We discussed last time the inclusion of more general background fields in spacetime where our strings are propagating. One important background to consider is a constant dilaton. The β-functions (and spacetime effective action) only depend on derivatives of Φ, so its zero-mode is a modulus (flat direction for potential).

Of course, we previously dismissed constant dilatons b/c they contribute a total derivative to the worldsheet action. But this was too fast! A constant dilaton appears as

$$S^{(\Phi)}[X,\gamma] = \frac{\Phi_0}{4\pi}\int d^2\sigma\sqrt{\gamma}\,R^{(2)}(\gamma)$$

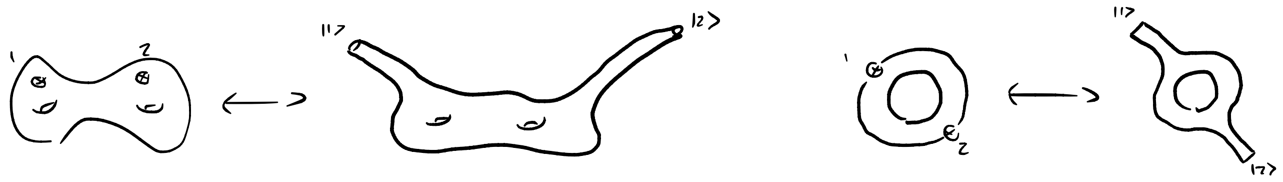
Now there is an important theorem of two-dimensional differential topology:

$$\text{Gauss-Bonet theorem:} \quad \frac{1}{4\pi}\int_{\Sigma_g} dA\, R^{(2)} = \chi(\Sigma_g) = 2-2g$$

extrinsic curvature, this term needed in presence of boundary to cancel boundary term from Ricci curvature in E-L derivation.

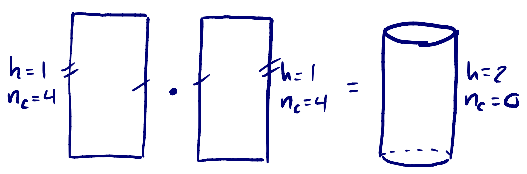
$$\frac{1}{4\pi}\int_{\Sigma_{g,h}} dA\, R^{(2)} + \frac{1}{2\pi}\int_{\partial\Sigma_{g,h}} K\,ds = \chi(\Sigma_{g,h}) = 2-2g-h$$

So if $\Phi = \Phi_0$, then closed-string loops get a factor of $e^{-2\Phi_0}$ and open string loops get a factor of $e^{-\Phi_0}$. More subtly, we should be able to think of incoming and outgoing lines as introducing extra boundaries/corners:



So an external closed string state gets a factor of $e^{-\Phi_0}$, and an external open-string state gets a factor of $e^{-\Phi_0/2}$.

gluing argument for open string dilaton dependence:



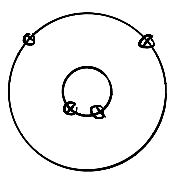
$$\Rightarrow e^{2\Phi_0(2-1+4\lambda_c)} = e^{\Phi_0(2-2)}$$

$$\Rightarrow \lambda_c = -\tfrac{1}{4} \quad \Rightarrow \text{external open string} \sim e^{2\lambda_c\Phi_0} = e^{-\Phi_0/2}$$

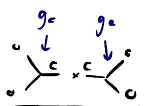
These weightings match with the appearance of the open- and closed-string coupling constants. Indeed, the effect of a shift in the dilaton zero-mode $\Phi_0 \to \Phi_0 + \lambda$ is a re-scaling of the string couplings.

$$g_c \to g_c e^{-\lambda} \qquad g_o \to g_o e^{-\lambda/2}$$

The string coupling constants are not actually parameters of the theory, they are dynamical! Related to expectation value of some field in spacetime. (The relative normalization is determined by unitarity of the annulus amplitude (exercise 7.9 of Polchinski))



