We have explicitly constructed the space of solutions to equations of motion ("phase space"), and it is an infinite-dim'l affine space w/ coordinates $(x^\mu, p^\mu, \alpha^\mu_n, \tilde{\alpha}^\mu)$ subject to quadratic constraints $\{L_n = \tilde{L}_n = 0 \ \forall n \in \mathbb{Z}\}$.

Of course, phase space is a Poisson manifold w/ Poisson brackets encoded in original Lagrangian $L = \frac{T}{2}\left[\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X\right]$

$$\Pi^\mu = \frac{\delta L}{\delta(\partial_\tau X^\mu)} = T\partial_\tau X^\mu \qquad \left\{\Pi^\mu(\sigma), X^\nu(\sigma')\right\}_{P.B.} = \eta^{\mu\nu}\delta(\sigma-\sigma')$$

We determine Poisson brackets of oscillator modes by extracting Fourier components:

$$\int_0^\pi e^{-2in\sigma} X^\mu(0,\sigma)\, d\sigma = \frac{\pi i \ell}{2n}\left(\alpha^\mu_n - \tilde{\alpha}^\mu_{-n}\right) \qquad \int_0^\pi X^\mu(\sigma,0)\, d\sigma = x^\mu$$

$$\int_0^\pi e^{-2in\sigma} \Pi^\mu(0,\sigma)\, d\sigma = \frac{1}{\ell}\left(\alpha^\mu_n + \tilde{\alpha}^\mu_{-n}\right) \qquad \int_0^\pi \Pi^\mu(\sigma,0)\, d\sigma = p^\mu$$

Taking + & − combinations, we have Poisson brackets for oscillators

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{P.B.} = im\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$\{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\}_{P.B.} = im\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$\{p^\mu, x^\nu\}_{P.B.} = \eta^{\mu\nu}$$

and similar for open string w/ one set of oscillators. We can now do an important calculation, which is the Poisson bracket of the constraints.

(In general in a constrained Hamiltonian system one needs a more delicate analysis (and "Dirac Brackets"). We are lucky here, and the naive analysis comes out right.)

$$\{L_m, L_n\}_{P.B.} = \frac{1}{4}\sum_{k,\ell=-\infty}^{\infty}\{\alpha_{m-k}\cdot\alpha_k, \alpha_{n-\ell}\cdot\alpha_\ell\} \underset{\substack{\uparrow \\ \text{exercise}}}{=} i(m-n)L_{m+n}$$

$$\{\tilde{L}_m, \tilde{L}_n\}_{P.B.} = i(m-n)\tilde{L}_{m+n}$$

In particular, our constraints are <u>first-class</u> constraints. This is a famous Lie algebra, the "Witt algebra" (not "Witten", not Virasoro yet). It is the algebra of infinitesimal conformal transformations on the world sheet.

A conformal transformation of a Riemannian/Lorentzian manifold M is a diffeomorphism that preserves the metric up to rescaling.

$$g(\sigma^\pm) \longrightarrow \tilde{g}(\tilde{\sigma}^\pm) = e^{2\omega(\tilde{\sigma}^\pm)} g(\tilde{\sigma}^\pm)$$

The existence of conformal transformations means our conformal/unit gauge fixing is incomplete, due to Weyl invariance.

$$\eta \xrightarrow{\text{Diff}} e^{2\omega(\tilde{\sigma}^\pm)} \eta \xrightarrow{\text{Weyl}} \eta$$
$$X \xrightarrow{\text{Diff}} \tilde{X} \xrightarrow{\text{Weyl}} \tilde{X}$$

The inf. conformal transformations can be determined explicitly. Consider a general infinitesimal diffeomorphism:

$$\sigma^\alpha \longrightarrow \sigma^\alpha + \xi^\alpha(\sigma^\pm)$$
$$\eta^{\alpha\beta} \longrightarrow \eta^{\alpha\beta} + \partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha$$

