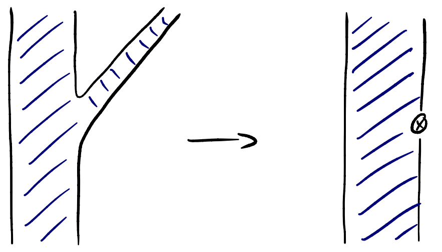
Recall from last time, we introduced "vertex operators" that describe the emission/absorption of physical string states from the perspective of a fixed string worldsheet



Thus we should have a state $\to$ operator map: $|\phi\rangle_{\text{open}} \longrightarrow V_\phi(\tau)$ with $V_\phi(\tau)$ a conformal primary of dimension $h=1$.
We saw some simple examples:

$$|0;k\rangle \xrightarrow{k\cdot k=2} :\exp(ik\cdot X):$$

$$|\zeta;k\rangle \xrightarrow{k\cdot k=0} \zeta\cdot\dot{X}\exp(ik\cdot X) \qquad w/\ \zeta\cdot k=0$$

(Recall $:e^{ik\cdot X}:$ has $h=\frac{1}{2}k\cdot k$)

We can continue at level 2:

$$|\gamma;k\rangle \xrightarrow{k\cdot k=-2} :\gamma_{\mu\nu}\dot{X}^\mu\dot{X}^\nu\exp(ik\cdot X): \qquad w/\ \gamma_\mu{}^\mu=0 \text{ and } k\cdot\gamma=0$$

Can get a little more clever and try to use $:\eta\cdot\ddot{X}\exp(ik\cdot X):$, but this isn't a primary so need correction terms (as w/ states in 2nd problem sheet). In $D=26$, above are enough.

We can see gauge invariance at this level as well. Consider photon vertex operator with $\zeta\sim k$:

$$\int d\tau\ k\cdot\dot{X}\exp(ik\cdot X) = \int d\tau\ \partial_\tau\left(\exp(ik\cdot X)\right) = 0 \qquad \text{(up to boundary terms)}$$

Extra gauge invariance @ $D=26$ more complicated to see; won't go into it here.

There's a superficial similarity b/w expressions for states and vertex operators. This is more than a coincidence. Consider the action of $:\exp(ik\cdot X):$ on the string vacuum state $|0;0\rangle$ ...

$$:e^{ik\cdot X(\tau)}:|0;0\rangle = e^{i\tau L_0}:e^{ik\cdot X(0)}:e^{-i\tau L_0}|0;0\rangle$$

$$= e^{i\tau L_0}\exp\left(\sum_{n>0}\frac{k\cdot\alpha_{-n}}{n}\right)|0;k\rangle$$

Now let us define $z=e^{i\tau}=e^{t}$ where $t=i\tau$, $\tau=-it$, so t is Euclidean worldsheet time

$$= z^{L_0}\exp\left(\sum_{n>0}\frac{k\cdot\alpha_{-n}}{n}\right)|0;k\rangle \qquad \left(L_0=\frac{p^2}{2}+N\right)$$

$$= z^{1+N}\left(1+(k\cdot\alpha_{-1})+\frac{1}{2}\left((k\cdot\alpha_{-2})+(k\cdot\alpha_{-1})^2\right)+\dots\right)|0;k\rangle$$

$$= z\left(|0;k\rangle + z(k\cdot\alpha_{-1})|0;k\rangle + \frac{z^2}{2}\left((k\cdot\alpha_{-2})+(k\cdot\alpha_{-1})^2\right)|0;k\rangle + \mathcal{O}(z^3)\right)$$

I.e., we can recover $|0;k\rangle$ by taking $\lim_{z\to 0}\frac{1}{z}V_T(k;\tau)|0;0\rangle = \lim_{t\to-\infty}e^{-t}V_T(k;it)|0;k\rangle$. Let's try it with the photon:

$$\zeta\cdot\dot{X}\exp(i\overset{k^2=0}{\overset{\downarrow}{k}}\cdot X)|0;0\rangle = z^N\sum_{n>0}(\zeta\cdot\alpha_{-n})\exp\left(\sum_{n>0}\frac{k\cdot\alpha_{-n}}{n}\right)|0;k\rangle$$

$$= \left(z\,(\zeta\cdot\alpha_{-1}) + z^2\left(\zeta\cdot\alpha_{-2}+(\zeta\cdot\alpha_{-1})(k\cdot\alpha_{-1})\right)+\dots\right)|0;k\rangle$$

And again we recover $|\zeta;k\rangle$ by taking $\lim_{z\to 0}\frac{1}{z}V_\zeta(k;it)|0;0\rangle$.

