Polyakov action in conformal gauge $S_P^{\text{conf. gauge}}[X^\mu] = -\frac{T}{2}\int d\tau d\sigma \left(\partial_a X \cdot \partial^a X\right)$

$\quad\quad (-\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X)$

This action is supplemented by E.O.M. for γ_{ab}: $T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma^{ab}} = 0$

Evaluating stress-energy tensor in conformal gauge: $T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2}\eta_{ab}\partial_c X \cdot \partial^c X$

In components: $T_{\tau\tau} = \frac{1}{2}(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X)$

$$T_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X$$

$$T_{\sigma\sigma} = \frac{1}{2}(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X)$$

Note $\eta^{ab}T_{ab} = T_{\sigma\sigma} - T_{\tau\tau} = 0$ (traceless stress tensor), due to Weyl invariance.

Aside on constraints

Here we see these constraints arising from gauge fixing (e.o.m. for metric that was fixed to conformal gauge). In a more careful treatment, we would recover these same constraints as "primary constraints" when performing the Legendre transform from Lagrangian to Hamiltonian formalism due to absence of kinetic terms for the worldsheet metric. This is similar to the appearance of Gauss' law in electromagnetism in Hamiltonian formalism.

Classical solutions

Equations of motion (from Euler-Lagrange) are just wave equations for the X^μ:

$$\frac{\partial}{\partial \xi^a}\left(\frac{\delta S}{\delta(\partial_a X^\mu)}\right) = T\left(\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu\right) = 0$$

We consider strings w/ finite spatial extent, so let $\sigma \in [0,\pi]$ and need to specify boundary conditions

$$\boxed{\begin{aligned}\frac{\delta S}{\delta x} &= \int\left(\frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta \partial x}\delta \partial x\right)\\ &= \int\left(\frac{\delta L}{\delta x} - \partial\left(\frac{\delta L}{\delta \partial x}\right)\right)\delta x + \left(\frac{\delta L}{\delta \partial x}\delta x\right)\Big|_{\text{boundaries}}\end{aligned}}$$

$$\frac{\delta \mathcal{L}}{\delta(\partial_a X^\mu)}\delta X^\mu \Big|_{\xi_a^-}^{\xi_a^+} = 0 \longrightarrow -T\partial_\sigma X^\mu \delta X^\mu \Big|_{\sigma=0}^{\sigma=\pi} = 0$$

There are a few natural choices (and some less natural):

- Periodic boundary conditions: $X^\mu(\tau,\pi) = X^\mu(\tau,0)$, $\partial_\sigma X^\mu(\tau,\pi) = \partial_\sigma X^\mu(\tau,0)$ } closed string, note π-periodicity! (thanks to GSW)

- Neumann boundary conditions: $\partial_\sigma X^\mu(\tau,\pi) = \partial_\sigma X^\mu(\tau,0) = 0$ } open string

- Dirichlet boundary conditions: $X^\mu(\tau,\pi) = c^\mu$, $X^\mu(\tau,0) = b^\mu$ } choice of b^μ's c^μ breaks spacetime Poincaré invariance. will come back to these; related to D-branes.

  $\quad$ constant spacetime vectors

General solution to wave equation is sum of "right-moving" and "left-moving" wave forms

$$X^\mu(\tau,\sigma) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$$

Closed String

X_R & X_L both separately periodic up to a zero mode, can expand in Fourier modes

$$X^\mu_R(\tau-\sigma) = \frac{x^\mu}{2} + \frac{\ell^2 p^\mu}{2}(\tau-\sigma) + \frac{i\ell}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\alpha^\mu_n}{n} e^{-2in(\tau-\sigma)}$$

ℓ is a spacetime length scale that we choose to be $\ell = (\pi T)^{-1/2}$.

$$X^\mu_L(\tau+\sigma) = \frac{x^\mu}{2} + \frac{\ell^2 p^\mu}{2}(\tau+\sigma) + \frac{i\ell}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\tilde{\alpha}^\mu_n}{n} e^{-2in(\tau+\sigma)}$$

X^μ real requires $x^\mu, p^\mu \in \mathbb{R}$, $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$, $\tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*$.