For this to be a conformal transformation, ξ must satisfy the conformal Killing equation: $\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda(\sigma^\pm)\eta^{\alpha\beta}$. In light-cone coordinates this takes an exceptionally simple form:

$$(++) \quad \partial^+ \xi^+ = 0 \;\Rightarrow\; \partial_- \xi^+ = 0 \;\Rightarrow\; \xi^+(\sigma^\pm) = \xi^+(\sigma^+)$$
$$(--) \quad \partial^- \xi^- = 0 \;\Rightarrow\; \partial_+ \xi^- = 0 \;\Rightarrow\; \xi^-(\sigma^\pm) = \xi^-(\sigma^-)$$
$$(+-) \quad \partial^+ \xi^- + \partial^- \xi^+ = \Lambda(\sigma^\pm) \;\Rightarrow\; \text{no further restriction}$$

We can find a (complex) basis for such diffeo's that are defined in a neighborhood of a fixed time (say $\tau=0$), which imposes periodicity in closed string (open string an exercise):

$$V_n = -\frac{1}{2} e^{2in\sigma^-} \frac{\partial}{\partial \sigma^-} \qquad \tilde{V}_n = -\frac{1}{2} e^{2in\sigma^+} \frac{\partial}{\partial \sigma^+}$$

Lie algebra given by commutator of differential operators:

$$[V_n, V_m] = \frac{1}{4} e^{2i(n+m)\sigma^-} (2im - 2in) \frac{\partial}{\partial \sigma^+}$$
$$= i(n-m) \left(-\frac{1}{2} e^{2i(n+m)\sigma^+}\right) \frac{\partial}{\partial \sigma^+}$$
$$= i(n-m) V_{n+m} \quad \text{and similar for } \tilde{V}_{n,m}\text{'s}.$$

Thus the L_m's generate residual gauge symmetries (constraints generating gauge symmetries is a standard feature of the Hamiltonian picture). A couple of remarks at this point.

(1) The appearance of conformal symmetry suggests an analysis based on conformal field theory. We will do this somewhat indirectly since CFT is a Trinity course. For a crash course, see Polchinski vol. 1 chapter 2. But this is not <u>just</u> CFT, since the conformal transformations are thought of as gauge transformations.

(2) Since the conformal symmetry is a residual gauge symmetry, we could imagine doing further gauge fixing. There is no spacetime-Lorentz invariant way to do this, but at the expense of covariance one can use lightcone gauge

(compare to electromagnetism: Lorenz vs. Coulomb gauge)

A further comment on conformal symmetry: this infinite-dimensional algebra is special to low dimensions. In general, one has

$$\text{conformal algebra}_{(1,d-1)} \cong so(2,d) \supseteq so(1,d-1)\times so(1,1)$$

for d=2: $so(2,2) \cong \underset{\{V_{0,\pm1}\}}{\overset{}{\underset{\uparrow}{sl(2,\mathbb{R})}}} \times \underset{\{\tilde{V}_{0,\pm1}\}}{\underset{\uparrow}{\widetilde{sl(2,\mathbb{R})}}}$ is the "global part" of the conformal algebra, which is $\text{Witt}\times\widetilde{\text{Witt}}$

This concludes our classical analysis and it's time to quantize. There are several (it turns out equivalent) approaches:

Approaches to quantization (a survey)

Path Integral: $$Z = \int \frac{[DX^\mu][D\gamma]}{\text{Vol}\,(\text{Diff}\times\text{Weyl})}\, e^{\frac{i}{\hbar}S_p[X^\mu,\gamma]}$$

Related: Covariant BRST quantization: $\left\{\begin{array}{l}\text{correct quantum mechanical}\\ \text{treatment of gauge invariance.}\\ \text{(Fadeev-Popov, etc.)}\end{array}\right\}$

our approach $\longrightarrow$ Old covariant quantization: $\left\{\begin{array}{l}\text{gauge fix to conformal gauge}\\ \text{quantize}\\ \text{Impose } T_{++}, T_{--} \text{ constraints in quantum Hilbert space.}\end{array}\right.$

$\longrightarrow$ Light cone quantization: $\left\{\begin{array}{l}\text{fix \underline{all} gauge symmetry (not Poincaré covariant)}\\ \text{quantize constrained theory}\\ \text{check for consistency}\end{array}\right.$

easy to get an answer, but hard to detect subtleties $\left(\sum_{n=1}^{\infty} n = -\frac{1}{12}\right)$