Scientific Programming with Mathematica 2020

Problems 3

Consider the space AESZ34 for which we have the Picard-Fuchs equation

$$\mathcal{L}\varpi \;=\; 0 \quad \text{with} \quad \mathcal{L} \;=\; \sum_{j=0}^{4} S_j \vartheta^j \;;\quad \vartheta \;=\; \varphi\frac{\partial}{\partial\varphi} \;.$$

In this operator, the coefficient functions of the Picard-Fuchs operator are the cubics

$$\begin{aligned}
S_4 &= (\varphi-1)(9\varphi-1)(25\varphi-1)\\
S_3 &= 2\varphi(675\varphi^2-518\varphi+35)\\
S_2 &= \varphi(2925\varphi^2-1580\varphi+63)\\
S_1 &= 4\varphi(675\varphi^2-272\varphi+7)\\
S_0 &= 5\varphi(180\varphi^2-57\varphi+1)\;.
\end{aligned}$$

1. First, let us solve the equation in a neighborhood of $\varphi = 0$: seek a solution

   $$\varpi \;=\; \sum_{n=0}^{\infty} A_n(\epsilon)\varphi^{n+\epsilon} \quad \text{with} \quad A_0(\epsilon) \;=\; 1 \;.$$

   Find a recurrence relation for the $A_n(\epsilon)$ and show that $\epsilon^4 = 0$. By expanding

   $$\varpi = \varpi_0(\varphi)+\epsilon\varpi_1(\varphi)+\frac{1}{2!}\epsilon^2\varpi_2(\varphi)+\frac{1}{3!}\epsilon^3\varpi_3(\varphi) \quad\text{and}\quad A_n(\epsilon) = a_n+\epsilon b_n+\frac{1}{2!}\epsilon^2 c_n+\frac{1}{3!}\epsilon^3 d_n\;,$$

   show that there are four solutions of the form

   $$\begin{aligned}
   \varpi_0 &= f_0\\
   \varpi_1 &= f_0\log\varphi+f_1\\
   \varpi_2 &= f_0\log^2\varphi+2f_1\log\varphi+f_2\\
   \varpi_3 &= f_0\log^3\varphi+3f_1\log^2\varphi+3f_2\log\varphi+f_3\;,
   \end{aligned}$$

   where the f_j are power series with coefficients a_n, b_n, c_n and d_n.

   Write routines that calculate the f_j, and so the ϖ_j, both as series to `nmax` terms and numerically to `nmax` terms. Check that $\mathcal{L}\varpi_3$ vanishes to the required order when `nmax` $= 100$, say.

2. Now let $\varphi = \psi$ be a regular point of the differential equation, so that there are four solutions that are power series in a neighborhood of ψ. Let us choose a basis of solutions of the form

$$
\begin{aligned}
\eta_0(\varphi,\psi) &= 1 + \mathcal{O}\left((\varphi-\psi)^4\right)\\
\eta_1(\varphi,\psi) &= (\varphi-\psi) + \mathcal{O}\left((\varphi-\psi)^4\right)\\
\eta_2(\varphi,\psi) &= (\varphi-\psi)^2 + \mathcal{O}\left((\varphi-\psi)^4\right)\\
\eta_3(\varphi,\psi) &= (\varphi-\psi)^3 + \mathcal{O}\left((\varphi-\psi)^4\right)
\end{aligned}
$$

Find a recurrence relation for coefficients A_n, B_n, C_n, D_n, which will depend on ψ, such that $\eta_0 = \sum_{n=0}^{\infty} A_n(\varphi-\psi)^n$, $\eta_1 = \sum_{n=0}^{\infty} B_n(\varphi-\psi)^n$ etc.. Note that the $A_n, .., D_n$ all satisfy the same recurrence relation. The difference is in the initial conditions.

3. Now let $\varpi = (\varpi_j)$ and $\eta = (\eta_j)$, that is we have defined vectors of solutions. If $|\psi| < \frac{1}{25}$, and φ is such that both ϖ and η converge then there will be a matrix M, independent of φ, such that $\varpi = M\eta$. Let W_ϖ and W_η be the Wronskian matrices

$$
W_{\varpi\, jk} = \vartheta^k \varpi_j \quad \text{and} \quad W_{\eta\, jk} = \vartheta^k \eta_j \ .
$$

Note that

$$
W_\varpi = M W_\eta \ .
$$

For the case $\psi = \frac{1}{50}$, calculate M numerically, to machine precision.

4. A little analytical work: Let y denote any vector of solutions to the differential equation and let W_y be the corresponding Wronskian. By considering the columns of ϑW_y show that

$$
W_y^{-1}\vartheta W_y = \begin{pmatrix} 0 & 0 & 0 & -S_0/S_4\\ 1 & 0 & 0 & -S_1/S_4\\ 0 & 1 & 0 & -S_2/S_4\\ 0 & 0 & 1 & -S_3/S_4 \end{pmatrix} .
$$

Now by means of the formulas, valid for a generic matrix A,

$$
\det A = \exp\left(\operatorname{Tr}\log A\right) \quad \text{and its consequence} \quad \vartheta \det A = (\det A)\ \operatorname{Tr}\left(A^{-1}\vartheta A\right) \ ,
$$

you can now show that

$$
\det W_y = \exp\left(-\int \frac{\mathrm{d}\varphi}{\varphi}\frac{S_3}{S_4}\right) = \frac{w_y}{S_4^2(\varphi)} \ ,
$$

where w_y is a constant that depends on the basis y.

Compute w_ϖ and w_η. Note that one can use this relation for $\det W$ to check on the accuracy of a numerical computation of W. Write a function that computes W to a given accuracy, using this check to increase `nmax` as necessary.

5. Now given a sequence of φ values $\{\psi_0 = 0, \psi_1, \psi_2, \ldots, \psi_f\}$ such that each ψ_{i+1} lies in a disk such that $\eta(\varphi, \psi_i)$ converges for $\varphi = \psi_{i+1}$. Write a code that evaluates $W_\varpi(\psi_f)$ to a specified accuracy.

   One way to choose the ψ_j proceeds as follows. Suppose $\psi_0 = 0$ and that the initial radius of convergence is r_0 $(= 1/25)$ and that there are no singularities in $\mathrm{Re}\,\varphi \leqslant 0$. Choose $\psi_1 = -r_0/2$. The new radius of convergence runs far as the singularity at $\varphi = r_0$, so is $r_1 = 3r_0/2$. Choose ψ_2 half way between the origin and the edge of the region of convergence so $\psi_2 = r_0 - r_1$, and so on. In this way we get a sequence of circles, that all pass through $\varphi = r_0$ and have radius $r_n = (\frac{3}{2})^n$, and we can take

$$\psi_n \;=\; r_0 - \left(\frac{3}{2}\right)^n r_0 \;.$$

6. Evaluate $\varpi(-1/7)$ to say 200 figures. Evaluate also $L(1)$ and $L(2)$ to the same accuracy, where $L(s)$ is the function introduced in Problem sheet 2, and seek linear relations between each of the quantities $\varpi_j(-1/7)/\pi^j$, $0 \leq j \leq 2$ and the quantities $L(k)/\pi^k$, $k = 1, 2$, using the function FindIntegerNullVector. The quantity ϖ_3 is special, and you should seek a linear relation between $\varpi_3(-1/7)/\pi^3$, $\zeta(3)\varpi_0(-1/7)/\pi^3$ and the $L(k)/\pi^k$.