Open String

X_L and X_R related through boundary conditions at $\sigma = 0$ and $\sigma = \pi$. Short calculation gives

$$X^\mu(\tau,\sigma) = x^\mu + \ell^2 p^\mu \tau + i\ell \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\alpha^\mu_n}{n} \cos(n\sigma)\, e^{-in\tau}$$

Reality conditions are similar: $x^\mu, p^\mu \in \mathbb{R}$, $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$.

Poincaré Charges

Can express conserved charges $P^\mu, M^{\mu\nu}$ associated w/ spacetime translations and Lorentz transformations in terms of oscillators, these are then interpreted as momentum and angular momentum of the string in spacetime.

For spacetime translation in the X^μ direction, $\delta X^\mu = \epsilon^\mu$ we compute the Noether current and conserved charge

$$J^\mu_a(\tau,\sigma) = T\,\partial_a X^\mu(\tau,\sigma)$$

$$P^\mu = \int_0^\pi J^\mu_\tau(\tau,\sigma)\,d\sigma = T\pi\ell^2 p^\mu = p^\mu$$

(this sets choice for ℓ).

Similarly, for spacetime Lorentz we have $\delta X^\mu = \underbrace{\epsilon^{\mu\nu}}_{\text{antisymmetric tensor}} X_\nu$ and we again compute the Noether current and conserved charge

$$J^{\mu\nu}_a(\tau,\sigma) = T\left(X^\mu\partial_a X^\nu - X^\nu\partial_a X^\mu\right)$$

$$M^{\mu\nu} = \int_0^\pi J^{\mu\nu}_\tau(\tau,\sigma)\,d\sigma = \begin{cases} \ell^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} & \text{(closed string)} \\ \ell^{\mu\nu} + E^{\mu\nu} & \text{(open string)} \end{cases}$$

Here we've defined $\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ (zero-mode angular momentum)

$$\left.\begin{aligned} E^{\mu\nu} &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{-i}{n}\left(\alpha^\mu_{-n}\alpha^\nu_{+n} - \alpha^\nu_{-n}\alpha^\mu_{+n}\right) \\ \tilde{E}^{\mu\nu} &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{-i}{n}\left(\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_{+n} - \tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_{+n}\right) \end{aligned}\right\} \text{oscillator angular momentum; relate to } \underline{\text{spin}}.$$

Finally we need to impose our constraints from the stress tensor: $\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X = 0$, $\partial_\tau X \cdot \partial_\sigma X = 0$.
This is simpler in "lightcone coordinates": $\sigma^\pm = \tau \pm \sigma$.

Useful dictionary: $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

$$\eta_{++} = \eta^{++} = \eta_{--} = \eta^{--} = 0$$

$$\eta^{+-} = \eta^{-+} = -2$$

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2}$$

Now in l.c. coordinates, the stress tensor obeys (automatically!) symmetry, conservation & tracelessness conditions

$$\eta^{\alpha\beta}\partial_\alpha T_{\beta\gamma} = 0 \longrightarrow \begin{array}{l} \partial_+ T_{--} + \partial_- T_{+-} = 0 \\ \partial_- T_{++} + \partial_+ T_{-+} = 0 \end{array}$$

$$\eta^{\alpha\beta} T_{\alpha\beta} = 0 \longrightarrow T_{+-} + T_{-+} = 0$$

$$T_{\alpha\beta} = T_{\beta\alpha} \longrightarrow T_{+-} = T_{-+}$$

Combining these gives $T_{+-} = 0$ and $\partial_+ T_{--} = \partial_- T_{++} = 0$. Last two are extremely powerful; Lorentzian version of holomorphicity / anti-holomorphicity. These give us an infinity of conserved charges! Concentrate on closed string first.

