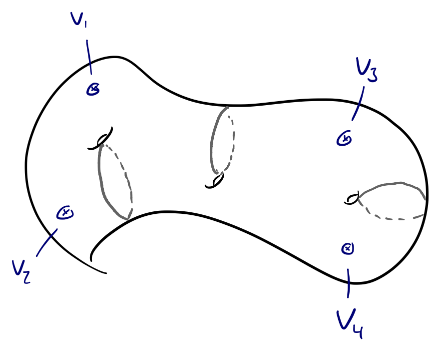
We will wrap up the discussion of string scattering with some comments of a formal (rather than computational) nature.

Physical states $\{\mathcal{V}_i\}$ $\longrightarrow$ Vertex operators $V_{\mathcal{V}_i}(\sigma_?)$ or $V_{\mathcal{V}_i}(\tau)$, primaries of dim. $(h,\tilde{h})=(1,1)$ or $h=1$.

Scattering Process (say closed strings)

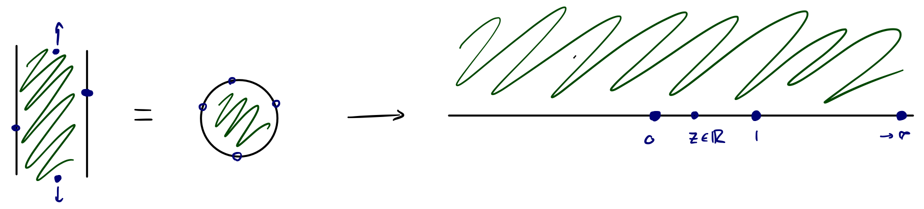


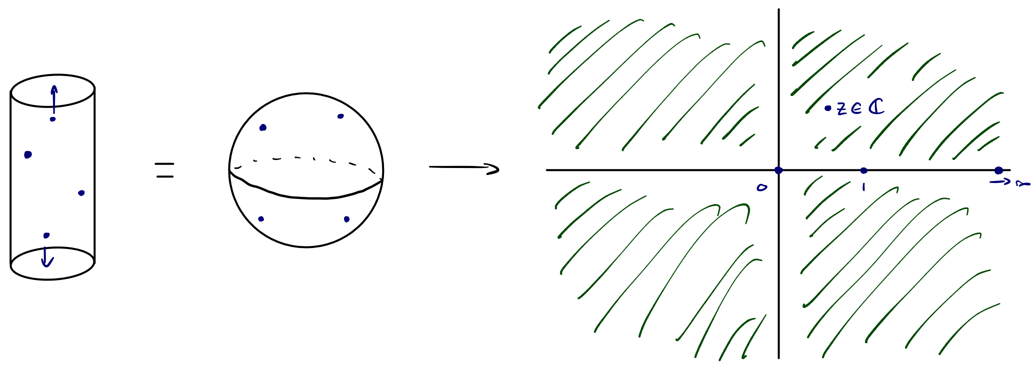
$$\mathcal{A}_g(\mathcal{V}_1,\dots,\mathcal{V}_4) = \int_{\mathcal{M}_{g,4}} [d\mu]\, \langle V_1V_2V_3V_4\rangle_{\Sigma_{g,4}(\mu)}$$

(i.e. Weyl) parameterizes conformal equiv. classes of metrics $\cong$ complex structures

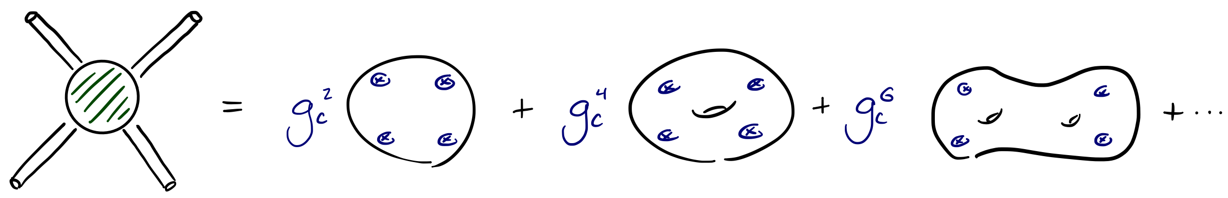
"Moduli space of Riemann surfaces"

In general, the description of $\mathcal{M}_\Sigma$ & $[d\mu]$ will get quite complicated, though at low genus it isn't so bad.



