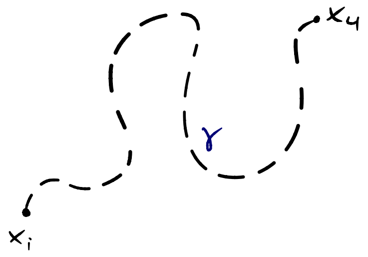
(Classical) Relativistic Point Particle

$$S = -m \int_\gamma ds$$



Treat as "0+1-dimensional field theory", τ local time parameter on worldline. $x^\mu(\tau) : \mathbb{R}_\tau \to \mathbb{R}$ worldline fields.

$$S[x^\mu(\tau)] = -m \int_{\tau_i}^{\tau_f} \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-h}$$

h induced (pull back) metric on $\mathbb{R}_\tau$

Euler - Lagrange equations $\longrightarrow$ trajectories are geodesics.

Symmetries of the action...

- if $g_{\mu\nu} = \eta_{\mu\nu}$ (flat Minkowski metric) then spacetime Poincaré invariance

$$x^\mu(\tau) \to \Lambda^\mu{}_\nu x^\nu(\tau) + b^\mu \quad \text{for } \Lambda \in SO(1, d-1), \; b \in \mathbb{R}$$

More generally, spacetime isometry group is realized as "internal" symmetries on the worldline theory.

- 1-dim'l reparameterization invariance

$$\tau \to \tilde{\tau}(\tau)$$

$$x^\mu(\tau) \to \tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau)$$

this is because action is really a function of worldline as a subset of spacetime; don't care about parameterization.

THIS IS A GAUGE SYMMETRY / REDUNDANCY OF DESCRIPTION.

This is a fine classical action, and can be analyzed as usual. Not very appealing if we want to quantize.

(Classical) Relativistic Point Particle

Introduce "einbein" $e : \mathbb{R}_\tau \to \mathbb{R}_{>0}$, a "square root of worldline metric", with new action:

$$\tilde{S} = \frac{1}{2}\int \left(e^{-1} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - e m^2 \right) d\tau$$

Imposing equations of motion for e gives $e^2 = \frac{-1}{m^2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$. Substitute back into $\tilde{S}$ to get S, so classically equivalent.

Symmetries of $\tilde{S}$...

- spacetime Poincaré/isometries ($e(\tau)$ invariant)
- reparameterization of worldline

$$\tau \mapsto \tilde{\tau}(\tau) \qquad\qquad \tau \mapsto \tau - \xi(\tau) \text{ for } \xi \text{ infinitesimal}$$

$$x^\mu(\tau) \mapsto \tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau) \qquad\qquad x^\mu \mapsto x^\mu + \xi \frac{dx^\mu}{d\tau}$$

$$e(\tau) \mapsto \tilde{e}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}} e(\tau) \qquad\qquad e \mapsto e + \xi \frac{de}{d\tau} + \frac{d\xi}{d\tau} e$$

Can use reparametrization to gauge fix and set $e(\tau) \equiv 1/m$.

$$\tilde{S}_{\text{fixed}} = \frac{1}{2}\int \left(m g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - m \right) d\tau$$

Equations of motion give geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Now there is an additional constraint from the einbein equation of motion

$$g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 1 = 0 \qquad \left(\text{so } \frac{dx^\mu}{d\tau} \text{ is 4-velocity; } \tau \text{ is proper time}\right).$$

Alternatively, if $m=0$ can gauge fix $e=1$ and recover geodesic equation plus null constraint

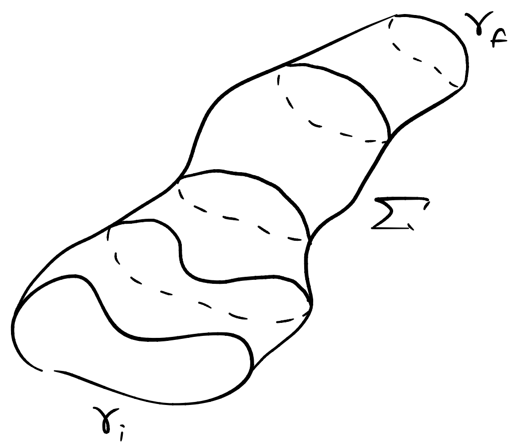
$$g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad (\text{so null geodesic}).$$

This would be a reasonable starting point for quantization (quadratic in time derivatives, etc.). Could even imagine starting to add interactions, building up Feynman diagrams in a first-quantized setting.

