

The Dedekind eta function is defined, for $\text{Im } \tau > 0$, by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) ; \quad q = e^{2\pi i \tau}$$

I have it on good authority that

$$f_4(\tau) = \frac{(\eta_2 \eta_7)^6}{(\eta_1 \eta_{14})^2} - 4(\eta_1 \eta_2 \eta_7 \eta_{14})^2 + \frac{(\eta_1 \eta_{14})^6}{(\eta_2 \eta_7)^2} ,$$

where $\eta_j = \eta(j\tau)$.

Check the q-expansion of this expression to 200 terms.

4. A Calabi-Yau manifold rejoices in the designation AESZ34. This is a one parameter family of manifolds with parameter φ . The periods of the manifold satisfy a fourth order differential equation corresponding to an operator

$$\mathcal{L} = R_4 \vartheta^4 + R_3 \vartheta^3 + R_2 \vartheta^2 + R_1 \vartheta + R_0 \quad ; \quad \vartheta = \varphi \frac{\partial}{\partial \varphi}$$

and the functions R_j are given by

$$R_{0} = -900\varphi^{3} + 285\varphi^{2} - 5\varphi$$

$$R_{1} = -2700\varphi^{3} + 1088\varphi^{2} - 28\varphi$$

$$R_{2} = -2925\varphi^{3} + 1580\varphi^{2} - 63\varphi$$

$$R_{3} = -1350\varphi^{3} + 1036\varphi^{2} - 70\varphi$$

$$R_{4} = -225\varphi^{3} + 259\varphi^{2} - 35\varphi + 1$$

Denote by ϖ_0 the power series solution to $\mathcal{L}\varpi = 0$, normalised such that $\varpi_0(0) = 1$. Find a recurrence relation for the coefficients of ϖ_0 . We wish to calculate $\varpi_0(-1/7)$. The value $\varphi = -1/7$ lies outside the radius of convergence of the series, which is 1/25, as is seen by factoring R_4 . In order to perform the calculation first calculate ϖ_0 , and its first three derivatives, at $\varphi = -1/50$ to, say, 100 figures, then integrate the differential equation to reach $\varphi = -1/7$. A good set of options for NDSolve is AccuracyGoal $\rightarrow 50$, WorkingPrecision $\rightarrow 55$,

Method \rightarrow {"ExplicitRungeKutta", "DifferenceOrder" \rightarrow 9 }.

Finally, compute

$$\frac{\pi^2\,\varpi_0}{L(2)}\;,$$

where L(s) is the function from the previous problem.