Before imposing constraints, canonical quantization is straightforward. $\{\cdot\,,\,\cdot\}_{PB} = i\,[\cdot\,,\,\cdot\,]$, so we get standard equal-time commutators for free scalar fields:

$$[\Pi^\mu(\sigma), X^\nu(\sigma')] = -i\delta(\sigma-\sigma')\eta^{\mu\nu}$$

Oscillator modes also follow immediately:

$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu} \quad ; \quad \hat{p}^\mu, \hat{x}^\mu \text{ Hermitian}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \quad ; \quad \alpha^\mu_{-m} = (\alpha^\mu_m)^\dagger$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \quad ; \quad \tilde{\alpha}^\mu_{-m} = (\tilde{\alpha}^\mu_m)^\dagger$$

This is an infinite collection of harmonic oscillators plus a standard Heisenberg pair (x,p) of zero modes, with sign flips for $\mu=0$. The Hilbert space can be constructed for the oscillators in the usual way:

Define oscillator vacuum $|0\rangle_{osc}$ obeying $\alpha^\mu_m|0\rangle_{osc} = \tilde{\alpha}^\mu_m|0\rangle_{osc}\ \ \forall m \geq 1$. On top of this state build "oscillator Fock spaces"

$$\mathcal{H}^{Fock}_{open} = \text{span}\left\{\prod_{i=1}^{k}\alpha^{\mu_i}_{-n_i}|0\rangle_{osc}\right\}\ n_i \geq 1 \qquad \mathcal{H}^{Fock}_{open} \cong \widetilde{\mathcal{H}}^{Fock}_{open}$$

$$\mathcal{H}^{Fock}_{closed} = \text{span}\left\{\prod_{i=1}^{k}\alpha^{\mu_i}_{-n_i}\prod_{j=1}^{k}\tilde{\alpha}^{\mu_j}_{-n_j}|0\rangle_{osc}\right\}\ n_{i,j} \geq 1 \cong \mathcal{H}^{Fock}_{right}\otimes\mathcal{H}^{Fock}_{left}$$

Useful to introduce oscillator number operators: $N = \sum_{k>0}\alpha_{-k}\cdot\alpha_k \qquad \tilde{N} = \sum_{k>0}\tilde{\alpha}_{-k}\cdot\tilde{\alpha}_k$, which count oscillators in a way,

$$N\left(\prod_{i=1}^{k}\alpha^{\mu_i}_{-n_i}|0\rangle_{osc}\right) = \sum_{i=1}^{k}n_i\left(\prod_{i=1}^{k}\alpha^{\mu_i}_{-n_i}|0\rangle_{osc}\right)$$

$$\tilde{N}\left(\prod_{i=1}^{k}\tilde{\alpha}^{\mu_i}_{-n_i}|0\rangle_{osc}\right) = \sum_{i=1}^{k}n_i\left(\prod_{i=1}^{k}\tilde{\alpha}^{\mu_i}_{-n_i}|0\rangle_{osc}\right)$$

Oscillator states organized into "levels" (N & $\tilde{N}$-eigenvalues). Focusing on open string/only right-movers

$N=0 \qquad |0\rangle_{osc}$

$N=1 \qquad \alpha^\mu_{-1}|0\rangle_{osc}$

$N=2 \qquad \alpha^\mu_{-2}|0\rangle_{osc},\ \ \alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle_{osc}$

$N=3 \qquad \alpha^\mu_{-3}|0\rangle_{osc},\ \ \alpha^\mu_{-2}\alpha^\nu_{-1}|0\rangle_{osc},\ \ \alpha^\mu_{-1}\alpha^\nu_{-1}\alpha^\rho_{-1}|0\rangle_{osc}$

$\vdots$

No states of negative level, obviously.

Treat zero modes as in non-relativistic quantum mechanics. $\mathcal{H}_{zero\text{-}mode} \cong L^2(\mathbb{R}^{1,D-1})$, use plane wave states $|k\rangle$, $k\in\mathbb{R}^{1,D-1}$ obeying $\hat{p}^\mu|k\rangle = k^\mu|k\rangle$ and $\langle k'|k\rangle = \delta^{(D)}(k'-k)$. Then full state spaces are

$$\mathcal{H}_{open} = \mathcal{H}^{Fock}_{open}\otimes L^2(\mathbb{R}^{1,D-1})$$

$$\mathcal{H}_{closed} = \widetilde{\mathcal{H}}^{Fock}_{left}\otimes\mathcal{H}^{Fock}_{right}\otimes L^2(\mathbb{R}^{1,D-1})$$

Immediately, there are problematic states, like $\alpha^0_{-1}|0;k\rangle$:

wrong sign! gives negative norm states when put into wave packets.

$$\langle 0;k|\alpha^0_{+1}\alpha^0_{-1}|0;k'\rangle = \eta^{00}\delta(k-k') = -\delta(k-k')$$

Good thing we still have to impose constraints!

Recall our Witt generators/constraints, given in terms of oscillators by (recall $\alpha_0 = \tilde{\alpha}_0 = \frac{\ell p}{2}$ (closed) or ℓp (open)).

