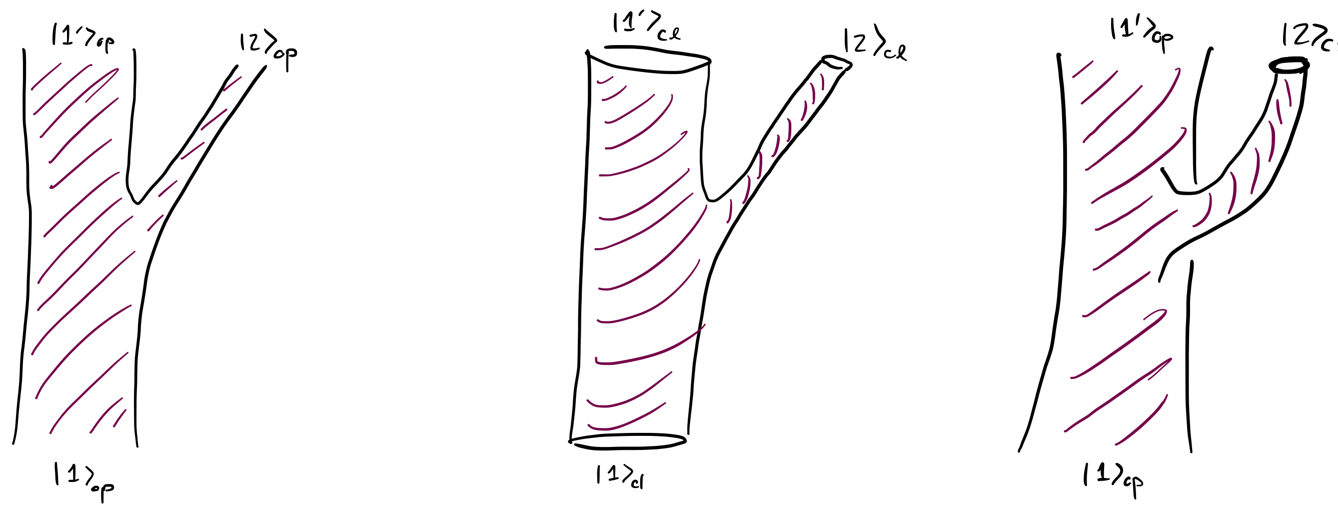
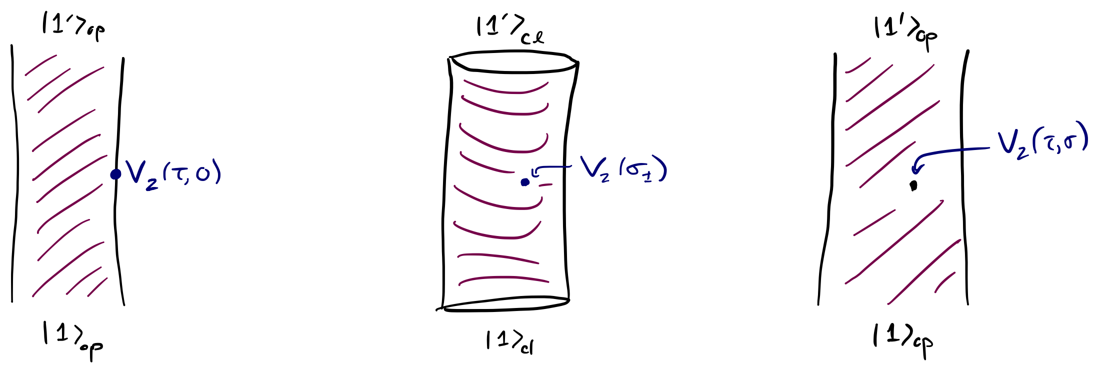
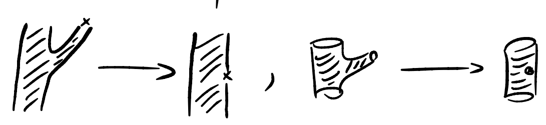
Now that we have some control over the string spectrum, we need to be able to study interactions. At a cartoon level, we want to describe the branching and joining of quantized strings.



In practice, we need to work in our first-quantized formalism, so what should we do? We model the emission or absorption of a given string state from a reference string worldsheet by the action of some local operator:



Two explanations:
- a given string state is "quantum-sized", so should be almost point-like! (Not that satisfying)
- After Wick rotating worldsheet theory,  by conformal transformation, so can choose this picture as a gauge choice.

What are these "vertex operators"?

