Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS TAKE HOME EXAM

Hilary Term 2020

Tuesday, 17th March 12 noon 2020, to Thursday, 19th March 12 noon 2020

You should submit answers to all questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. This question explores the physics of magnetic mirror confinement systems. We consider the confinement of particles within a cylindrical volume of radius a and length $2L$.

   (a) [5 marks] Consider a magnetic field of the form

$$\mathbf{B} = B_r \hat{\mathbf{r}} + B_z \hat{\mathbf{z}}, \tag{1}$$

   with

$$B_r = 2B_0 b \frac{rz}{L^2}\left(\frac{z^2}{L^2} - 1\right), \quad B_z = B_0\left(1 + 2b\frac{z^2}{L^2} - b\frac{z^4}{L^4}\right), \tag{2}$$

   for $b > 0$, $-L < z < L$, and $0 < r < a$. Here we express $\mathbf{B}$ in terms of the usual cylindrical coordinates (r, θ, z). Show that $\mathbf{B}$ may be written in the form

$$\mathbf{B} = \frac{\nabla\psi \times \hat{\boldsymbol{\theta}}}{r}, \tag{3}$$

   where $\psi(r, z)$ is a function to be determined. Sketch the form of B_z for $-L < z < L$ and sketch the contours of $\psi(r, z)$ for $-L < z < L$, and $0 < r < a$.

   (b) [5 marks] Assume that there is an externally imposed electric field $\mathbf{E} = -\nabla\phi$, where $\phi = \phi(\psi)$. Take the size of the electric potential $\phi \sim T/e\rho_*$, where T is the temperature of the particles, e is the unit charge, and ρ_* is the ratio of the thermal gyroradius to the length scale a. Using the leading-order equations for the motion of particle guiding centres; show that the externally imposed $\mathbf{E}$ drives an $\mathbf{E} \times \mathbf{B}$ velocity $\mathbf{v}_E = r\Omega\hat{\boldsymbol{\theta}}$, with $\Omega = d\phi/d\psi$; and show that ψ is conserved along the paths followed by particle guiding centres.

   (c) [5 marks] Show that single particle motion conserves an additional quantity

$$U = \frac{mv_\parallel^2}{2} + m\mu B - \frac{m|\mathbf{v}_E|^2}{2}, \tag{4}$$

   where m is the particle mass, $v_\parallel$ is the particle velocity in the direction parallel to the magnetic field, μ is the magnetic moment, and B is the magnetic field strength.

   (d) [5 marks] Find $U(\psi, z, v_\parallel, \mu)$ for a magnetic mirror with $a \ll L$ and $a\Omega \sim \sqrt{2T/m}$. Decompose $U = mv_\parallel^2/2 + m\Phi(\psi, z, \mu)$, where $\Phi(\psi, z, \mu)$ is an effective potential, and hence describe the trajectories of particles within the magnetic mirror, for a given $\mu \neq 0$. What are the minimum and maximum values of U for particles that are confined to remain in the region $-L < z < L$? Sketch $\Phi(\psi, z, \mu)$ for fixed ψ and $\mu \neq 0$. *[Hint: Use the expansion in $a/L \ll 1$ to obtain an approximate expression for B which is a function of z only.]*

   (e) [5 marks] Using your results from part (d), consider the confinement of particles with $\mu = 0$. Show that when $\Omega = 0$ particles with $\mu = 0$ and $v_\parallel \neq 0$ are not confined to the region $-L < z < L$. Find the function $v_{\parallel,\max}(\mu, \psi)$ such that particles with $v_\parallel(z = 0) < v_{\parallel,\max}(\mu, \psi)$ are confined. Hence, show that imposing $\Omega \neq 0$ confines particles with $\mu = 0$ and $v_\parallel(z = 0) < v_{\parallel,\max}(\mu = 0, \psi)$. Comment on this result.

   (f) [10 marks] Use the drift kinetic equation to find the distribution function $\langle f\rangle_\varphi$ for particles in the magnetic mirror, assuming that unconfined particles are lost to the system. For the confined piece of velocity space, take the boundary condition for the problem to be

$$\langle f\rangle_\varphi(v_\parallel, \mu, r, z = 0) = n_0\left(\frac{m}{2\pi T_0}\right)^{3/2} \exp\left[-\frac{mv_\parallel^2}{2T_0} - \frac{m\mu B(r, z = 0)}{T_0}\right], \tag{5}$$

where n_0 and T_0 are constant densities and temperatures, respectively. Assume that $a/L \ll 1$ and $a\Omega \sim \sqrt{2T_0/m}$. Show that the density of confined particles is given by

