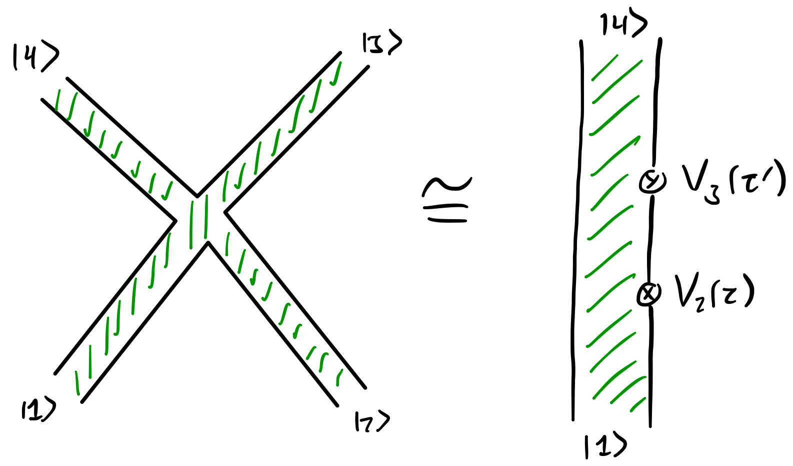
Let's take a look at the four-point tachyon amplitude



$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{\tau' > \tau} d\tau' d\tau \, \langle 4 | V_3(\tau') V_2(\tau) | 1 \rangle \Big/ \mathrm{Vol(Conf)}$$

can act w/ L_0 to translate simultaneously

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 4 | V_3(0) V_2(\tau) | 1 \rangle$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 4 | V_3(0) e^{i\tau L_0} V_2(0) e^{-i\tau \overset{1}{L_0}} | 1 \rangle$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 4 | V_3(0) e^{i\tau(L_0 - 1)} V_2(0) | 1 \rangle$$

$$= g_0^2 \langle 4 | V_3(0) \left[\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0-1)} \right] V_2(0) | 1 \rangle \longrightarrow g_0^2 \langle 4 | V_3(0) \int_{-\infty}^{0} d\tau \, e^{i\tau(L_0 - 1 - i\epsilon)} V_2(0) | 1 \rangle$$

$$\overset{?}{=} g_0^2 \langle 4 | V_3(0) \frac{-i + (\text{oscillatory})}{L_0 - 1} V_2(0) | 1 \rangle \; *$$

$\Big\downarrow$ rotate contour $t = i\tau$

$$g_0^2 \langle 4 | V_3(0) \int_{-\infty}^{\infty} dt \, e^{t(L_0 - 1)} V_2(0) | 1 \rangle$$

We now have a very natural Euclidean worldsheet calculation!

*

There is an analogy in ordinary QFT:

"Schwinger proper-time" formalism

$$\frac{-i}{p^2 + m^2 - i\epsilon} = \int_{-\infty}^{0} d\tau \, e^{i\tau(p^2 + m^2 - i\epsilon)}$$

$i\epsilon$ avoids poles due to on-shell production

$\Big\downarrow$ rotate contour, $\tau = -it$, $t = i\tau$

$\left\{ \text{as long as } p^2 + m^2 > 0, \text{ i.e., below threshold for production} \right\}$

$$\frac{1}{p^2 + m^2} = \int_{-\infty}^{0} dt \, e^{t(p^2 + m^2)}$$

Continuing with rotated contour:

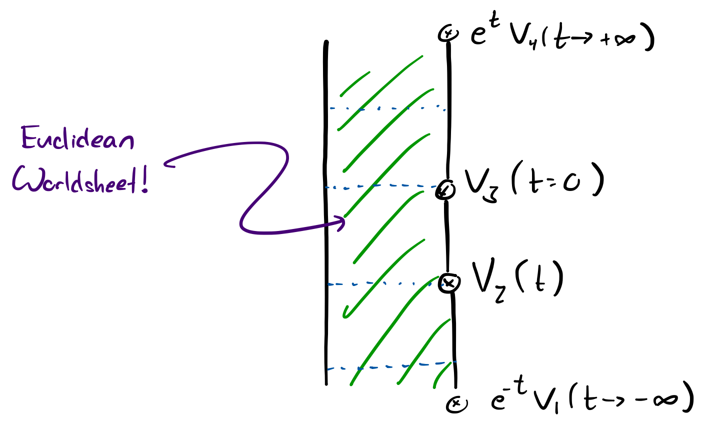
$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_o^2 \langle 4|V_3(0)\left[\int_{-\infty}^{0} e^{t(L_o-1)}dt\right]V_2(0)|1\rangle$$

$$= g_o^2 \int_{-\infty}^{0} dt \, \langle 4|V_3(0)\, e^{t(L_o-1)} V_2(0)|1\rangle$$