The pattern is clear: for physical state $|\psi\rangle$ w/ vertex operator $V_\psi(\tau)$, we should have

$$|\psi\rangle = \lim_{z\to 0} \frac{1}{z} V_\psi(it)|0;0\rangle$$

An analogous statement holds for "out" states. Looking at the tachyon, we have

$$\langle 0;0|V_T(k;it) = \langle 0;k|\exp\left(\sum_{n>0}\frac{\alpha_n\cdot k}{n}\right)z^{-L_0}$$

$$= \frac{1}{z}\langle 0;k|\left(1 + z^{-1}(\alpha_1\cdot k) + \frac{z^{-2}}{2}(\alpha_2\cdot k + (\alpha_1\cdot k)^2) + \dots\right)$$

$$\lim_{z\to\infty} z\,\langle 0;0|V_T(k,\tau) = \lim_{t\to\infty} e^{t}\langle 0;0|V_T(k,it)$$

The picture is that we prepare states by acting in infinite Euclidean past (or future) with vertex operators, giving us an operator $\to$ state map.

Euclidean time evolution operator e^{-tL_0} suppresses states with larger values of L_0, so infinite t-evolution projects onto lowest-energy state (after rescaling). This is frequently a useful mechanism in QFT.

This is part of a totally general part of Conformal Field Theory, the "operator-state correspondence". The general construction would be

$$A(\tau) \longrightarrow |\psi_A\rangle = \lim_{t\to-\infty} z^{h_A} A(it)\underbrace{|\Omega\rangle}_{\text{vacuum state}}$$

$$\longrightarrow \langle\psi_A| = \lim_{t\to+\infty} z^{-h_A}\langle\Omega|A(it)$$

Remark: closed string version is analogous. Now a primary operator of dimension $(h,\tilde{h})$ is an operator transforming according to

$$A(\sigma_+,\sigma_-) \to \tilde{A}(\tilde{\sigma}_+,\tilde{\sigma}_-) = \left(\frac{d\sigma_+}{d\tilde{\sigma}_+}\right)^{\tilde{h}}\left(\frac{d\sigma_-}{d\tilde{\sigma}_-}\right)^{h} A(\sigma_+,\sigma_-)$$

Which gives infinitesimal transformation $\quad \delta A(\tau,\sigma) = -\partial_+(\tilde{\epsilon}A) - (\tilde{h}-1)(\partial_+\tilde{\epsilon})A - \partial_-(\epsilon A) - (h-1)(\partial_-\epsilon)A$

$$= \text{total derivatives if } h = \tilde{h} = 1.$$

For $\epsilon = \frac{i}{2}e^{2im\sigma_+}$ this gives the action of L_m:

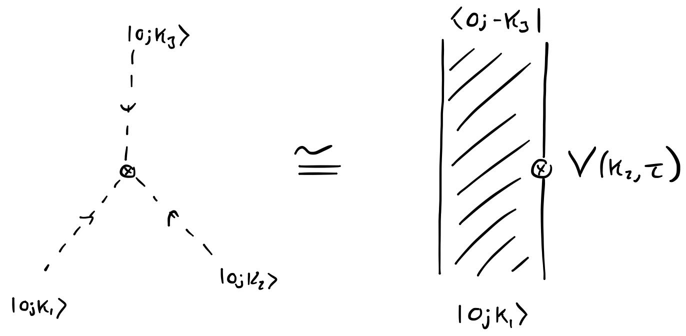
$$[L_m, A(\sigma_\pm)] = \frac{1}{2}e^{2im\sigma_+}\left(-i\partial_+ + 2mh\right)A(\sigma_\pm)$$

$$[\tilde{L}_m, A(\sigma_\pm)] = \frac{1}{2}e^{2im\sigma_-}\left(-i\partial_- + 2m\tilde{h}\right)A(\sigma_\pm)$$

For closed strings, $:e^{ik\cdot X(\sigma_\pm)}:$ is a primary w/ $h = \tilde{h} = \frac{k\cdot k}{8}$, so $V_{T,\text{closed}}(k;\sigma_\pm) = :e^{ik\cdot X(\sigma_\pm)}:$ w/ $k^2 = 8$, and similar for excited states. The map to states is now given by

$$|\psi\rangle = \lim_{t\to-\infty}(z\bar{z})^{-1}V_\psi(it,\sigma)|0;0\rangle \quad \text{with } z = e^{2(t-i\sigma)},\ \bar{z} = e^{2(t+i\sigma)}$$

