Want to consider "S^1-compactification" of the bosonic string

$$\mathbb{R}^{1,25} \longrightarrow \mathbb{R}^{1,24}\times S^1_R$$

We can try to understand this a bit from two different perspectives

- Spacetime EFT ("Kaluza-Klein reduction")

  The spacetime background is still flat, so our leading effective action seems like a good bet. However, if $R\sim \ell_s$, this feels a little bit wrong, since there is now an $O(1)$ dimensionless ratio.

- Worldsheet CFT ($X^M \to X^i,\ X^{25}\sim X^{25}+2\pi R$)

  The worldsheet action will look the same as in flat $\mathbb{R}^{1,25}$, but now there is some nontrivial topology in field space. Will have to work out the consequences.

We'll look at the EFT approach first. A quick-and-dirty approach is to just assume all our massless fields are independent of X^{25} and reinterpret in $(1+24)$-dimensional terms ("dimensional reduction").

$$G_{\mu\nu}(x) \longrightarrow G_{\mu\nu}(x^i) : \Big\{ \overset{\text{25-d graviton}}{G_{ij}(x^i)},\ \overset{\text{graviphoton}}{G_{i,25}(x^i)},\ \overset{\text{radion}}{G_{25,25}(x^i)} \Big\}$$

$$B_{\mu\nu}(x) \longrightarrow B_{\mu\nu}(x^i) : \Big\{ \underset{\text{25d Kalb-Ramond field}}{B_{ij}(x^i)},\ \underset{\text{Kalb-Ramond photon}}{B_{i,25}(x^i)} \Big\}$$

$$\Phi(x) \longrightarrow \Phi(x^i) : \Big\{ \underset{\text{25d Dilaton}}{\Phi(x^i)} \Big\}$$

To be more careful, we should write a "Kaluza-Klein Ansatz" for our fields and rewrite the effective action as a 25-d action for the new fields:

$$G_{\mu\nu}dx^\mu dx^\nu = G_{ij}dx^i dx^j + G_{25,25}(dX^{25}+A_i dx^i)^2\ ,\quad G_{25,25}=e^{2\sigma}$$

$$B_{\mu\nu}dx^\mu\wedge dx^\nu = B_{ij}\,dx^i\wedge dx^j + \tilde{A}_i\, dx^i\wedge dx^{25}$$

$$\Phi_{(25)} = \Phi - \frac{\sigma}{2}$$

A long calculation shows that one gets a reasonable action for a metric, KR-field, two $U(1)$ gauge fields, and two scalars (though there are interesting details). [Exercise!] Of course, there is more to life than zero-modes with respect to X^{25}. Take dilaton, for example:

$$\Phi(x^M) = \sum_{n\in\mathbb{Z}} e^{inX^{25}/R}\,\phi_n(x^i) \qquad (\phi_n = \phi^*_{-n})$$

At non-interacting level, $\Box_{26}\Phi = 0 \longrightarrow \sum_{n\in\mathbb{Z}} e^{\frac{inX^{25}}{R}}\left(\Box_{25}+\frac{n^2}{R^2}\right)\phi_n(x^i) = 0 \longrightarrow M^2_{\phi_n} = \frac{n^2}{R^2}$. These higher modes are the "Kaluza-Klein modes", or "KK modes". We get a discrete, infinite tower of massive modes of this type for each massless field in $d=26$. We can make an interesting observation that under a reparameterization of the X^{25} direction.

$$\left.\begin{aligned} X^{25}\to X^{25}+\xi(x^i) \ &\Rightarrow\ A_i \longrightarrow A_i+\partial_i\xi \\ &\Rightarrow\ \phi_n \longrightarrow e^{\frac{in\xi}{R}}\phi_n \end{aligned}\right\}\ \text{i.e., KK momentum is charge for the graviphoton.}$$

On the other, we don't get any excitations charged under the $U(1)$ associated with the Kalb-Ramond photon. Will see why in a moment.

As we've introduced a new scale $M_{KK} \sim \frac{1}{R}$, we should be skeptical of our EFT analysis for $M_{KK} \sim M_s$ ($R \sim 1$ in string units). Luckily, in this case we can perform an exact analysis of the worldsheet CFT.

Worldsheet theory is as in $\mathbb{R}^{1,25}$, but now X^{25} an S^1-valued field. This periodicity has a couple of effects.

- $e^{2\pi i R \hat{p}_{25}}$ should act as identity (spacetime translation by $2\pi R$)

  $$e^{2\pi i R \hat{p}_{25}} |\ldots, k_{25}\rangle = e^{2\pi i R k_{25}} |\ldots, k_{25}\rangle = |\ldots, k_{25}\rangle \text{ iff } k_{25} = \frac{m}{R} \quad w/ \, m \in \mathbb{Z} \quad (\text{as in EFT analysis}).$$

- X^{25} only needs to return to itself up to $2\pi R$ shifts around closed string.

  $$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R \omega \quad (\omega \in \mathbb{Z}).$$ Winding is a <u>stringy effect</u>.