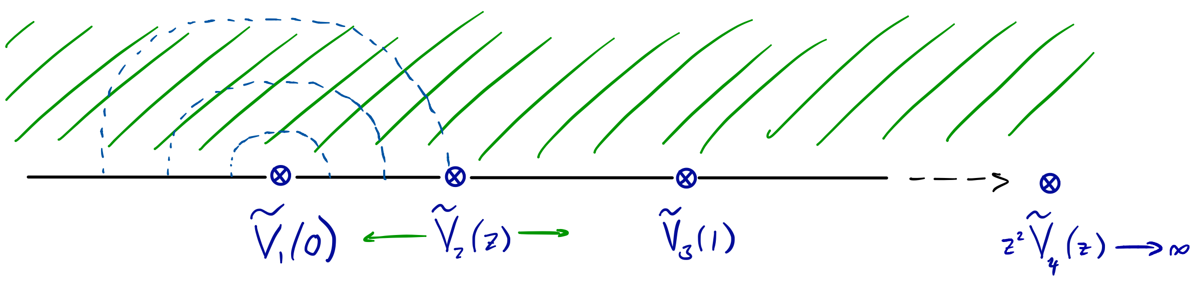
$$= g_o^2 \int_{-\infty}^{0} dt \, \langle 4|V_3(0)\, e^{tL_o} V_2(0) e^{-tL_o}|1\rangle$$

$$= g_o^2 \int_{-\infty}^{0} dt \, \langle 4|V_3(0) V_2(-it)|1\rangle$$

All of our operators are now displaced in Euclidean worldsheet time, so we discover a Euclidean worldsheet interpretation of our amplitude:



Now let's do a Euclidean conformal map: take $z = e^{t-i\sigma}$, $\bar{z} = e^{t+i\sigma}$, so $dz d\bar{z} = e^{2t}(dt^2+d\sigma^2) \overset{\text{Weyl equivalent}}{\approx} dt^2+d\sigma^2$



Conformal transformation modifies vertex operators according to $\tilde{V}(z=\bar{z}) = \left(\frac{dt}{dz}\right)V(t) = \frac{1}{z}V(t)$, so

- $\lim_{t\to-\infty} e^{-t}V_1(t) = \lim_{z\to0}\tilde{V}_1(z)$
- $V_2(t)dt = \tilde{V}_2(z)dz$
- $V_3(0) = \tilde{V}_3(1)$
- $\lim_{t\to\infty} e^{t}V_4(t) = \lim_{z\to\infty} z^2\tilde{V}_4(z)$

We can be a bit more coherent about conformal transformations and gauge fixing now. The conformal transformations of the upper half plane are well known to be given by $PSL(2,\mathbb{R})$:

$$z \mapsto \frac{az+b}{cz+d} \;;\quad a,b,c,d \in \mathbb{R}\;;\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

This is a 3-dimensional group of residual gauge symmetries. It acts triply transitively on the UHP (while preserving cyclic ordering of points on the boundary. In particular, for any four points $z_1 < z_3 < z_4$ on boundary:

$$z \mapsto \frac{z_{34}(z_1 - z)}{z_{13}(z - z_4)} \quad \text{with } z_{ij} = z_i - z_j.$$

This maps $z_1 \to 0$, $z_3 \to 1$, $z_4 \to \infty$, and corresponds to our "gauge fixing" for the 3-pt. amplitude. For a fourth point $z_1 < z_2 < z_3$, this maps

$$z_2 \mapsto \frac{z_{12} z_{34}}{z_{13} z_{24}} \in (0,1)\,, \text{ the "conformal cross ratio".}$$

Fixing 0 & ∞, can send two additional points $(z_2, z_3) \mapsto (z, 1)$. In fancier terms, we have that $(0,1)$ is the moduli space of conformal structures on the UHP with four marked points, and this is what we're integrating over in our scattering amplitude calculation.

A bit more on gauge fixing. Can write our amplitude more covariantly:

$$\int dz_1 dz_2 dz_3 dz_4 \left\langle \tilde{V}_1(z_1)\tilde{V}_2(z_2)\tilde{V}_3(z_3)\tilde{V}_4(z_4) \right\rangle_{UHP} \Big/ \mathrm{Vol}(SL(2,\mathbb{R}))$$