$\alpha_0 = \frac{\ell p}{2}$ or ℓp

$\ell^2 = 2\alpha' = \frac{1}{\pi T}$

$$L_m = \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{m-k}\cdot\alpha_k \,,\quad L_m^+ = L_{-m} \qquad m\neq 0$$

$$\tilde{L}_m = \frac{1}{2}\sum_{k=-\infty}^{\infty} \tilde{\alpha}_{m-k}\cdot\alpha_k \,,\quad \tilde{L}_m^+ = \tilde{L}_{-m} \qquad m\neq 0$$

There is a problem for L_0 & $\tilde{L}_0$: nontrivial operator ordering! Take naive version:

$$L_0^{\text{naive}} = \frac{1}{2}\sum_k \alpha_{-k}\cdot\alpha_k \longrightarrow L_0^{\text{naive}}|0;p\rangle = \left(\frac{\ell^2p^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty} k\underbrace{(D-2)}_{\mathrm{Tr}(\eta^{\mu\nu})}\right)|0;p\rangle$$

$$= \left(\frac{\ell^2p^2}{4} - \frac{D-2}{24}\right)|0;p\rangle \quad \text{using zeta-fn regularization.}$$

A more reasonable approach is to define the quantum L_0 to be normal ordered, and keep in mind we made a choice that could require some correction.

closed string
$$\begin{cases} L_0 := \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k = \frac{\ell^2p^2}{8} + N \longrightarrow L_0|0;p\rangle = \frac{\ell^2p^2}{8}|0;p\rangle \\ \tilde{L}_0 := \frac{1}{2}\tilde{\alpha}_0\cdot\tilde{\alpha}_0 + \sum_{k=1}^{\infty}\tilde{\alpha}_{-k}\cdot\tilde{\alpha}_k = \frac{\ell^2p^2}{8} + \tilde{N} \end{cases}$$

open string
$$\left\{ L_0 = \frac{\ell^2p^2}{2} + N \right.$$

Commutator algebra of $\{L_m\}$'s needs to be checked, can receive quantum corrections! Indeed direct computation (exercise on PS2) gives

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n,0}$$

linear term can be shifted by re-defining L_0 up to a constant, but we've settled on our definition for L_0.

"Virasoro algebra with central charge D"

This is a central extension of the Witt algebra: $0 \longrightarrow \mathfrak{h} \longrightarrow \mathrm{Vir} \longrightarrow \mathrm{Witt} \longrightarrow 0$. More accurately, we have a rep^{n} of Virasoro where $\hat{c} \equiv D$.

$\hat{c};[\hat{c},\hat{c}]=0$ as above w/ $\hat{c}=D$ $\mathrm{Vir}/\langle\hat{c}\rangle$

This central extension is a kind of anomaly; in fact related to Weyl invariance. We will touch on this more later. Observe that the "global" sl(2) generated by $L_{\pm1,0}$ & $\tilde{L}_{\pm1,0}$ are non-anomalous.

Our next project is to impose our constraints in the quantum theory to identify "physical states" in naive state-space. Here we see a big problem with the Virasoro anomaly! We naively want to impose

$$\forall m \quad L_m|\psi\rangle = 0 \quad \text{for } |\psi\rangle \in \mathcal{H}_{\text{phys}}$$

But this leads to a contradiction immediately

$$\boxed{\begin{array}{ccc} & [L_m, L_{-m}]|\psi\rangle & \\ \rotatebox{90}{=} & & \rotatebox{90}{=} \\ 2mL_{2m}\overset{0}{\cancel{|\psi\rangle}} + \frac{D}{12}(m^3-m)|\psi\rangle & & L_mL_{-m}\overset{0}{\cancel{|\psi\rangle}} - L_{-m}L_m\overset{0}{\cancel{|\psi\rangle}} \\ \| & & \| \\ \frac{D}{12}(m^3-m)|\psi\rangle & = & 0 \end{array}}$$

So we can't impose all of the constraints if we want to get a non-trivial Hilbert space of states for $D\neq 0$.

We can try to get away with demanding less: $\langle\psi|L_m|\phi\rangle \overset{!}{=} 0$ for $\phi,\psi \in \mathcal{H}_{phys}$. This we can do without killing all states

Def: a state $|\phi\rangle$ is physical if:

- $\forall m>0,\ L_m|\phi\rangle = 0$
- $(L_0 - a)|\phi\rangle = 0$ for fixed $a \in \mathbb{R}$ (TBD), the "normal ordering constant".

Observe then that for $\phi,\psi \in \mathcal{H}_{phys}$,

$$\langle\psi|L_m|\phi\rangle = 0 \text{ for } m>0$$
$$\langle\psi|L_m|\phi\rangle = \overline{\langle\phi|L_{-m}|\psi\rangle} = 0 \text{ for } m<0$$
$$\langle\psi|(L_0-a)|\phi\rangle = 0$$

(For closed string, require same of $\tilde{L}_m$'s for a state to be physical.)