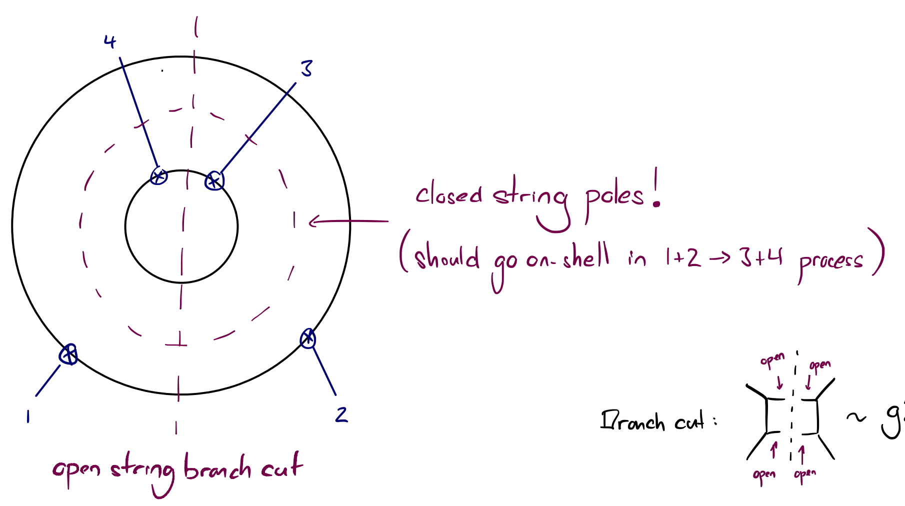


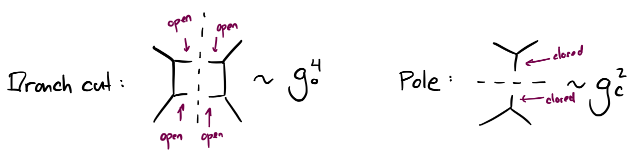
The torus is also an interesting case, though we won't spend time with it. The string perturbation series will be a genus expansion:



Some immediate observations about this set-up:
- ▷ One "diagram" per order in perturbation theory
- ▷ Degeneration limits look like many Feynman diagrams

Magical annulus amplitude:





So $g_c \sim g_o^2$ (also $g_c\sim$ gravitational coupling & $g_o\sim U(1)$ gauge coupling)

Our next topic is strings in background fields. We've identified various massless fields in the bosonic string spectrum, and we should be able to describe the dynamics of string excitations propagating in nontrivial backgrounds for those fields.

Firstly, we identified a graviton in the string spectrum, so presumably target spacetime should be allowed a non-trivial metric (or topology even!). We can write down an action for such a configuration:

$$S_P^{flat}[\gamma, X] = \frac{-1}{2\pi}\int d^2\sigma \sqrt{\gamma}\, \gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \longmapsto S_P^{(G)}[\gamma, X] = \frac{-1}{2\pi}\int d^2\sigma \sqrt{\gamma}\, \gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)$$

still classically Weyl-invariant; take $\gamma_{ab} = e^{2\phi(\sigma)}\delta_{ab}$

$$S_P^{(G)}[X] = \frac{-1}{2\pi}\int d^2\sigma\, G_{\mu\nu}(X)\,\partial_a X^\mu \partial^a X^\nu$$

For general $G_{\mu\nu}(X)$, this defines an interacting worldsheet QFT. This makes things complicated, but first let's take the case where $G_{\mu\nu}$ describes a small perturbation on flat space.

