Recap of open string unitarity: we saw that for $a>1$ or $D>26$, there are negative-norm physical states in open-string spectrum. $a=1$, $D=26$ is the "critical string", and there we found two infinite families of extra null states.

Comment: since we only imposed $\frac{1}{2}$ the Virasoro constraints by hand, it is natural to hope for the remaining Virasoro charges to give redundancies associated to residual gauge transformations. In the critical string it looks like this is indeed what is happening.

Regarding the subcritical case $(D\leq 25,\ a\leq 1)$, one sees quickly that there can be no inconsistencies at the level of the free string spectrum (so no ghosts).

Consider critical string, look at states with

$$K=(K^0,K^1,\dots,K^{24},\delta)\quad \text{for fixed } \delta\in\mathbb{R}$$

$$\left.\begin{aligned} N_{25} &= \sum_{n=1}^{\infty}\alpha^{25}_{-n}\alpha^{25}_{+n}\equiv 0\\ \tilde N_{25} &= \sum_{n=1}^{\infty}\tilde\alpha^{25}_{-n}\tilde\alpha^{25}_{+n}\equiv 0\end{aligned}\right\}\ \text{so no } X^{25} \text{ oscillators activated.}$$

These states obey $\alpha' K^2 = \underbrace{\alpha' K_a K^a}_{a=0,1,\dots,25} + \alpha'\delta^2 = 1-N = 1-N_{0-24}$ ~ number operator w/ 25-oscillators omitted.

so $\alpha' K_a K^a = (1-\alpha'\delta^2)-N$, which is identical to $D=25$ mass shell condition for $a=1-\alpha'\delta^2$. This gives us an embedding of the $D\leq 25$, $a\leq 1$ statespace in the critical string state space, so no ghost theorem for critical string applies to sub-critical as a Corollary.

Now let's look at the low-lying physical states of the closed string.

$$\mathcal{H}_{\text{closed}} \cong L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}_{\text{Fock}} \otimes \widetilde{\mathcal{H}}_{\text{Fock}}$$

Physical state conditions:
- $(L_0 + \tilde{L}_0 - 2a)|\phi\rangle = 0 \implies \alpha' k^2 = 4a - 2(N + \tilde{N})$
- $(L_0 - \tilde{L}_0)|\phi\rangle = 0 \implies N = \tilde{N}$
- $L_{+1}, L_{+2}, \tilde{L}_{+1}, \tilde{L}_{+2}$ annihilate $|\phi\rangle$

Level zero: $|0;k\rangle$ w/ $\alpha' k^2 = 4a = 4$ in the critical string.

Level one: $|\Omega;k\rangle = \Omega_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0;k\rangle$ w/ $\alpha' k^2 = 4a - 4 = 0$ for critical string.

Can further decompose this level-one state in terms of space-time Lorentz representations: $\Omega_{\mu\nu} = \underset{\text{symmetric traceless}}{\underset{\uparrow}{\Upsilon_{(\mu\nu)}}} + \underset{\text{anti-symmetric}}{\underset{\uparrow}{b_{[\mu\nu]}}} + \underset{\text{trace part}}{\underset{\uparrow}{\varphi\,\eta_{\mu\nu}}}$

Need to impose L_{+1} & $\tilde{L}_{+1}$ conditions (and no further Virasoro constraints.)

- $L_{+1}|\Upsilon;k\rangle = \frac{\ell}{2}\Upsilon_{\mu\nu}k^{\mu}\tilde{\alpha}_{-1}^{\nu}|0;k\rangle = 0$ if $k^{\mu}\Upsilon_{\mu\nu} = 0$
- $\tilde{L}_{+1}|\Upsilon;k\rangle = \frac{\ell}{2}\Upsilon_{\mu\nu}k^{\nu}\alpha_{-1}^{\mu}|0;k\rangle = 0$ if $k^{\nu}\Upsilon_{\mu\nu} = 0$

} $\Upsilon_{\mu\nu}$ polarization is "transverse traceless"

