We want to study σ-model perturbation theory for the curved-space sigma model

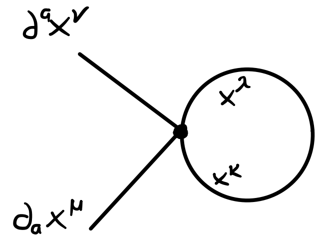
$$S_P^{(G)}[X] = \frac{-1}{2\pi}\int d^2\sigma\, G_{\mu\nu}(X)\,\partial_a X^\mu \partial^a X^\nu$$

quartic interaction term

$$= \frac{-1}{4\pi\alpha'}\int d^2\sigma \left(\partial_\alpha x^\mu \partial^\alpha x_\mu - \frac{1}{3}R_{\mu\lambda\nu\kappa}(X_0)\,x^\lambda x^\kappa \partial_\alpha x^\mu \partial^\alpha x^\nu + O(x^5)\right)$$

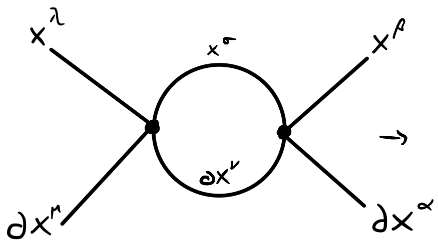
Consider 1-loop renormalization using dimensional regularization $(d = 2+\epsilon)$

Kinetic term:



$$\rightarrow\; R_{\mu\lambda\nu\kappa}\int \frac{d^{2+\epsilon}p}{(2\pi)^{2+\epsilon}}\,\frac{\eta^{\lambda\kappa}}{p^2} \;\rightarrow\; \text{C.T.}:\; \frac{-1}{2\epsilon}R_{\mu\nu}(X_0)\,\partial x^\mu \partial x^\nu$$

Interaction vertex:



$$\rightarrow\; \frac{1}{9}R_{\mu\lambda\nu\sigma}R_{\alpha\rho}{}^{\nu\sigma}\int \frac{d^{2+\epsilon}p}{(2\pi)^{2+\epsilon}}\,\frac{p^2}{p^4} \;\longrightarrow\; \text{C.T.} \sim \frac{1}{\epsilon}(R^2)_{\mu\lambda\nu\kappa}\,\partial x^\mu \partial x^\nu x^\lambda x^\kappa$$

An important (and not entirely easy to verify) fact is that these divergences (and more if we want to higher loops) can be absorbed entirely in a combination of wave-function renormalization of the x's and a functional renormalization on $G_{\mu\nu}(X)$. This is not obvious because our naive approach to this calculation is not manifestly covariant with respect to target space reparameterizations. A more sophisticated approach uses the "background field method" to maintain spacetime covariance.

- $x^\mu \to x^\mu + \frac{1}{6\epsilon}R^\mu{}_\nu x^\nu + O(x^2)$
- $G_{\mu\nu}(X) \to G_{\mu\nu}(X) - \frac{1}{2\epsilon}R_{\mu\nu}(X)$

This gives rise to a one-loop β-functional: $\beta_{G_{\mu\nu}}(X) = \frac{-1}{2\pi}R_{\mu\nu}(X)$.

Conformal invariance is a consistency condition for the string worldsheet. Thus we see that as a consistency condition, we recover (at leading order in α'), vacuum Einstein's equations in spacetime!

$$R_{\mu\nu} = 0$$

Worldsheet consistency condition $\quad$ Spacetime equation of motion

That's quite remarkable! We've recovered spacetime dynamics as a pure consistency condition! In retrospect, given our worldsheet approach, this seems pretty much the only way this could have worked. Nevertheless it is a remarkable result that gives us much more confidence that the spacetime interpretation of string theory can give something sensible.

We can also include backgrounds for the other massless fields. Anti-symmetric Kalb-Ramond field $B_{\mu\nu}(X)dx^\mu \wedge dx^\nu$:

$$S^{(B)}[X] = -\frac{1}{2\pi}\int d^2\sigma\, \epsilon^{\alpha\beta} B_{\mu\nu}(X)\,\partial_\alpha X^\mu \partial_\beta X^\nu$$

Under spacetime gauge transformation $B_{(2)} \to B_{(2)} + d\Lambda_{(1)}$, this action changes by a total divergence (exercise). Including a dilaton background is more subtle. The right answer turns out to be

$$S^{(\Phi)}[X;\gamma] = \frac{1}{4\pi}\int d^2\sigma \sqrt{\gamma}\, \Phi(X)\, R^{(2)}(\gamma)$$

We ignored this term previously because it was a total derivative when $\Phi(X) = \text{const}$. For general Φ it is not a total derivative. However, it is also not Weyl invariant!

$$\gamma \to e^{2\omega(\sigma)}\gamma \quad \Longrightarrow \quad R^{(2)} \longrightarrow e^{-2\omega}\left(R^{(2)} - 2\nabla^2\omega\right), \text{ not a total derivative if } \Phi(X) \neq \text{constant}$$

This is related to subtlety in dilaton vertex operator. Need to modify G when perturbing Φ to preserve Weyl invariance, but hard to see on flat worldsheet where $R^{(2)} = 0$ (cf. Polchinski vol. 1 §3.6).