Importantly, $[P^\mu, L_m] = [M^{\mu\nu}, L_m] = 0\ \forall m$, so spacetime Poincaré preserves physicality. (Exercise!) Thus $\mathcal{H}_{phys}$ will decompose into $ISO(1, D-1)$ representations (hence "covariant" quantization).

Returning to L_0 constraints we get quantum versions of mass shell & level-matching conditions:

Closed string mass shell

$$(L_0 + \tilde{L}_0 - 2a)|\varphi;k\rangle = 0$$
$$\left(\frac{\ell^2 p^2}{4} + N + \tilde{N} - 2a\right)|\varphi;k\rangle = 0$$
$$\frac{\ell^2 M^2}{4} = \underbrace{N+\tilde{N}}_{\text{using same name for eigenvalues as number operators}} - 2a \longrightarrow \alpha' M^2 = 2N + 2\tilde{N} - 4a \quad \left(\text{recall } \alpha' = \frac{\ell^2}{2}\right)$$

Level-matching

$$(L_0 - \tilde{L}_0)|\varphi;k\rangle = 0$$
$$(N-\tilde{N})|\varphi;k\rangle = 0$$
$$N = \tilde{N}$$

Open string mass shell

$$(L_0 - a)|\varphi;k\rangle = 0$$
$$\left(\frac{\ell^2 p^2}{2} + N - a\right)|\varphi;k\rangle = 0$$
$$\alpha' M^2 = N - a$$

It's useful to note (exercise) that $[N, L_m] = -m L_m$, so L_m shifts N-level by $-m$ (similar for $\tilde{N}$ & $\tilde{L}_m$). Thus at level zero (Fock vacuum) we only need to check the L_0-conditions.

open: $\alpha' M^2 = -a$

closed: $\alpha' M^2 = -4a$

so we have an important physical dependence on the sign of a!

- $a<0$: massive ground state
- $a=0$: massless ground state
- $a>0$: tachyonic ground state

There was a problem w/ negative-norm states ("ghosts") at level-one of the open string before. Do the constraints help? We'll work covariantly.

General level-one open-string state: $\zeta \cdot \alpha_{-1}|0;k\rangle =: |\zeta;k\rangle \quad (\zeta \in \mathbb{R}^{1,D-1})$

Now we need to impose both L_0 & L_{+1} physical state conditions.

- Mass-shell (L_0): $\alpha' k^2 = -\alpha' m^2 = a - 1$

$[L_m, \alpha^\mu_n] = -n\alpha^\mu_{m+n}$

- L_{+1} condition: $L_{+1}\eta_{\mu\nu}\zeta^\mu \alpha^\nu_{-1}|0;k\rangle = \eta_{\mu\nu}\zeta^\mu [L_{+1}, \alpha^\nu_{-1}]|0;k\rangle$

$$= \eta_{\mu\nu}\zeta^\mu \underbrace{\alpha^\nu_0}_{\ell p^\nu}|0;k\rangle$$
$$= \ell\,\eta_{\mu\nu}\zeta^\mu k^\nu|0;k\rangle$$
$$= \ell\,\zeta\cdot k|0;k\rangle$$
$$= 0 \text{ for physical state, so } \zeta\cdot k = 0$$

- L_{+n} conditions for $n \geq 2$ satisfied automatically by level considerations.

Compare with norms of general states: $\langle \zeta;k|\zeta';k'\rangle = \langle 0;k|(\zeta\cdot\alpha_{+1})(\zeta'\cdot\alpha_{-1})|0;k'\rangle$

$= \langle 0;k|\zeta\cdot\zeta'|0;k'\rangle = (\zeta\cdot\zeta')\,\delta(k-k')$ so for $\zeta=\zeta'$, get

$\zeta^2\,\delta(k-k')$, and require $\zeta^2 \geq 0$ to avoid ghosts.

So we need: $\zeta\cdot k = 0$ (transverse polarization)

$\zeta^2 \geq 0$ (polarization light-like or space-like).

$\alpha' k^2 = a - 1$ (so k is spacelike if $a>1$, lightlike if $a=1$, timelike if $a<1$)

if k is spacelike, can choose transverse ζ time-like $\Rightarrow$ ghosts! So we require $a \leq 1$ at least.

The threshold case is interesting ($a=1$). Now $k^2=0$, so we have a massless state with a spacetime vector index. Additionally, there is now a physical state with zero norm:

$|k;k\rangle = (k\cdot\alpha_{-1})|0;k\rangle$ is "longitudinally polarized", physical because $k^2=0$.

In fact, $|k;k\rangle$ is orthogonal to all physical states

$\langle k;k|\zeta;k'\rangle = (k\cdot\zeta)\,\delta(k-k') = 0$ since $\zeta\cdot k'=0$ for physical state.

So the longitudinal polarization decouples, leaving $D-2$ physical polarizations (like the photon!)

Observe: $L_{-1}|0;k\rangle = \ell\, k\cdot\alpha_{-1}|0;k\rangle = \ell|k;k\rangle$, so our null state is created by the action of L_{-1}. This is a "pure gauge" state, hence decoupling.

However, for $a=1$, ground state is a <u>tachyon</u>! We will have to make peace with this.