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \underbrace{\chi_{\mu\nu}}_{\text{small perturbation}} e^{ik\cdot X}$$

If we compute string amplitudes in this background and treat $\chi_{\mu\nu}$ as a perturbative parameter, then we will have

$$\langle A_1 \cdots A_n \rangle_\chi = \langle A_1 \cdots A_n \rangle_0 - \frac{1}{2\pi}\langle A_1 \cdots A_n \underbrace{\int d^2z\, \chi_{\mu\nu} e^{ik\cdot X}\partial X^\mu \bar{\partial} X^\nu}_{\text{graviton vertex operator!}} \rangle_0 + \mathcal{O}(\chi_{\mu\nu}^2)$$

We know there are important physicality conditions for $\chi_{\mu\nu}e^{ik\cdot X}$ in order for this to be a physical vertex operator. This is the infinitesimal version of some rather strong consistency conditions on $G_{\mu\nu}(X)$. We require the interacting worldsheet theory to be Weyl invariant at the quantum level, which in particular implies that the conformal-gauge theory is conformally invariant.

Is it? Focusing on scale invariance, we know that quantum effects can lead to the running of couplings as length scale changes, which means a classically conformally invariant theory may not be so after quantization. For our action above, the couplings are encoded in the metric function $G_{\mu\nu}(X)$. How to compute?

Worldsheet theory is not easy to analyze in presence of general background. We use some kind of perturbation theory.

$$X^\mu = \underset{\substack{\uparrow \\ \text{"background value"}}}{X_0^\mu} + \underset{\substack{\uparrow \\ \text{dynamical}}}{x^\mu}$$

$\left\{ \begin{array}{l} \text{for our purposes, set } X_0^\mu = \text{const.} \\ \text{more generally, could have } X_0^\mu(\tau,\sigma) \\ \text{solving e.o.m.} \end{array} \right\}$

Expand action for fluctuations about $x^\mu = 0$ (consider pure geometric σ-model)

using Riemann normal coordinates near X_0 $\longrightarrow$ $G_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\kappa}(X_0)\,x^\lambda x^\kappa + \mathcal{O}(x^3)$

Corresponding expansion of the action:

$$S = \frac{-1}{4\pi\alpha'}\int d^2\sigma \Big(\partial_\alpha x^\mu \partial^\alpha x_\mu - \underbrace{\tfrac{1}{3}R_{\mu\lambda\nu\kappa}(X_0)\,x^\lambda x^\kappa \partial_\alpha x^\mu \partial^\alpha x^\nu}_{\text{quartic interaction term}} + \mathcal{O}(x^5)\Big)$$

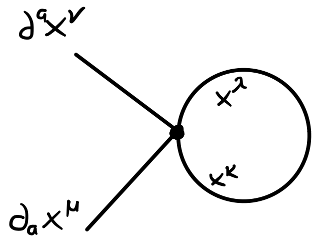
To understand what this expansion really means to us, note that $G_{\mu\nu} \to \lambda^2 G_{\mu\nu}$ in the action is the same as $\alpha' \to \lambda^{-2}\alpha'$. α' appears as an $\hbar$-like parameter, so a small α' expansion is a large-distance (in spacetime) expansion. Said differently, our dimensionless parameters are of the form $\sqrt{\alpha'}/r$, where r represents a typical radius of curvature/length scale associated with $R_{\mu\lambda\nu\kappa}$.

This is the "large radius" or "large volume" expansion, corresponding in spacetime to an EFT-like expansion w/cutoff $M_s \sim (\alpha')^{-1/2}$.

Can now develop "σ-model perturbation theory" in the usual perturbative QFT framework, computing one-loop divergences that contribute to renormalization of the couplings.