$\sim g_o^4(\ldots) \xrightarrow[\text{string pole}]{\text{closed}}$



$$\frac{\cdots}{s - M^2_{\text{closed}}}$$

$$\frac{g_o^2}{g_c} = 2^{18}\pi^{25/2}(\alpha')^6 = 2^{12}\pi^{25/2} \text{ in string units}$$

This is an instance of what turns out to be a general principle: in string theory there are no continuous parameters; when they seem to appear, they ultimately turn out to be related to expectation values of dynamical spacetime fields. Of course, this leaves a question of how, say, the dilaton expectation value gets set — this is an instance of the problem of "moduli stabilization". Not easy to solve, requires quite a bit more machinery.

Now recall the spacetime effective action that encoded the one-loop β-functions:

String frame: $$S_{26}^{(S)} = \frac{1}{2\kappa_0^2}\int d^{26}X\sqrt{-G}\,e^{-2\Phi}\left(R - \tfrac{1}{12}|H|^2 + 4|D\Phi|^2\right)$$

$\tilde{G} = \exp\left(\tfrac{1}{6}\tilde{\Phi}\right)G$

$\tilde{\Phi} = \Phi - \Phi_0$

Einstein frame: $$S_{26}^{(E)} = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde{G}}\left(\tilde{R} - \tfrac{1}{12}e^{-\frac{\tilde{\Phi}}{3}}|H|^2 - \tfrac{1}{6}|D\tilde{\Phi}|^2\right)$$

$\kappa = \kappa_0 e^{\Phi_0}$ indices raised & lowered w/ $\tilde{G}$

In Einstein frame, the Riemann-Hilbert term takes the canonical form with gravitational coupling $\kappa = (8\pi G_N)^{1/2}$. We can make some helpful observations now about spacetime energy scales.

"Gravitational coupling": $\kappa = \kappa_0 e^{\Phi_0} \sim G_N^{1/2} \sim (M_{pl})^{\frac{2-D}{2}}$ (controls spacetime quantum effects)

"String scale": $\alpha' \sim (M_s)^{-2}$ (controls "stringy corrections")

Dimensionless ratio: $\frac{M_s}{M_{pl}} \sim e^{\frac{2\Phi_0}{D-2}}$ ($e^{\Phi_0} \to 0$ gives "classical" limit in spacetime)

This clarifies a bit our effective action, which is an α'-expansion. This is the effective action for $E \ll M_s$ in the limit $\frac{M_s}{M_{pl}} \to 0$, so we suppress spacetime quantum effects and keep stringy effects (note: world-sheet quantum corrections (controlled by α') are spacetime classical effects!)

This whole business of the α'-expansion is perturbative on the worldsheet, but we've implicitly already seen what should, in some sense, be the "exact" version of this analysis, at least for static spacetimes.

$$\mathbb{R}^{1,25}_\eta \xrightarrow{\text{is a special case of}} \underset{\uparrow\ \text{Ricci flat to leading order}}{\mathbb{R}^{1,25-n}_\eta \times \mathbb{R}^n_G} \xrightarrow{\text{is a special case of}} \underset{\uparrow\ \text{Ricci flat to leading order}}{\mathbb{R}^{1,25-n}_\eta \times \mathcal{M}^n_G} \xrightarrow{\text{is a special case of}} \underset{\uparrow\ \text{non-perturbative in } \alpha'!}{\mathbb{R}^{1,25-n}_\eta \times \left(CFT^{2d}_{c_{vir}=n}\right)}$$

The machinery from the free bosonic string carries over to the abstract CFT case, including the no ghost theorem and the formalism of vertex operators. In this way, string theory goes beyond target spaces described by semi-classical geometry, but this requires more formal development of CFT (so take the course next term!)

This is all before taking into account effects of spacetime loops. For this reason, people sometimes say that 2d unitary CFTs (w/appropriate central charge) are classical string backgrounds, or classical solutions of the string theory E.O.M.'s.

We'll now start looking at our first non-trivial example.