$$n(r,z) = n_0 \left(\frac{m}{2\pi T_0}\right)^{3/2} 2\pi \int_{-\infty}^{\infty} dv_\parallel \int_{\mu_{\min}(\psi,z,v_\parallel)}^{\infty} d\mu B(r,z) \exp\left[-\frac{mv_\parallel^2}{2T_0} - \frac{m\mu B(r,z)}{T_0}\right] N(r,z), \tag{6}$$

with

$$N(r,z) = \exp\left[\frac{mr^2\Omega^2(1-f(z))}{T_0}\right], \tag{7}$$

and where $\mu_{\min}(\psi, z, v_\parallel)$ and $f(z)$ are functions to be determined. Evaluate the integral for $n(r,z)$ in the limits $a\Omega \ll \sqrt{2T_0/m}$ and $a\Omega \gg \sqrt{2T_0/m}$, and comment on the results. *[Hint: To obtain $\mu_{\min}$, use your expression from part (d) for the maximum value of U for particles that are confined to the region $-L < z < L$.]*

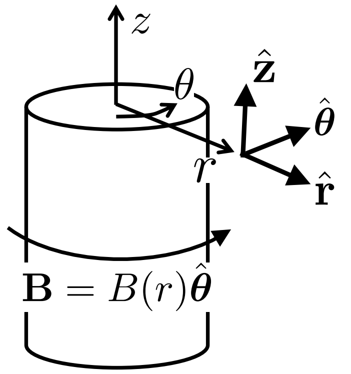

Figure 1: A diagram showing z-pinch geometry, and cylindrical coordinates.

2. Consider a low-β, electrostatic plasma confined in a z-pinch geometry, illustrated in figure 1 with the standard cylindrical coordinates (r, θ, z). Assume that the plasma contains only 2 species, ions with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e. Take the magnetic field to be $\mathbf{B} = B(r)\hat{\boldsymbol{\theta}}$. Assume that the plasma has reached a time-independent equilibrium, with an equilibrium distribution function for the species s of the form

$$f_M(r, v_\parallel, \mu) = \frac{n_s(r)}{\pi^{3/2}} \left(\frac{m_s}{2T_{\parallel,s}(r)}\right)^{1/2} \frac{m_s}{2T_{\perp,s}(r)} \exp\left[-\frac{m_s v_\parallel^2}{2T_{\parallel,s}(r)} - \frac{m_s \mu B(r)}{T_{\perp,s}(r)}\right], \tag{8}$$

where $v_\parallel$ is the particle velocity in the direction parallel to the magnetic field, μ is the magnetic moment. Take the equilibrium electric field to vanish, i.e., $\mathbf{E} = 0$.

(a) [2 marks] Write down the condition which $\mathbf{B}$ satisfies if $\mathbf{B}$ is sourced entirely by external magnets. Deduce the form of $B(r)$.

(b) [8 marks] Consider the stability of equilibrium distribution function in this geometry. Consider perturbations to the equilibrium distribution function $\delta\langle f\rangle_\varphi$ and electric potential $\delta\phi$ of the forms

$$\delta\langle f\rangle_\varphi(t, r, \theta, z, v_\parallel, \mu) = \widetilde{g}(r, v_\parallel, \mu) \exp\left[\mathrm{i}M\theta + \mathrm{i}kz - \mathrm{i}\omega t\right], \tag{9}$$

and

$$\delta\phi(t, r, \theta, z) = \widetilde{\phi}(r) \exp\left[\mathrm{i}M\theta + \mathrm{i}kz - \mathrm{i}\omega t\right], \tag{10}$$

respectively. Find the linearised, low-flow drift kinetic equation for these fluctuations in terms of $\widetilde{g}$ and $\widetilde{\phi}$. Write down the linearised form of the quasineutrality equation.

(c) [5 marks] For $M \neq 0$ fluctuations consider the ordering

$$\frac{kT}{\omega r e B} \sim \sqrt{\frac{T}{m_i}}\frac{M}{r} \ll \frac{\omega}{\omega'_{*,e}} \sim \frac{\omega}{\omega'_{*,i}} \sim \eta \sim 1 \ll \sqrt{\frac{T}{m_e}}\frac{M}{r}, \tag{11}$$

where $\omega'_{*,s} = T_{\parallel,s}k/ZeBL_{n_s}$, with $L_{n_s} = -dr/d\ln n_s$, and where

$$T \sim T_{\perp,s} \sim T_{\parallel,s}, \qquad \text{and } \eta \sim \eta_{\parallel,s} = d\ln T_{\parallel,s}/d\ln n_s \sim \eta_{\perp,s} = d\ln T_{\perp,s}/d\ln n_s. \tag{12}$$

