To summarize the situation from last time, we had state-spaces & constraints as follows in quantum theory

$$\longrightarrow \mathcal{H}_{open} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}_{Fock}$$

$$\longrightarrow \mathcal{H}_{closed} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}_{Fock} \otimes \widetilde{\mathcal{H}}_{Fock}$$

$$\longrightarrow \mathcal{H}_{Fock} = \text{span}\left\{ \prod_{i=1}^{K} \alpha_{-n_i}^{\mu_i} |0\rangle \right\} = \bigoplus_{N=1}^{\infty} \mathcal{H}_{Fock}[\overset{\text{level}}{\overset{\downarrow}{N}}]$$

$$\longrightarrow L_m = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{m-k} \cdot \alpha_k \qquad m \neq 0$$

$$\longrightarrow L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \overset{\text{number operator}}{\overset{\downarrow}{\hat{N}}}$$

(idem for $\tilde{L}$'s.)

The algebra of the $\{L_n\}$ operators in the quantum theory are modified

$$\rightarrow [L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0}$$

$$\rightarrow \left([L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \right)$$

(Virasoro algebra w/ central charge "c")

Due to central term, can't impose vanishing of all L_m's on physical states; use "Gupta-Bleuler"-like method:

$$\mathcal{H}_{phys} = \left[\bigcap_{k=1}^{\infty} \text{Ker}(L_k) \right] \cap \text{Ker}(L_0 - \overset{\text{normal ordering constant}}{\overset{\downarrow}{a}})$$

For a state at a given level, only need to check a finite # of constraints because L_m's lower level by m (for $m > 0$), and there are no negative-level states:

$$[N, L_m] = -m L_m \implies L_m : \mathcal{H}_{Fock}[N] \longrightarrow \mathcal{H}_{Fock}[N-m]$$

In fact, the situation is simpler still: $\left.\begin{array}{l} [L_{+2}, L_{+1}] = L_{+3} \\ [L_{+2}, L_{+2}] = L_{+4} \\ \vdots \end{array}\right\}$ L_{+1}, L_{+2} generate every positive L_m — so — only need $L_{+1}|\phi\rangle = L_{+2}|\phi\rangle = 0$

So the space of physical states is a triple intersection:

$$\boxed{\mathcal{H}_{phys} = \left[\bigcap_{k=1}^{2} \text{Ker}(L_k) \right] \cap \text{Ker}(L_0 - \overset{\text{normal ordering constant}}{\overset{\downarrow}{a}})}$$

We started a level-by-level analysis of the space of physical states. At level-zero, there is just $|0;k\rangle$ and the L_0 constraint requires $\alpha' k^2 = a$ $(\alpha' = \frac{1}{2}\ell^2)$. So $a<0$: massive, $a=0$: massless, $a>0$: tachyon.

At level one, we had the general state $|\zeta;k\rangle = \zeta_\mu \alpha^\mu_{-1}|0;k\rangle$ obeying $\alpha' k^2 = a-1$, $\zeta\cdot k = 0$ and with norm $\sim \zeta^2$, so to avoid negative-norm states, need no transverse, time-like polarizations, so k should be time-like or light-like.

$$a \leq 1 \text{ to avoid "ghosts"}$$

The threshold case is interesting $(a=1)$. Now $k^2=0$, so we have a massless state with a spacetime vector index. Additionally, there is now a physical state with zero norm:

$$|k;k\rangle = (k\cdot\alpha_{-1})|0;k\rangle \text{ is "longitudinally polarized", physical because } k^2=0.$$

In fact, $|k;k\rangle$ is orthogonal to all physical states

$$\langle k;k|\zeta;k'\rangle = (k\cdot\zeta)\,\delta(k-k') = 0 \quad \text{since } \zeta\cdot k' = 0 \text{ for physical state.}$$

So the longitudinal polarization decouples, leaving $D-2$ physical polarizations (like the photon!)

Observe: $L_{-1}|0;k\rangle = \ell k\cdot\alpha_{-1}|0;k\rangle = \ell|k;k\rangle$, so our null state is created by the action of L_{-1}. This is a "pure gauge" state, hence decoupling.

However, for $a=1$, ground state is a tachyon! We will have to make peace with this.

Def: A state is called spurious if it is orthogonal to all physical states and obeys $(L_0 - a)|\phi\rangle = 0$.

We observe that if a spurious state is itself physical, it must have zero norm (orthogonal to itself). A physical, spurious state is called a Null State. The longitudinal polarization state above was exactly such a state. One expects to encounter these in gauge theories with residual gauge symmetry as "pure gauge" states. Then we want to quotient by these.

$$\mathcal{H}_{red} = \mathcal{H}_{phys}/\mathcal{H}_{null}, \text{ so } |\Psi\rangle_{phys} \sim |\Psi\rangle_{phys} + |\Phi\rangle_{null}$$

We see that precisely for $a=1$, we have states given by the action of L_{-1} that are pure gauge and can be quotiented. This is a hint that $a=1$ is the "right value" for conformal invariance to be present quantum mechanically.