- $L_{+1}|b;k\rangle = \frac{\ell}{2}b_{\mu\nu}k^{\mu}\tilde{\alpha}_{-1}^{\nu}|0;k\rangle = 0$ if $k^{\mu}b_{\mu\nu} = 0$
- $\tilde{L}_{+1}|b;k\rangle = \frac{\ell}{2}b_{\mu\nu}k^{\nu}\alpha_{-1}^{\mu}|0;k\rangle = 0$ if $k^{\nu}b_{\mu\nu} = 0$

} $b_{\mu\nu}$ polarization is transverse, antisymmetric

- $L_{+1}|\varphi;k\rangle = \frac{\ell}{2}\varphi\, k\cdot\tilde{\alpha}_{-1}|0;k\rangle = 0$ if $k \equiv 0$
- $\tilde{L}_{+1}|\varphi;k\rangle = \frac{\ell}{2}\varphi\, k\cdot\alpha_{-1}|0;k\rangle = 0$ if $k \equiv 0$

} unphysical requirement! return to this soon.

As in open-string, there is some extra gauge invariance associated to null states.

- $L_{-1}\left(\xi_{\mu}\tilde{\alpha}_{-1}^{\mu}|0;k\rangle\right) = \frac{\ell}{2}\left(\xi_{\mu}k_{\nu}\tilde{\alpha}_{-1}^{\mu}\alpha_{-1}^{\nu}\right)|0;k\rangle$ spurious if $k^2 = 0$, physical if $k\cdot\xi = 0$
- $\tilde{L}_{-1}\left(\xi_{\nu}\alpha_{-1}^{\nu}|0;k\rangle\right) = \frac{\ell}{2}\left(k_{\mu}\xi_{\nu}\tilde{\alpha}_{-1}^{\mu}\alpha_{-1}^{\nu}\right)|0;k\rangle$ spurious if $k^2 = 0$, physical if $k\cdot\xi = 0$

Taking symmetric and anti-symmetric combinations, we have gauge invariances:

$$\Upsilon_{\mu\nu} \to \Upsilon_{\mu\nu} + \xi_{\mu}k_{\nu} + \xi_{\nu}k_{\mu}$$
$$b_{\mu\nu} \to b_{\mu\nu} + \xi_{\mu}k_{\nu} - \xi_{\nu}k_{\mu} \qquad (\xi_{\mu} \sim \xi_{\mu} + k_{\mu})$$

two separate gauge invariances

These two string states have an elegant space-time interpretation. The traceless-symmetric state acts like a graviton in traceless, harmonic gauge:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \Upsilon_{\mu\nu}(x)\,, \quad \Upsilon_{\mu\nu}(x) \sim \Upsilon_{\mu\nu}(x) + \partial_{\mu}\xi_{\nu}(x) + \partial_{\nu}\xi_{\mu}(x)$$

in momentum space, $\Upsilon_{\mu\nu}(k) \sim \Upsilon_{\mu\nu}(k) + k_{\mu}\xi_{\nu} + k_{\nu}\xi_{\mu}$ with $k\cdot\xi = 0$

The anti-symmetric state acts like a 2-form gauge field ("Kalb-Ramond field").

$$b_{\mu\nu}(x)dx^{\mu}dx^{\nu} = b^{(2)} \sim b^{(2)} + da^{(1)} \qquad (a^{(1)} \sim a^{(1)} + d\lambda)$$

in momentum space, $b_{\mu\nu}(k) \sim b_{\mu\nu}(k) + k_{\mu}\xi_{\nu} - k_{\nu}\xi_{\mu}$ with $\xi_{\mu} \sim \xi_{\mu} + k_{\mu}\phi$

The scalar is a little subtle. Let us define a state

$$|\varphi_{\xi,\tilde{\xi}};k\rangle = \varphi\left[(\xi\cdot\alpha_{-1})\left(\tfrac{\ell k}{2}\cdot\tilde{\alpha}_{-1}\right) + \left(\tfrac{\ell k}{2}\cdot\alpha_{-1}\right)(\tilde{\xi}\cdot\tilde{\alpha}_{-1}) + (\alpha_{-1}\cdot\tilde{\alpha}_{-1})\right]|0;k\rangle$$