Using the Fadeev-Popov trick, we gauge fix by imposing $z_1 = z_1^{(0)}$, $z_3 = z_3^{(0)}$, $z_4 = z_4^{(0)}$

$$= \int dz_1 dz_2 dz_3 dz_4\, \delta(z_1 - z_1^{(0)})\delta(z_3 - z_3^{(0)})\delta(z_4 - z_4^{(0)}) \left\langle \tilde{V}_4(z_4)\tilde{V}_3(z_3)\tilde{V}_2(z_2)\tilde{V}_1(z_1)\right\rangle_{UHP} \left| \mathrm{Det}\left(\underbrace{\frac{\partial(z_1,z_3,z_4)}{\partial(\lambda_1,\lambda_2,\lambda_3)}}_{=(z_1-z_3)(z_3-z_4)(z_4-z_1)}\right)\right|$$

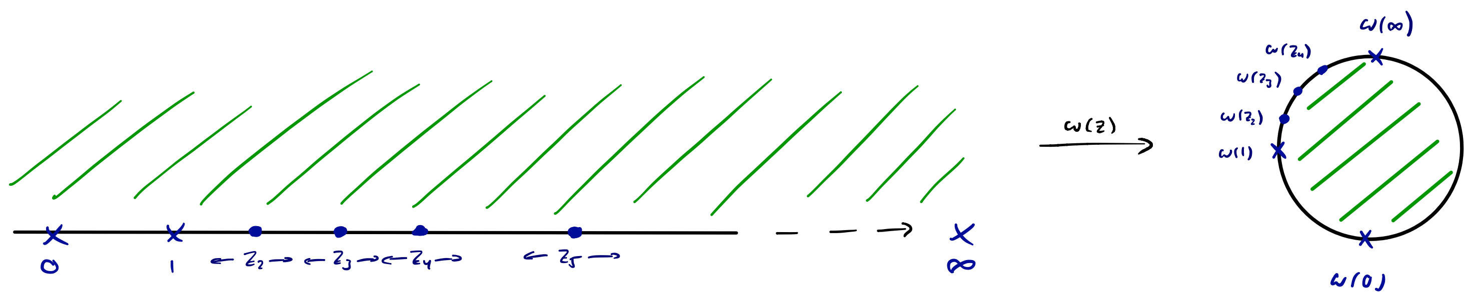
λ: generators of $SL(2,\mathbb{R})$

$z_1^{(0)} = 0$, $z_3^{(0)} = 1$, $z_4^{(0)} = \Lambda \to \infty$

$$= \int dz_2\, (\Lambda^2 - \Lambda)\left\langle \tilde{V}_4(\Lambda)\tilde{V}_3(1)\tilde{V}_2(z_2)\tilde{V}_1(0)\right\rangle_{UHP}$$

$$\xrightarrow{\Lambda\to\infty} \int dz \left\langle 4 \right| \tilde{V}_3(1)\tilde{V}_2(z)\left| 1 \right\rangle$$

Remark: defining $\omega = \frac{z-i}{z+i}$ maps the UHP to the unit disk!

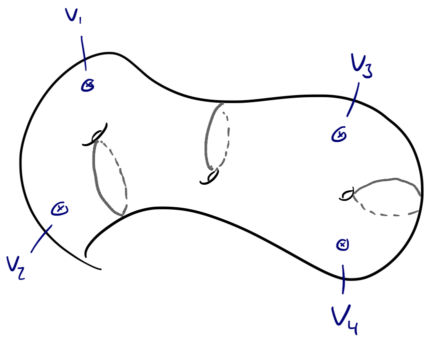


This shows how cyclic symmetry should manifest.

Let's take stock of the general picture for string scattering, then.

Physical states $\{\psi_i\}$ $\longrightarrow$ Vertex operators $V_{\psi_i}(\sigma_\pm)$ or $V_{\psi_i}(\tau)$, primaries of dim. $(h,\tilde{h})=(1,1)$ or $h=1$.

Scattering Process (say closed strings)