Let's look at this last statement in more general language. The state $L_{-1}|0;k\rangle$ is manifectly $\perp$ to physical states. To be spurious, need to adjust k so that

$$(L_0-a)L_{-1}|0;k\rangle = L_{-1}(L_0-a+1)|0;k\rangle = 0 \quad \text{if} \quad L_0|0;k\rangle = (a-1)|0;k\rangle$$

State is then physical if we have

$$L_{+1}\left(L_{-1}|0;k\rangle\right) = (2L_0 - L_{-1}L_{+1})|0;k\rangle = 2L_0|0;k\rangle = 2(a-1)|0;k\rangle = 0 \quad \text{iff } a=1.$$

$$L_{+2}\left(L_{-1}|0;k\rangle\right) = (3L_{+1} + L_{-1}L_{+2})|0;k\rangle = 0$$

Can see the start of a more general argument. Any state of the form

$$|\xi\rangle = \sum_{m=1}^{\infty} L_{-m}|\chi_m\rangle \quad \text{w/ } L_0|\chi_m\rangle = (a-m)|\chi_m\rangle$$

is spurious. In fact, one can argue (I won't here) that this is the form of the most general spurious state. But equivalently, this is just

$$|\xi\rangle = L_{-1}\underset{\substack{\uparrow\\ L_0=a-1}}{|\tilde{\chi}_1\rangle} + L_{-2}\underset{\substack{\uparrow\\ L_0=a-2}}{|\tilde{\chi}_2\rangle}$$

Let's examine physical state condition, first w/ $|\tilde{\chi}_2\rangle = 0$ and assume $L_{+m}|\tilde{\chi}_1\rangle = 0$ $(m>0)$. Then

$$L_{+1}|\xi\rangle = (2L_0 + L_{-1}L_{+1})|\tilde{\chi}_1\rangle = (a-1)|\tilde{\chi}_1\rangle = 0 \quad \text{iff } a=1$$

$$L_{+2}|\xi\rangle = (3L_{+1} + L_{-1}L_{+2})|\tilde{\chi}_1\rangle = 0$$

So we get an infinite class of null states generalizing $|k;k\rangle$. Now consider state w/ L_{-2} action:

$$|\xi\rangle = (L_{-2} + \gamma L_{-1}^2)|\tilde{\chi}_2\rangle \quad , \text{assume } L_{+m}|\tilde{\chi}_2\rangle = 0 \ (m>0).$$

Physical state conditions are more laborious now.

$$\begin{aligned}
L_{+1}|\xi\rangle &= \left(3L_{-1} + L_{-2}L_{+1} + 2\gamma(L_0L_{-1} + L_{-1}L_0) + \gamma L_{-1}^2 L_{+1}\right)|\tilde{\chi}_2\rangle \\
&= \left(3 + 2\gamma(a-1+a-2)\right)|\tilde{\chi}_2\rangle \\
&= \left(3 + \gamma(4a-6)\right)|\tilde{\chi}_2\rangle = 0 \text{ iff } \gamma = \frac{3}{6-4a} \quad \left(=\frac{3}{2} \text{ for } a=1\right)
\end{aligned}$$

$$\begin{aligned}
L_{+2}|\xi\rangle &= \left(4L_0 + \frac{D}{12}\cdot 6 + 6\gamma L_0\right)|\tilde{\chi}_2\rangle \\
&= \left[(4+6\gamma)(a-2) + \frac{D}{2}\right]|\tilde{\chi}_2\rangle = 0 \text{ iff } D = (2-a)(8+12\gamma) = 26 \text{ for } a=1
\end{aligned}$$

$D=26$ known as "critical dimension", $(D=26, a=1)$ is the critical bosonic string.

For normal ordering constant, critical value was at threshold beyond which we found ghosts in the physical spectrum. An analogous result holds for the critical dimension. (We set $a=1$ for now).

Take level-2 state: $|\xi_K\rangle = \left[c_1 \alpha_{-1}\cdot\alpha_{-1} + c_2\,\alpha_{-2}\cdot\alpha_0 + c_3\,(\alpha_{-1}\cdot\alpha_0)\,(\alpha_{-1}\cdot\alpha_0)\right]|0;K\rangle$, $\alpha' K^2 = -1$

L_{+1}: $c_1 + c_2 - 2c_3 = 0$

L_{+2}: $Dc_1 - 4c_2 - 2c_3 = 0$

$\left([L_m, \alpha_n^\mu] = -n\alpha_{m+n}^\mu\right)$

$$|\xi_K\rangle = \left[\alpha_{-1}\cdot\alpha_{-1} + \frac{D-1}{5}\alpha_{-2}\cdot\alpha_0 + \frac{D+4}{10}(\alpha_{-1}\cdot\alpha_0)^2\right]|0;K\rangle$$ physical for $a=1$, any D.

Computing the norm, $\langle \xi_K | \xi_{K'}\rangle = \frac{2}{25}\delta(K-K')(D-1)(D-26)$, so $1 \le D \le 26$ to avoid ghosts. When $D=1$ or $D=26$, this becomes exactly our null state.

So, at $a=1$ & $D=26$ there is a double enhancement of null states due to being "on the edge" of inconsistency. But now we're out of variables to play with, so if there are left over ghosts, we're out of luck.

(1972) Brower; Goddard, Thorn, "No ghost theorem": For $a=1$, $D=26$ the physical space of states is free of ghosts.

What about $a<1$, $1 \le D \le 26$? It turns out there are no ghosts for $a \le 1$, $D < 25$. However, away from criticality there are inconsistencies @ level of string loops.

Remarks on other approaches:

- in light-cone gauge, manifestly ghost free, but spacetime Lorentz requires $a=1$, $D=26$.
- in BRST quantization, $a=1$, $D=26$ required for quantum gauge (conformal) invariance.

Next week: closed string, scattering.