Show that the ordering (11) yields waves which travel with the phase velocity

$$\mathbf{v}_{*,e} = -\frac{T_{\parallel,e}}{en_e}\frac{\hat{\mathbf{b}}}{B} \times \nabla n_e. \tag{13}$$

Use the dispersion relation which you obtain to give a physical picture for these waves. *[Hint: Find a fluid equation for δn_i. Plot the amplitude of the density fluctuation with z at fixed θ, and consider how this fluctuation evolves with time.]*

(d) [8 marks] For $M \neq 0$ fluctuations show that the ordering

$$\frac{kT}{\omega reB} \sim \eta^{-1} \ll \sqrt{\frac{T}{m_i}}\frac{M}{r} \sim \eta^{-1/2} \ll \frac{\omega}{\omega'_{*,e}} \sim \frac{\omega}{\omega'_{*,i}} \sim 1 \ll \eta \ll \sqrt{\frac{m_i}{m_e}} \tag{14}$$

in the low-flow drift kinetic system of equations yields waves which satisfy the dispersion relation

$$1 - \frac{\omega'_{*,e}}{\omega} + \left(\frac{\omega'_{*,e}}{\omega}\right)^2 \frac{L_n(T_{\perp,i}\eta_{\perp,i} + T_{\parallel,i}\eta_{\parallel,i})}{ZT_{\parallel,e}r} - \frac{\omega'_{*,e}}{\omega}\left(\frac{M}{r\omega}\right)^2 \frac{T_{\parallel,i}\eta_{\parallel,i}}{m_i} = 0. \tag{15}$$

[Hint: first perform an expansion in $\sqrt{m_e/m_i}$ to find expressions for $\widetilde{g}_e$ and $\widetilde{g}_i$. Then, perform the expansion in η to simplify the expression for $\widetilde{g}_i$ needed to compute the dispersion relation.]

(e) [2 marks] Solve the dispersion relation (15) in the subsidiary limit that $T_{\perp,s} = T_{\parallel,s} = T_s$ and

$$\frac{T_i\eta_i}{m_i}\left(\frac{M}{r\omega'_{*,e}}\right)^2 \sim \frac{L_n\eta_i T_i}{ZT_e r} \sim \epsilon^{-2} \gg \frac{\omega}{\omega'_{*,e}} \sim \epsilon^{-1} \gg 1. \tag{16}$$

Find the leading-order non-zero expressions for the real and imaginary parts of ω.

(f) [5 marks] Using your results from (d), construct a physical picture for the instability found in (e). Again, assume the subsidiary ordering (16) and take $T_{\perp,s} = T_{\parallel,s} = T_s$. By taking moments, or otherwise, obtain leading-order expressions for the fluctuating ion temperature $\widetilde{T}_i$ and density $\widetilde{n}_i$. Convert these equations into moment equations of the form

$$\frac{\partial \delta T_i}{\partial t} + \delta\mathbf{v}_1 \cdot \nabla f_1 = 0, \tag{17}$$

and

$$\frac{\partial \delta n_i}{\partial t} + \delta\mathbf{v}_2 \cdot \nabla f_2 = 0, \tag{18}$$

and identify $\delta\mathbf{v}_1$, $\delta\mathbf{v}_2$, f_1 and f_2. Using these moment equations describe the physical picture for the growth of the instability. *[Hint: Plot the amplitude of a temperature fluctuation with z at fixed θ, and consider how this fluctuation evolves with time.]*

3. Consider an inhomogeneous plasma consisting of electrons, with mass m_e and charge $-e$, and ions, with mass m_i and charge Ze, in a cartesian slab geometry described by the cartesian coordinates (x, y, z). The plasma is magnetised by an externally imposed magnetic field $\mathbf{B} = B(x)\hat{\mathbf{z}}$. Assume that $dB/dx > 0$ everywhere. The equilibrium plasma density is a function of x only, i.e., $n_e = n_e(x)$, and $dn_e/dx > 0$ within the plasma. An antenna is positioned at $x = x_0$, and is able to launch waves towards $x > x_0$. Assume throughout that $Zm_e/m_i \ll 1$.