(Classical) Relativistic String

Analogue of first action

$$S[\Sigma] = -T\int_\Sigma dA$$



Introduce worldsheet parameters (σ,τ), treat as 1+1-dimensional fieldtheory with fields $X^\mu(\tau,\sigma):\Sigma\to\mathbb{R}$

$$S_{NG}[X^\mu(\tau,\sigma)] = -T\int_\Sigma \sqrt{-(\partial_\tau X\cdot\partial_\tau X)(\partial_\sigma X\cdot\partial_\sigma X)+(\partial_\tau X\cdot\partial_\sigma X)^2} = -T\int_\Sigma \sqrt{-h}\, d\tau\, d\sigma$$

Nambu-Goto action

$\boxed{V\cdot W = g_{\mu\nu}V^\mu W^\nu \text{ for spacetime vectors}}$

Euler-Lagrange equations $\longrightarrow$ minimal area surfaces.

T interpreted as string tension

$$\left\{\begin{array}{l} \text{Set } X^0=\tau,\ \partial_\tau X^{\mu\neq 0}=0,\ g_{0\mu}=0 \text{ for } \mu\neq 0, \text{ so static string in static geometry} \\ S = -T\int d\tau\, d\sigma \sqrt{\partial_\sigma X\cdot\partial_\sigma X} = -T\int d\tau\ (\text{string length}) \\ \qquad = \int d\tau\ (\text{Kinetic} - \text{Potential}), \text{ so potential energy} \sim T\times\text{length} \end{array}\right\}$$

Symmetries of the Nambu-Goto action:

- spacetime Poincaré / isometries as internal symmetries on worldsheet

$$X^\mu(\tau,\sigma)\longmapsto \Lambda^\mu{}_\nu X^\nu(\tau,\sigma)+b^\mu$$

- 2-dim'l reparameterization invariance (often referred to as diffeomorphism invariance)

$$(\tau,\sigma)\to(\tilde\tau(\tau,\sigma),\tilde\sigma(\tau,\sigma))$$

$$X^\mu(\tau,\sigma)\to \tilde X^\mu(\tilde\tau,\tilde\sigma)=X^\mu(\tau,\sigma)$$

Again, reparameterizations are a gauge symmetry.

Once again, a fine classical theory (see problem sheet 1). Not clear how to quantize.

(Classical) Relativistic String

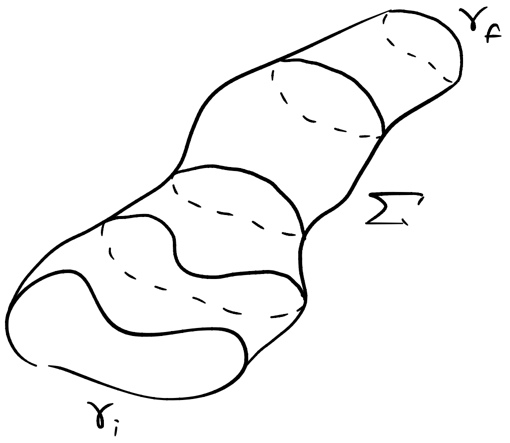
Analogue of second action

$$S_p[\gamma_{ab}, X^\mu] = -\frac{T}{2}\int_\Sigma d\tau d\sigma \sqrt{-\gamma}\, \gamma^{ab} \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X_\mu}{\partial \xi^b}$$

auxiliary Lorentzian world-sheet metric

det (γ_{ab})

Polyakov Action (Brink-di Vecchia-Howe-Deser-Zumino)



As in point particle case, prove classical equivalence by solving E.O.M. for worldsheet metric γ_{ab}.

Symmetries of the Polyakov action

- spacetime Poincaré (γ doesn't transform)
- worldsheet reparameterizations

$$\gamma_{ab}(\xi) \to \tilde{\gamma}_{ab}(\tilde{\xi}) = \gamma_{cd}(\xi)\frac{\partial \xi^c}{\partial \tilde{\xi}^a}\frac{\partial \xi^d}{\partial \tilde{\xi}^b} \qquad (\xi = (\sigma, \tau))$$

- Weyl invariance (local scale transformations).

$$\gamma_{ab} \mapsto e^{2\omega(\sigma,\tau)}\gamma_{ab}$$

special to 2-dimensional worldvolume; $\sqrt{|\gamma|}$ picks up factor of $\exp(D\omega(\xi))$ while γ^{ab} picks up $e^{-2\omega(\xi)}$.

worldvolume dimensions (D=2 for string)

Weyl transformations are now also a gauge symmetry, so $D=2$ has extra gauge invariance. This is, in a way, what is special about strings (vs. membranes, extended objects of greater dimension, etc).