The Virasoro conditions gives:

$$L_{+1}|\varphi_{\xi,\tilde{\xi}};k\rangle = \frac{\ell\varphi}{2}\left[(\xi\cdot k)\left(\tfrac{\ell k}{2}\cdot\tilde{\alpha}_{-1}\right) + k\cdot\tilde{\alpha}_{-1}\right]|0;k\rangle = 0 \text{ if } \xi\cdot k = -\frac{2}{\ell}$$

$$\tilde{L}_{+1}|\varphi_{\xi,\tilde{\xi}};k\rangle = \frac{\ell\varphi}{2}\left[(\tilde{\xi}\cdot k)\left(\tfrac{\ell k}{2}\cdot\alpha_{-1}\right) + k\cdot\alpha_{-1}\right]|0;k\rangle = 0 \text{ if } \tilde{\xi}\cdot k = -\frac{2}{\ell}$$

Superficially, we seem to have mixed a spacetime scalar with spacetime vectors. However, let us consider dependence on choice of ξ & $\tilde{\xi}$:

$$|\varphi_{\xi,\tilde{\xi}};k\rangle - |\varphi_{\xi',\tilde{\xi}};k\rangle = \varphi\left[\left((\xi-\xi')\cdot\alpha_{-1}\right)\left(\tfrac{\ell k}{2}\cdot\tilde{\alpha}_{-1}\right)\right]|0;k\rangle \quad \text{\& analogously for } \tilde{\xi} \text{ vs. } \tilde{\xi}'.$$

$$= \varphi\,\tilde{L}_{-1}\left((\xi-\xi')\cdot\alpha_{-1}\right)|0;k\rangle$$

$(\xi-\xi')\cdot k = \xi\cdot k - \xi'\cdot k = 0$, so this state is null! Up to gauge equivalence, state is a scalar (exercise!)

An aside on the lightcone:

The fastest way to construct the physical/null states of the quantized string is to further gauge fix to lightcone gauge. This proceeds by defining $X^{\pm}(\tau,\sigma) = \frac{X^0 \pm X^{D-1}}{2}$, and using the residual gauge symmetry to set $X^+ = x^+ + p^+\tau$ (no oscillators). One can then solve the Virasoro constraints for X^-, and the result is that the physical state space is constructed from

- $D-2$ spacelike "transverse" oscillators α^i_n ; $i=1,\dots,D-2$ (and $\tilde{\alpha}^i_n$; $i=1,\dots,D-2$ for closed string). No negative norms!
- subject to a zero mode condition $M^2 = \frac{1}{\alpha'}(N^\perp - 1)$ ($M^2 = \frac{1}{\alpha'}(N^\perp + \tilde{N}^\perp - 2)$ for closed string)

"transverse" number operator $N^\perp = \sum_{n=1}^{\infty} \delta_{ij}\, \alpha^i_{-n} \alpha^j_n$

It is very fast to count states now: level-1 open: $\alpha^i_{-1}|0;k\rangle$ $i=1,\dots,D-2$, i.e., $D-2$ states transforming as a vector of $SO(D-2)$. This only makes sense if this is a massless state (for which little group is $SO(D-2)$). Otherwise need an extra state to get a rep^{n} of the massive-particle little group $SO(D-1)$.

Thus $a=1$ for Lorentz-invariance. A more detailed calculation (see typed notes or GSW) shows $D=26$ necessary as well.

Example: level-2 open string (from homework): $\alpha^i_{-1}\alpha^j_{-1}|0;k\rangle$ & $\alpha^i_{-2}|0;k\rangle$ give $\frac{(D-1)(D-2)}{2} + (D-2) = \frac{(D+1)(D-2)}{2}$ states. But how to organize as an $SO(D-1)$ representation? Maybe not immediately obvious!

$$\frac{(D+1)(D-2)}{2} = \frac{(D)(D-1)}{2} - 1$$

subtract trace

symmetric 2-index tensor of $SO(D-1)$

massive spin-2

(Under full $SO(1,D-1)$ this is a transverse, traceless, symmetric 2-tensor)