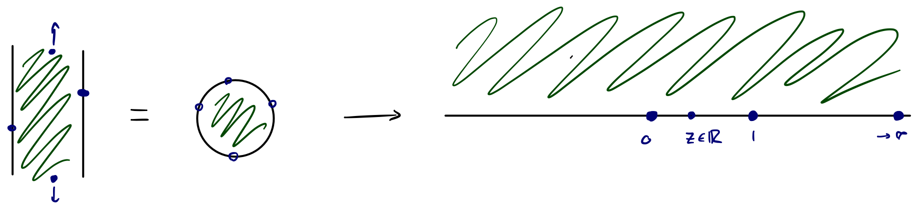


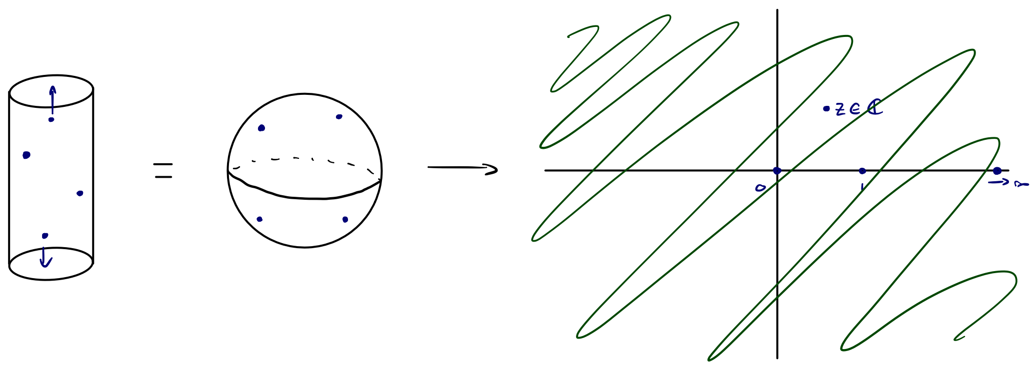
$$\longrightarrow \quad \mathcal{A}_g(\psi_1,\dots,\psi_4) = \int_{\mathcal{M}_{g,4}} [d\mu]\, \langle V_1 V_2 V_3 V_4 \rangle_{\Sigma_{g,4}(\mu)}$$

(i.e. Weyl)
parametrizes conformal equiv. classes of metrics
$\cong$ complex structures

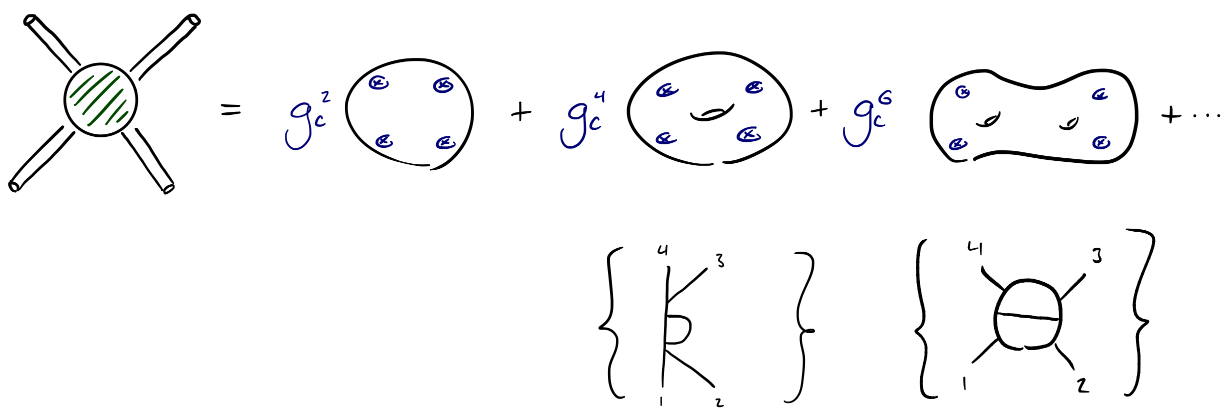
"Moduli space of Riemann surfaces"

In general, the description of $\mathcal{M}_\Sigma$ & $[d\mu]$ will get quite complicated, though at low genus it isn't so bad.





The string perturbation series will be a genus expansion:



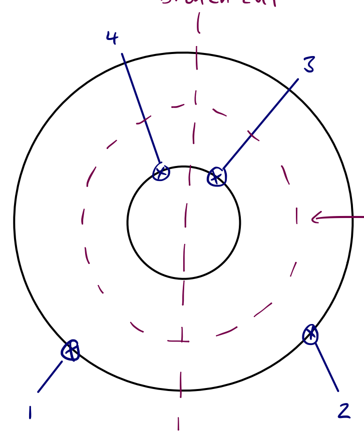
Some immediate observations about this set-up:

- ▷ One "diagram" per order in perturbation theory
- ▷ Degeneration limits look like many Feynman diagrams
- ▷ This gives massive generalization of DHS duality

A particularly interesting generalization of DHS duality is open/closed duality:

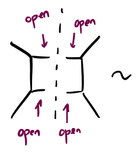
open string branch cut

Annulus amplitude

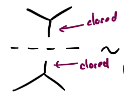


closed string poles!
(should go on-shell in $1+2 \to 3+4$ process)

This is actually a magical amplitude

Branch cut: 

$\sim g_o^4$

Pole: 

$\sim g_c^2$

So $g_c \sim g_o^2$ (also $g_c \sim$ gravitational coupling & $g_o \sim U(1)$ gauge coupling)

Finally, the worldsheet picture gives an heuristic explanation of good UV behaviour. The source of possible divergences in our string amplitudes is the boundaries of $\mathcal{M}_\Sigma$.

get singularities from on-shell state propagating for long times "all divergences are IR divergences"