Should consider what power-counting renormalizable terms we could add to the action consistent with symmetries.

$$S^{(1)} = \lambda \int_\Sigma \sqrt{-\gamma}\, d\tau d\sigma$$

"cosmological constant" term $\longrightarrow$ not Weyl invariant.

$$S^{(2)} = \frac{1}{2\pi}\int \sqrt{-\gamma}\, R^{(2)}(\gamma)\, d\tau d\sigma$$

worldsheet Ricci scalar

"Einstein-Hilbert term" $\longrightarrow$ total derivative in 2d, no effect locally.
(2d gravity is "trivial")

A note on dimensional analysis: worldsheet & spacetime both have notion of length scale/units.

worldsheet: $[X^\mu] = 0$
$[\frac{d}{d\xi}] = 1$
$[\gamma] = 0$
$[T] = 0$
$[d\tau d\sigma] = -2$

spacetime: $[X^\mu] = -1$
$[\frac{d}{d\xi}] = 0$
$[\gamma] = 0$
$[T] = 2$
$[d\tau d\sigma] = 0$

α' will play role of $\hbar$ on the world sheet

Conventional to write $T = \frac{1}{2\pi\alpha'}$, $[\alpha'] = -2$

Gauge Fixing the Polyakov String

Use reparameterization & Weyl gauge invariances to simplify the action further.

Reparam. : $\gamma^{ab} \to e^{2\omega(\tau,\sigma)}\eta^{ab}$ "conformal gauge" $\left\{\begin{array}{l}\text{naive counting: 2 functions of}\\ \text{coordinates in reparam., setting}\\ \text{two metric entries to zero}\end{array}\right\}$

$\eta = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

Weyl : $e^{2\omega(\tau,\sigma)}\gamma^{ab} \to \gamma^{ab}$ "unit gauge" $\left\{\begin{array}{l}\text{one more function to cancel; this}\\ \text{one clearly works (not naive).}\end{array}\right\}$

Comments: room for some concern here; can we do these gauge fixings globally on Σ or just locally? For γ^{ab} Lorentzian/Semi-Riemannian, things could get pretty messy.

The secret is that ultimately when the global structure of Σ is relevant, we will "Wick rotate" to Euclidean signature on the worldsheet. Here the story is cleaner (though still interesting). For now we press on.

The Polyakov action in conformal gauge simplifies dramatically

$$S_P^{\text{conf. gauge}}[X^\mu] = -\frac{T}{2}\int d\tau d\sigma \left(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right)$$

This is just the theory of D massless scalar fields in flat 1+1-dimensional space. (One with wrong-sign kinetic term)

In addition, supplement with constraints from equations of motion for worldsheet metric.

$$T_{ab} := -\frac{2}{T}\cdot\frac{1}{\sqrt{-\gamma}}\frac{\delta S_P}{\delta\gamma^{ab}} \stackrel{!}{=} 0$$

world-sheet stress tensor

E.O.M. constraint

$$T_{ab} = \partial_a X\cdot\partial_b X - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X\cdot\partial_d X \quad \text{in conformal gauge.}$$

$$\left.\begin{aligned}T_{\tau\tau} &= \tfrac{1}{2}\partial_\tau X\cdot\partial_\tau X + \tfrac{1}{2}\partial_\sigma X\cdot\partial_\sigma X\\ T_{\tau\sigma} &= \partial_\tau X\cdot\partial_\sigma X\\ T_{\sigma\sigma} &= \tfrac{1}{2}\partial_\tau X\cdot\partial_\tau X + \tfrac{1}{2}\partial_\sigma X\cdot\partial_\sigma X\end{aligned}\right\}\ \gamma^{ab}T_{ab} = T_{\sigma\sigma} - T_{\tau\tau} = 0$$

(this is due to Weyl invariance)

So only two constraints to impose, with third satisfied identically due to Weyl symmetry.

Gauge Fixing the Nambu-Goto String

Note for good measure, can use reparameterizations to fix Nambu-Goto action to conformal gauge as well.

Induced metric $h_{ab} = \partial_a X \cdot \partial_b X$

Fix $h_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X = 0$

$$h_{\tau\tau} + h_{\sigma\sigma} = \partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X = 0$$

Use constraints to rewrite action

$$S_{NG} = -T \int \sqrt{-h} = -T \int_\Sigma \sqrt{(\partial_\sigma X \cdot \partial_\tau X)^2 - (\partial_\tau X \cdot \partial_\tau X)(\partial_\sigma X \cdot \partial_\sigma X)}\, d\tau d\sigma$$

$$= -T \int_\Sigma \sqrt{0 - \tfrac{1}{2}(\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X) \cdot \tfrac{1}{2}(\partial_\sigma X \cdot \partial_\sigma X - \partial_\tau X \cdot \partial_\tau X)}\, d\tau d\sigma$$

$$= \frac{-T}{2} \int_\Sigma d\tau d\sigma \left(-\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X \right)$$

So subject to some constraints, recover same simple free field action.

(More on NG action on problem sheet 1)

Plan for next week: General classical solutions

Constraints

Residual gauge symmetry & conformal transformations

Quantization