We're ready to look at our first string interactions : 3-pt. vertices.



$$\mathcal{A}(k_1,k_2,k_3) \stackrel{?}{=} g_o \int d\tau \, \langle 0;-k_3|V_T(\tau,k_2)|0;k_1\rangle \Big/ \text{Vol (conf.)} \qquad \left(g_o : \text{open string coupling constant}\right)$$

$$= g_o \int d\tau \, \langle 0;-k_3| \underset{=1}{e^{i\tau L_0}} V_T(0;k_2) \underset{=1}{e^{-i\tau L_0}} |0;k_1\rangle \Big/ \text{Vol (conf.)}$$

$$= g_o \int d\tau \, \langle 0;-k_3|V_T(0;k_2)|0;k_1\rangle \Big/ \text{Vol (conf.)}$$

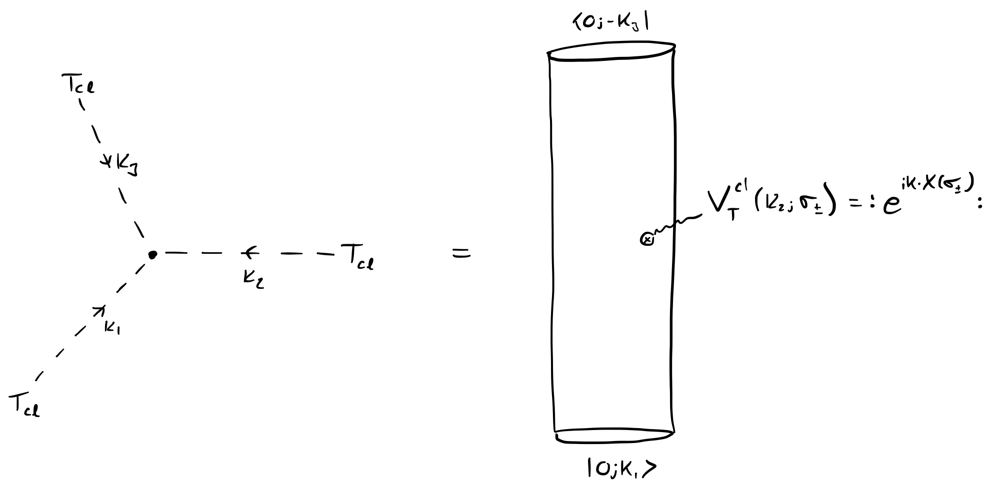
$$= g_o \int d\tau \, \langle 0;-k_3| e^{ik_2\cdot x} |0;k_1\rangle \Big/ \text{Vol (conf.)}$$

$$= g_o \int d\tau \, \langle 0;-k_3|0;k_1+k_2\rangle \Big/ \text{Vol (conf.)}$$

$$= g_o \, \delta(k_1+k_2+k_3) \int_{-\infty}^{\infty} d\tau \Big/ \text{Vol (conf.)}$$

$$= g_o \, \delta(k_1+k_2+k_3)$$

Here we had to divide out by a divergent "volume" integral. This should have been expected, because τ- translations of $V_T(k;\tau)$ are residual gauge symmetries (they leave the past & future states invariant). Hence we "divide by infinite volume of gauge group", or alternatively, gauge fix to $\tau = 0$.



$$\mathcal{A}_3(k_1,k_2,k_3) = g_c \int d^2\sigma_\pm \, \langle 0;-k_3| e^{2i\sigma_- L_0 + 2i\sigma_+ \tilde{L}_0} \underbrace{V_T^{cl}(k_2;0)}_{(\text{creation})\; e^{ik_2\cdot x}\; (\text{annihilation})} e^{-2i\sigma_- L_0 - 2i\sigma_+ \tilde{L}_0} |0;k_1\rangle \Big/ \text{Vol (Conf.)} \qquad \left(g_c : \text{closed string coupling constant}\right)$$

$$= g_c \int d^2\sigma_\pm \, \langle 0;-k_3| e^{ik_2\cdot x} |0;k_1\rangle \Big/ \text{Vol (Conf.)}$$

$$= g_c \int d^2\sigma_\pm \, \delta(k_1+k_2+k_3) \Big/ \text{Vol (Conf.)}$$

$$= g_c \, \delta(k_1+k_2+k_3)$$