Why is it okay that we just added a Weyl non-invariant term to the action? The situation is clarified if we re-introduce the factors of α' into our action.

$$S^{(G)} \sim \frac{1}{4\pi\alpha'}, \quad S^{(B)} \sim \frac{1}{4\pi\alpha'}, \quad S^{(\Phi)} \sim \frac{1}{4\pi}$$

So a classical Weyl variation of $S^{(\Phi)}$ can be cancelled by an $O(\alpha')$ variation of $S^{(G)}$ and $S^{(B)}$. Indeed, a (quite involved) extension of the one-loop calculation we discussed before leads to the following β-functions for the full string action:

- $$\beta^G_{\mu\nu} = \alpha'\Big(\overbrace{R_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma}}^{\text{one loop } G+B} + \overbrace{2D_\mu D_\nu \Phi}^{\text{classical } \Phi}\Big)$$
- $$\beta^B_{\mu\nu} = \alpha'\Big(\overbrace{-\tfrac{1}{2}D^\lambda H_{\lambda\mu\nu} + D^\lambda\Phi H_{\lambda\mu\nu}}^{\text{one loop } G+B}\Big)$$
- $$\beta^\Phi = \overbrace{\frac{D-26}{6}}^{\text{one loop } G+B} + \alpha'\Big(\overbrace{D_\mu\Phi D^\mu\Phi - \tfrac{1}{2}D^2\Phi}^{\text{one loop } \Phi} - \overbrace{\tfrac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}}^{\text{two loop } G+B}\Big) \qquad H = dB \quad \left(H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\right)$$

c.f. Friedan thesis
Callan & Thorlacius "Sigma models & string theory"
Tseytlin "Conformal anomaly in a 2-dimensional sigma model..."

I need to comment on the $\frac{D-26}{6}$ in β^Φ, because I just sort of snuck it in. When $R^{(2)} \neq 0$, even our simple free worldsheet theory develops an anomalous scale dependence; it is determined by the Virasoro central charge! (Usually phrased as "Weyl anomaly").

$$\beta^\Phi_{CFT} = \frac{c}{6}$$

That's bad! How can we set $c = 0$? Luckily, we've been sloppy elsewhere, too, when we imposed conformal gauge fixing without doing Fadeev-Popov. Had we been more careful we would have introduced the diffeomorphism ghost CFT. Amazingly, this is a CFT with $c = -26$, so this is another way to deduce $D = 26$!

(b,c) ghost CFT of type (2,1).

Coming back to the β-functions, based on experience with metric we'd like to interpret these as spacetime equations of motion. Indeed, after some work, we can show that they all arise from Euler-Lagrange equations for the effective action

$$S_{26}^{(S)} = \frac{1}{2\kappa_0^2}\int d^{26}X\sqrt{-G}\,e^{-2\Phi}\left(R - \frac{1}{12}|H|^2 + 4|D\Phi|^2\right)$$

This is called the "string frame" action since G, B, Φ are what appear in string σ-model action. For doing spacetime calculations, it is often useful to go to "Einstein frame", defining $\tilde\Phi = \Phi - \Phi_0$ and $\tilde G = \exp\left(\frac{1}{6}\tilde\Phi\right)G$:

$$S_{26}^{(E)} = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde G}\left(\tilde R - \frac{1}{12}e^{-\frac{\tilde\Phi}{3}}|H|^2 - \frac{1}{6}|D\tilde\Phi|^2\right)$$

$\kappa = \kappa_0 e^{\Phi_0}$

indices raised & lowered w/ $\tilde G$

Now the Riemann-Hilbert term takes the canonical form with gravitational coupling $\kappa = (8\pi G_N)^{1/2}$. Our action should capture the classical limit when $E \ll M_S$. To see stringy corrections, need to correct β-functions:

$$\beta = \beta^{(0)} + \alpha'\beta^{(1)} + (\alpha')^2\beta^{(2)} + \dots$$

It gets challenging, but can be done. For example, $\beta^G_{\mu\nu} \sim \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2}R_{\mu\lambda\rho\sigma}R_\nu{}^{\lambda\rho\sigma} + \dots$. Then higher order corrections get reinterpreted as E-L equations for a corrected action.

$$S_{26} = S_{26}^{(0)} + \underbrace{\alpha'}_{1/M_S^2}\underbrace{S_{26}^{(1)}}_{\text{4-deriv. terms}} + \underbrace{(\alpha')^2}_{1/M_S^4}\underbrace{S_{26}^{(2)}}_{\text{6-deriv. terms}} + \dots$$

This is an EFT expansion w/ cutoff scale M_S, and can be interpreted as the effective action obtained upon integrating out massive string modes.