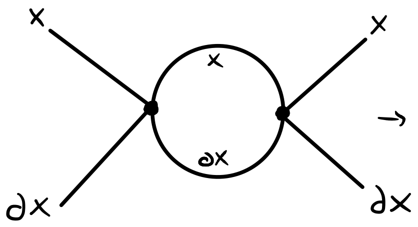
One loop logarithmic divergences & counterterms in dimensional regularization:

Kinetic term:



$$\to R_{\mu\lambda\nu\kappa} \int \frac{d^{2+\epsilon}p}{(2\pi)^{2+\epsilon}} \frac{\eta^{\lambda\kappa}}{p^2} \to \text{C.T.: } \frac{-1}{2\epsilon} R_{\mu\nu}(x_0)\, \partial x^\mu \partial x^\nu$$

Interaction vertex:



$$\to \frac{1}{9} R_{\mu\lambda\nu\sigma} R_{\rho}{}^{\lambda}{}_{\kappa}{}^{\sigma} \int \frac{d^{2+\epsilon}p}{(2\pi)^{2+\epsilon}} \frac{p^2}{p^4} \to \text{C.T.: } \frac{-1}{18\epsilon} (R^2)_{\mu\lambda\nu\kappa}\, \partial x^\mu \partial x^\nu x^\lambda x^\kappa$$

An important (and not entirely easy to verify) fact is that these divergences (and more if we went to higher loops) can be absorbed entirely in a combination of wave-function renormalization of the x's and a functional renormalization on $G_{\mu\nu}$:

- $x^\mu \to x^\mu + \frac{1}{6\epsilon} R^\mu{}_\nu x^\nu + O(x^2)$
- $G_{\mu\nu}(X) \to G_{\mu\nu}(X) - \frac{1}{2\epsilon} R_{\mu\nu}(X)$

This gives rise to a one-loop β-functional: $\beta_{G_{\mu\nu}}(X) = \frac{-1}{2\pi} R_{\mu\nu}(X)$.

Conformal invariance is a consistency condition for the string worldsheet. Thus we see that as a consistency condition, we recover (at leading order in α'), vacuum Einstein's equations in spacetime!

$$R_{\mu\nu} = 0$$

Worldsheet consistency condition — Spacetime equation of motion

That's quite remarkable! We've recovered spacetime dynamics as a pure consistency condition! If we go to higher orders, we get "stringy corrections", e.g.

$$R_{\mu\nu} + \frac{\alpha'}{2} R_{\mu\kappa\lambda\tau} R_\nu{}^{\kappa\lambda\tau} = 0 \quad \text{to order } (\alpha')^2.$$

These look like corrections to E.O.M. from terms in EFT suppressed by cutoff scale $M_s \sim \sqrt{\alpha'}$. String theory predicts specific (small) corrections to Einstein (in D=26 @ large radius).

This whole discussion has been perturbative on the worldsheet, but we've implicitly already seen what should, in some sense, be the "exact" version of this analysis, at least for static spacetimes.

$$\mathbb{R}^{1,25}_{\eta} \xrightarrow{\text{is a special case of}} \mathbb{R}^{1,25-n}_{\eta} \times \underset{\substack{\uparrow \\ \text{Ricci flat to leading order}}}{\mathbb{R}^{n}_{G}} \xrightarrow{\text{is a special case of}} \mathbb{R}^{1,25-n}_{\eta} \times \underset{\substack{\uparrow \\ \text{Ricci flat to leading order}}}{\mathcal{M}^{n}_{G}} \xrightarrow{\text{is a special case of}} \mathbb{R}^{1,25-n}_{\eta} \times \underset{\substack{\uparrow \\ \text{non-perturbative in } \alpha'!}}{\left(\mathrm{CFT}^{2d}_{c_{vir}=n}\right)}$$

The machinery from the free bosonic string carries over to the abstract CFT case, including the no ghost theorem and the formalism of vertex operators. In this way, string theory goes beyond target spaces described by semi-classical geometry, but this requires more formal development of CFT (so take the course next term!)

Next time we'll come back to the rest of our massless fields and consider the most general string σ-model.