For any $f(\sigma_-)$, consider: $Q_f = \int d\sigma\, f(\sigma_-) T_{--}(\sigma_-)$

$$\frac{\partial}{\partial \tau} Q_f = \int d\sigma\, (2\partial_+ - \partial_\sigma) f(\sigma_-) T_{--}(\sigma_-)$$

$$= -\int d\sigma \frac{\partial}{\partial \sigma}\left(f(\sigma_-) T_{--}(\sigma_-)\right) = f(\sigma_-) T_{--}(\sigma_-)\Big|^{\sigma=0,\ \text{fixed } \tau}_{\sigma=\pi,\ \text{fixed } \tau}$$

$$= 0 \text{ if } f(\sigma_-)\ \pi\text{-periodic}$$

Get a complete set of conserved charges by taking $f(\sigma_-) = e^{2im\sigma_-}$

$$L_m = \frac{T}{2}\int d\sigma\, e^{2im\sigma_-} T_{--}(\sigma_-) \overset{\tau=0}{=} \frac{T}{2}\int d\sigma\, e^{-2im\sigma} \partial_- X_R \cdot \partial_- X_R$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot \alpha_n \quad \text{w/ } \alpha_0^\mu = \frac{\ell p^\mu}{2}.$$

Note that $L_{-m} = (L_m)^*$ (due to reality of T_{--}).

Similarly w/ T_{++} and $g(\sigma_+) = e^{2im\sigma_+}$

$$\tilde{L}_m = \frac{T}{2}\int d\sigma\, e^{2im\sigma_+} T_{++}(\sigma_+) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \quad \text{w/ } \tilde{\alpha}_0^\mu = \frac{\ell p^\mu}{2}.$$

These L_m's & $\tilde{L}_m$'s are just Fourier components of T_{--} & T_{++} respectively, so setting them to zero imposes the full stress tensor constraints.

Amongst the (only numerous) quadratic constraints on the oscillators, the L_0 constraint is particularly interesting.

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_n = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = \frac{\ell^2 p^2}{8} + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = 0$$

$$\tilde{L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = \frac{\ell^2 p^2}{8} + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = 0$$

Recall spacetime momentum $P^\mu = p^\mu$, so we have a Mass Shell Condition

$$\alpha' = \frac{1}{2\pi T} = \frac{\ell^2}{2}$$

$$M^2 = -p^2 = 4\pi T\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n)$$

In addition, by virtue of shared zero mode, we have the Level-Matching Condition

$$\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$$

(but don't forget there are still the $L_{m\neq 0}$ and $\tilde{L}_{m\neq 0}$ constraints to impose.)

Open string constraints

Need to revisit boundary contributions to conservation equation in this case:

$$\partial_\tau Q_f^{(-)} = -f(\sigma_-)T_{--}(\sigma_-)\Big|_0^\pi$$

$$\partial_\tau Q_g^{(+)} = g(\sigma_+)T_{++}(\sigma_+)\Big|_0^\pi$$

At $\sigma = 0$ or π, $\partial_\sigma X^\mu = 0$ by B.C., so $\partial_- X^\mu = \partial_+ X^\mu$ and $T_{--} = T_{++}$, so consider $Q_f^{(-)} + Q_g^{(+)}$ with

$$g(\tau) - f(\tau) = 0 \longrightarrow g \equiv f$$
$$g(\tau+\pi) - f(\tau-\pi) = 0 \longrightarrow g(\tau+2\pi) = g(\tau)$$

So our complete set of charges is given by

$$L_m = \frac{T}{2}\int_0^\pi \left(e^{im\sigma_+}T_{++} + e^{im\sigma_-}T_{--}\right)d\sigma \overset{\tau=0}{=} \frac{T}{2}\int_0^\pi \left(e^{im\sigma}T_{++} + e^{-im\sigma}T_{--}\right)d\sigma$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}}(\alpha_{m-n}\cdot\alpha_n)$$

L_0 gives open-string mass-shell condition

$$M^2 = -p^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n$$