   (a) [3 marks] Write down the cold plasma dispersion relation for waves that propagate in a homogeneous plasma and homogeneous background magnetic field with a wave vector $\mathbf{k}$ that is perpendicular to the magnetic field direction $\hat{\mathbf{b}}$. Assume that the wave frequency ω is of order the plasma frequency ω_{pe} and of order the electron cyclotron frequency Ω_e, i.e., $\omega \sim \omega_{pe} \sim \Omega_e$. Show that for waves with polarisation vectors $\mathbf{e}$ perpendicular to the magnetic field direction, i.e., $\mathbf{e} \cdot \hat{\mathbf{b}} = 0$, the wave number $k = |\mathbf{k}|$ satisfies

$$kc = \sqrt{\frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{(\omega^2 - \omega_{UH}^2)}}, \tag{19}$$

   with ω_R, ω_{UH}, and ω_L frequencies to be determined. Prove that ω_R, ω_{UH}, and ω_L satisfy the inequality $\omega_R > \omega_{UH} > \omega_L$ when the plasma frequency $\omega_{pe} = \sqrt{e^2 n_e/m_e \epsilon_0} \gtrsim \Omega_e = eB/m_e$. Show that the polarisation $\mathbf{e}$ satisfies

$$\mathbf{e} = \mathrm{i}g\hat{\mathbf{k}} - \epsilon_\perp \hat{\mathbf{b}} \times \hat{\mathbf{k}}, \tag{20}$$

   where $\hat{\mathbf{k}} = \mathbf{k}/k$, and g and $\epsilon_\perp$ should be determined. Modes with a dispersion relation (19) and polarisation given by (20) are called X modes.

   (b) [5 marks] Assume now that the antenna launches an X mode into the inhomogeneous plasma with $\mathbf{k} = k_0\hat{\mathbf{x}}$, $k_0 > 0$, at $x = x_0$. Assume that the frequency of the wave ω satisfies $\omega > \omega_R(x_0)$, and assume that $kL \gg 1$, with $L^{-1} \sim d\ln\Omega_e/dx \sim d\ln\omega_{pe}/dx$. Use the ray tracing equation to show that the ray propagates in the $\hat{\mathbf{x}}$ direction. Hence, or otherwise, determine $\mathbf{k}$ along the path of the ray. A reflected wave will be received by the antenna. Determine the position x_c where the wave reflects.

   (c) [8 marks] Use the WKB ansatz for the wave to show that in the region $x_0 \leqslant x < x_c$ the leading-order solution for the wave electric field $\delta\mathbf{E}$ is

$$\delta\mathbf{E} = A(x)\mathbf{e}\exp\left[\mathrm{i}\int_{x_0}^{x} |k_x(x')|dx' - \mathrm{i}\omega t\right] + R(x)\mathbf{e}\exp\left[-\mathrm{i}\int_{x_0}^{x} |k_x(x')|dx' - \mathrm{i}\omega t\right], \tag{21}$$

   where $k_x = \mathbf{k} \cdot \hat{\mathbf{x}}$, and $A(x)$ and $R(x)$ are functions to be determined. You may assume that the amplitude of the wave launched at $x = x_0$, $A(x_0)$, is known. Give a physical interpretation to the equations which determine $A(x)$ and $R(x)$. Explain why the solution (21) contains one undetermined complex constant, and explain why this solution breaks down at $x = x_c$.

   (d) [7 marks] Argue that to complete the leading-order solution (21) we need to solve the general cold plasma equation for the wave electric field near $x = x_c$. Argue that near $x = x_c$ we should assume that

$$\delta\mathbf{E} = E(x)\mathbf{e}\exp[-\mathrm{i}\omega t], \tag{22}$$

   and show that $E(x)$ is determined by

$$\frac{c^2}{\omega^2}\frac{d^2E(x)}{dx^2} + \left.\frac{dF(x)}{dx}\right|_{x=x_c}(x - x_c)E(x) = 0, \tag{23}$$

   with $F(x) = \epsilon_\perp - g^2/\epsilon_\perp$.

(e) [8 marks] Solve equation (23) assuming non-diverging solutions, and match the solution (22) to the solution (21) found in (c). Show that the ratio between the complex amplitudes of the reflected and launched wave at $x = x_0$ is

$$\frac{R(x_0)}{A(x_0)} = \exp\left[\mathrm{i}\Psi(x_c)\right] = \exp\left[2\mathrm{i}\int_{x_0}^{x_c} |k_x(x')|dx' - \mathrm{i}\frac{\pi}{2}\right] \tag{24}$$

[Hint: Limits of special functions that are proved in the appendices of the printed lecture notes may be used in your solution without proof, provided that the result is clearly stated and referenced.]

(f) [4 marks] Describe the qualitative relationships between $\Psi = \Psi(x_c)$, x_c, and ω. Assuming that $\Psi(x_c)$ could be inverted to obtain $x_c(\Psi)$, what could be experimentally determined by a measurement of Ψ? Briefly comment on why it is desirable to launch a wave with $\omega > \omega_R(x_0)$, as we assumed in part (b).