Two key requirements

- time evolution on worldsheet is a gauge transformation, so the position of the vertex operator shouldn't be meaningful:

  Open string vertex operator: $\int d\tau\, V_2(\tau)$ ∼ inserted on the boundary

  Closed string vertex operator: $\int d\tau d\sigma\, V_2(\tau,\sigma)$ ∼ inserted in the interior

- We need the absorption/emission of strings to map $\mathcal{H}_{phys} \to \mathcal{H}_{phys}$ and $\mathcal{H}_{null} \to \mathcal{H}_{null}$. How to arrange this? (Let's focus on the open string for now).

$$L_m \left[\int V(\tau,0)\, d\tau\, |\phi_{phys}\rangle \right] = \int d\tau\, [L_m, V(\tau,0)]\, |\phi_{phys}\rangle$$

$$= 0 \text{ if } [L_m, V(\tau,0)] \sim \partial_\tau(\cdots)$$

$$\int V(\tau,0)\, d\tau \left[L_{-m} |\phi\rangle \right] = \int d\tau\, [V(\tau,0), L_{-m}]\, |\phi\rangle + L_{-m}(\cdots)$$

null again if $[V(\tau,0), L_{-m}] \sim \partial_\tau(\cdots)$

More generally, conformal transformations of the open string act as $\tau \to \tilde{\tau}(\tau)$, and we want (morally)

$$\int V(\tau,0)d\tau \to \int \tilde{V}(\tilde{\tau},0)d\tilde{\tau} \overset{?}{=} \int V(\tau,0)\left(\frac{d\tau}{d\tilde{\tau}}\right)d\tilde{\tau}$$

We define an operator $A(\tau)$ to be a primary operator of weight h if under the transformation $\tau \to \tilde{\tau}(\tau)$, one has

$$\boxed{A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(\frac{d\tau}{d\tilde{\tau}}\right)^h}$$

For such an operator with $h=1$, we have $\int \tilde{A}(\tilde{\tau})d\tilde{\tau} = \int A(\tau)d\tau$, so the integrated operator is conformally invariant. If we study the infinitesimal case, $\tau \to \tilde{\tau} = \tau + \epsilon(\tau)$, we then have

$$A(\tau) \to \tilde{A}(\tilde{\tau}) = A(\tau)\left(1 - h\partial_\tau \epsilon\right) \quad \text{and} \quad \tilde{A}(\tilde{\tau}) = \tilde{A}(\tau+\epsilon) \approx \tilde{A}(\tau) + \epsilon\,\partial_\tau A(\tau)$$

We find the variation

$$\begin{aligned}\delta A(\tau) = \tilde{A}(\tau) - A(\tau) &= -\epsilon\partial_\tau A - h(\partial_\tau \epsilon)A \\ &= -\partial_\tau(\epsilon A) - (h-1)(\partial_\tau\epsilon)A \\ &= \text{total derivative if } h=1.\end{aligned}$$

Setting $\epsilon = ie^{im\tau}$, we get the action of the Virasoro operators

$$[L_m, A(\tau)] = e^{im\tau}(-i\partial_\tau + mh)A(\tau)$$

So the problem is to identify primaries of weight $h=1$ corresponding to the physical states in the string Hilbert space, use these to compute string amplitudes!

Remark: closed string version is analogous. Now a primary operator of dimension $(h,\tilde{h})$ is an operator transforming according to

$$A(\sigma_+,\sigma_-) \to \tilde{A}(\tilde{\sigma}_+,\tilde{\sigma}_-) = \left(\frac{d\sigma_+}{d\tilde{\sigma}_+}\right)^{\tilde{h}}\left(\frac{d\sigma_-}{d\tilde{\sigma}_-}\right)^{h} A(\sigma_+,\sigma_-)$$

Which gives infinitesimal transformation

$$\begin{aligned}\delta A(\tau,\sigma) &= -\partial_+(\tilde{\epsilon}A) - (\tilde{h}-1)(\partial_+\tilde{\epsilon})A - \partial_-(\epsilon A) - (h-1)(\partial_-\epsilon)A \\ &= \text{total derivatives if } h=\tilde{h}=1.\end{aligned}$$

For $\epsilon = \frac{i}{2}e^{2im\sigma_+}$ this gives the action of L_m:

$$[L_m, A(\sigma_\pm)] = \frac{1}{2}e^{2im\sigma_+}\left(-i\partial_+ + 2mh\right)A(\sigma_\pm)$$

$$[\tilde{L}_m, A(\sigma_\pm)] = \frac{1}{2}e^{2im\sigma_-}\left(-i\partial_- + 2m\tilde{h}\right)A(\sigma_\pm)$$

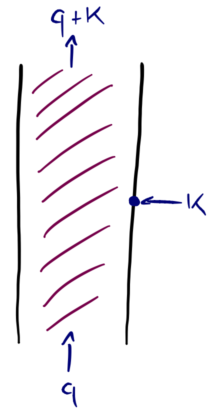
Today we restrict to open-string vertex operators, so all operators are boundary local operators with (say) $\sigma=0$.

Consider boundary scalar operator $X^\mu(\tau,0)$. We can check conformal transformations:

$$X^\mu(\tau,0) = x^\mu + \tau p^\mu + i\sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \qquad L_m = \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n$$

Direct computation: $\left[L_m, X^\mu(\tau,0)\right] = -i\sum_n \alpha_n^\mu e^{-i(n-m)\tau} = -ie^{im\tau}\frac{d}{d\tau}\left(X^\mu(\tau,0)\right)$, so $h=0$ (matches engineering dimension).

