String Theory I: Problem Sheet 4

Hilary Term 2020

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Please submit your solutions to **Problem 1** *only*. Problem 2 is for your own consideration and may be discussed in class. You need not produce a detailed solution.

This is a *long* problem sheet, and I don't expect that most people will finish it. However, many of these are classic problems, so I am still including them all! You should feel free to make liberal use of references where these calculations are at least sketched and pick your battles when it comes to carrying things out to the bitter end.

1. Low energy effective theory in spacetime

The conditions for Weyl invariance of the bosonic string sigma model at leading order in the α' expansion take the form $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$ with

$$\begin{aligned}
\beta^G_{\mu\nu} &= R_{\mu\nu} + 2D_\mu D_\nu \Phi - \frac{1}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} \ , \\
\beta^B_{\mu\nu} &= -\frac{1}{2} D^\lambda H_{\lambda\mu\nu} + D^\lambda \Phi H_{\lambda\mu\nu} \ , \\
\beta^\Phi &= \frac{D-26}{6} + \alpha' \left(-\frac{1}{2} D^2 \Phi + D_\lambda \Phi D^\lambda \Phi - \frac{1}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) \ .
\end{aligned}$$

In this problem you will check a number of good features of these equations, which we interpret as equations of motion for the spacetime fields G, B, and Φ.

1. Show that these Weyl invariance conditions are equivalent to the Euler-Lagrange equations for the spacetime effective action

   $$S_{26} = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[-\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \Phi \partial^\mu \Phi + O(\alpha') \right] \ .$$

   [*It will be helpful for you to change the action from string frame to Einstein frame. You will need to use a formula for the transformation of the Ricci curvature under a Weyl rescaling to go back to string frame when all is said and done. You should feel free to look up this formula on Wikipedia or in your favorite differential geometry text.*]

2. The conditions, $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = 0$ imply (one-loop) conformal invariance of the sigma model on a flat worldsheet. Show that for a solution of these equations, one has automatically that β^Φ is constant.

   [*You will need to massage these equations a bit to get what you need. You should use that* $\beta^G_{\mu\nu} = 0 \Rightarrow D^\mu \beta^G_{\mu\nu} = 0$, *as well as the Bianchi identities for the Ricci curvature and* H*-field,*

   $$D^\mu R_{\mu\nu} = \frac{1}{2} D_\nu R \ , \qquad H_{\mu\nu\lambda} D^\mu H^{\rho\nu\lambda} = \frac{1}{6} D^\rho H^2$$

   This result is an important consistency check since in this case β^Φ *should be identified with the central charge of the conformal field theory.*]

3. Consider the simple "linear dilaton" background,

   $$G_{\mu\nu}(X) = \eta_{\mu\nu} \ , \qquad B_{\mu\nu}(X) = 0 \ , \qquad \Phi(X) = V_\mu X^\mu \ .$$

This background can solve the Weyl invariance conditions with $D < 26$, even though on a flat worldsheet it is identical to the usual free string action.

Derive this as an exact statement (as opposed to at leading order in α') by computing the stress tensor and Virasoro central charge for the (free!) linear dilaton theory.

[*You will need to incorporate the dilaton coupling when computing the flat-space stress tensor, since the stress tensor is obtained by taking a variation with respect to the worldsheet metric.*]

4. What is the on-shell three-point dilaton vertex implied by the spacetime action S_{26}? Verify that the result agrees with the worldsheet worldsheet computation.

   [*This is a tedious world-sheet computation, but the result is very simple in the end.*]

2. Circle compactification of the bosonic string

In lecture we (will have) considered compactification of the 26-dimensional closed bosonic string on a spacetime circle of radius R: $X^{25} \sim X^{25} + 2\pi R$. In this problem you will work out some details and extensions of that story.

1. At generic compactification radius, the massless spectrum of physical string states matches our expectation from dimensional reduction of the particle spectrum of the low energy effective theory.

   Following the discussion in Polchinski Volume 1 Section 8.1, perform the dimensional reduction of the *action* S_{26} to 25 dimensions (*i.e.*, take all 26-dimensional fields to be independent of the X^{25} coordinate and rewrite the action in terms of 25-dimensional fields).

   [*If you endeavour to do this calculation honestly, you will have to find the dimensional reduction of the Ricci scalar. This is not easy to derive, it is probably best done in a vielbein formalism. The final result, however, takes the form*

$$R_{26} = R_{25} - 2e^{-\sigma} D^2 e^{\sigma} - \frac{1}{4} e^{2\sigma} F^2$$

   where $G_{25,25} = e^{2\sigma}$ *and* F *is the graviphoton field strength.*]

2. Show that for the special choice $R^2 = 1/2$ in string units (*i.e.*, $R = \sqrt{\alpha'}$) there are additional massless states in the physical string spectrum. What are their space-time quantum numbers? How can we interpretation these additional massless states?

3. The Weinberg soft photon theorem guarantees that in the limit of zero photon momentum, the three-point vertex between a photon and any particle is proportional to the charge of that particle. Using vertex operators, show that the charge of a state as measured by the graviphoton and the B-field photon is, respectively, the Kaluza-Klein momentum and the winding number.

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