How should we build vertex operators associated to (say) a tachyon of momentum k?



Focus on effect on spacetime momentum: $V_k(\tau)|0;q\rangle \sim |\phi;q+k\rangle$, so demand $[P^\mu, V_k(\tau)] = (k)^\mu V_k(\tau)$. We know how to arrange this:

$$V_k(\tau) \sim \exp(ik\cdot x)$$ completed to give boundary local operator.

How to accomplish this? Naive guess is $V_k(\tau) = \exp\left(ik\cdot X(\tau)\right)$, but here we are beset by ordering questions. Relatedly, conformal dimension of $X^\mu(\tau)$ is zero, so naive dimension of this operator is also zero. But need to be more careful.

$$:\exp\left(ik\cdot X(\tau)\right): = \exp\left(-ik\cdot\sum_{n>0}\frac{\alpha_{-n}}{n}e^{in\tau}\right)e^{ik\cdot(x+\tau p)}\exp\left(ik\cdot\sum_{n>0}\frac{\alpha_n}{n}e^{-in\tau}\right)$$

As with L_0, there is room for some finite re-ordering here, especially with treatment of zero modes $\left(e^{ik\cdot(x+\tau p)} \text{ vs. } e^{ik\cdot x}e^{ik\cdot\tau p} \text{ vs. other ordering}\right)$. We've made a conventional choice.

Important observation: $\exp\left(ik\cdot\alpha_m\right)\exp\left(ik\cdot\alpha_n\right) = \exp\left(ik\cdot(\alpha_m+\alpha_n) - \frac{1}{2}(k\cdot k)\delta_{m+n,0}\right)$, so re-ordering is free if $k^2=0$!

An important computation (see PS3) is the conformal transformation of the normally-ordered exponential operator:

$$\left[L_m, :e^{ik\cdot X(\tau)}:\right] = e^{im\tau}\left(-i\frac{d}{d\tau} + \frac{1}{2}m(k\cdot k)\right):e^{ik\cdot X(\tau)}:$$

from re-ordering, gives $h=\frac{1}{2}(k\cdot k)$

Thus we get a good vertex operator for $k^2 = 2$ $(\alpha' k^2 = 1)$, same as tachyon mass-shell condition!

Things get more interesting for the level-one states. Here we have $(k\cdot k)=0$ and we are looking for the photon emission/absorption vertex.

$$V_\xi(\tau) = \underbrace{(\ \cdots\ )}_{h=1,\text{ no zero mode dependence}} \underbrace{:e^{ik\cdot X(\tau)}:}_{h=0} \qquad \text{w/no "short-distance singularities"}$$

To eliminate zero mode dependence, should differentiate X^μ:

$$\begin{aligned}[L_m, \partial_\tau X^\mu(\tau)] &= \frac{\partial}{\partial\tau}\left([L_m, X^\mu(\tau)]\right)\\ &= \frac{\partial}{\partial\tau}\left(e^{im\tau}(-i\partial_\tau X^\mu)\right)\\ &= e^{im\tau}\left(-i\partial_\tau + m\right)(\partial_\tau X^\mu)\end{aligned}$$

So $\partial_\tau X^\mu \equiv \dot{X}^\mu(\tau)$ is a primary of dimension $h=1$. So perhaps

$$V_\xi(\tau) \overset{?}{=} (\xi\cdot\dot{X}):e^{ik\cdot X}:$$

Need to normal-order the whole thing, and then conformal dimensions don't necessarily add. But in $(\xi\cdot\dot{X})$, each oscillator contracted w/ ξ, and in $:e^{ik\cdot X}:$ each oscillator contracted with k.

$$[\alpha_m\cdot\xi, \alpha_n\cdot k] \sim \delta_{m+n,0}\,(\xi\cdot k)$$

So for physical polarization $\xi\cdot k=0$, no corrections from normal ordering! So indeed

$$V_\xi(\tau) = \xi\cdot\dot{X}\, e^{ik\cdot X} \qquad \text{w/ } k^2=0 \ \&\ \xi\cdot k = 0$$

Decoupling the longitudinal mode: $k\cdot\dot{X}\,e^{ik\cdot X} = -i\partial_\tau\left(e^{ik\cdot X}\right)$, which formally vanishes after integrating. So rules for constructing these vertex operators closely parallels the construction of physical states. This is not a coincidence of course!

Continuing to level 2: $\gamma_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0;k\rangle \longleftarrow - - \rightarrow \gamma_{\mu\nu}\dot{X}^\mu\dot{X}^\nu :e^{ik\cdot X}:$ w/ $k^2=-2$ (so $h_{exp}=-1$)

Normal ordering corrections? in $\gamma_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$, oscillator re-orderings proportional to $\gamma^\mu{}_\mu = 0$ for traceless.

b/w $\gamma_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$ & $:e^{ik\cdot X}:$, proportional to $k\cdot\gamma = 0$ for transverse.

Need to do some more work to show this is the only possibility, but these match the D=26 physical/null states.