Phase space has disconnected components labelled by ω.

$$X^{25}(\sigma_\pm) = x^{25} + \tau p^{25} + 2\omega R \sigma + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^{25} e^{-2in\sigma_-} + \tilde{\alpha}_n^{25} e^{-2in\sigma_+} \right) = X_R^{25}(\sigma_-) + X_L^{25}(\sigma_+)$$

$$X_R^{25}(\sigma_-) = x_R^{25} + \underbrace{\left(\frac{p^{25}}{2} - R\omega \right)}_{p_R/2} \sigma_- + (\text{osc.}) \qquad X_L^{25}(\sigma_+) = x_L^{25} + \underbrace{\left(\frac{p^{25}}{2} + R\omega \right)}_{p_L/2} \sigma_+ + (\widetilde{\text{osc.}})$$

This is mostly as in the previous case of $\mathbb{R}^{1,25}$, but now $\alpha_0^{25} = \frac{1}{2} p_R$, $\tilde{\alpha}_0^{25} = \frac{1}{2} p_L$: $\alpha_0^{25} - \tilde{\alpha}_0^{25} = -2R\omega$ & $\alpha_0^{25} + \tilde{\alpha}_0^{25} = p^{25}$. Our compact-boson Hilbert (Fock) space now takes the form

$$\mathcal{H}_{\text{compact}} \cong \bigoplus_{(m,\omega) \in \mathbb{Z} \times \mathbb{Z}} \mathcal{H}_{\text{Fock}}^{(m,\omega)}, \quad \mathcal{H}_{\text{Fock}}^{(m,\omega)} \cong \text{span} \left\{ \prod_i \alpha_{-n_i}^{25} |0; \underset{\substack{\uparrow \uparrow \\ \text{determine zero-mode actions}}}{m, \omega}\rangle \right\}$$

Virasoro constraints are as usual, with some reorganizing.

- $$L_0 = \frac{1}{2} \Big(\underset{\substack{\uparrow \\ 24+1\text{-dimensional inner product}}}{\alpha_0 \cdot \alpha_0} + (\alpha_0^{25})^2 \Big) + \overbrace{\sum_{n>0} \underset{\substack{\uparrow \\ 24+1\text{-dimensional inner product}}}{\alpha_{-n} \cdot \alpha_n} + \sum_{n>0} \alpha_{-n}^{25} \alpha_n^{25}}^{N}$$

- $$L_m = \frac{1}{2} \sum_n \underset{\substack{\uparrow \\ 24+1\text{-dimensional inner product}}}{\alpha_{m-n} \cdot \alpha_n} + \frac{1}{2} \sum_n \alpha_{m-n}^{25} \alpha_n^{25}$$

The main novelty is in the mass-shell and level-matching conditions.

- $$L_0 - 1 = \frac{p \cdot p}{8} + \frac{p_R^2}{8} + N - 1 = 0 \longrightarrow M_{(25)}^2 = \underset{\frac{m}{R} - 2R\omega}{p_R^2} + 8(N-1)$$

- $$\tilde{L}_0 - 1 = \frac{p \cdot p}{8} + \frac{p_L^2}{8} + \tilde{N} - 1 = 0 \longrightarrow M_{(25)}^2 = \underset{\frac{m}{R} + 2R\omega}{p_L^2} + 8(\tilde{N}-1)$$

Reorganizing as usual:

$$M_{(25)}^2 = \frac{m^2}{R^2} + 4R^2\omega^2 + 4(N + \tilde{N} - 2) \quad (\text{mass shell})$$

$$m\omega = N - \tilde{N} \quad (\text{level-mismatching condition})$$

At $\omega = 0$, this is compatible with our EFT intuition. But for $R \sim 1$, "stringy" winding modes will be important.

Analysis of the light string spectrum on S^1:

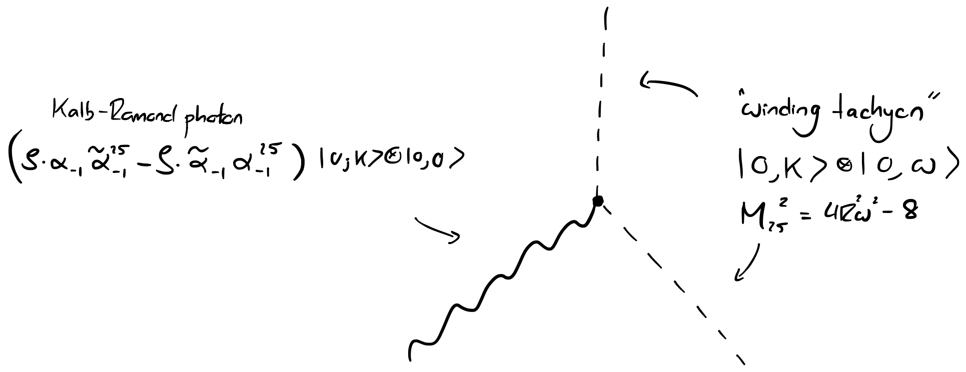
$$\gamma_{ij}\,\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0,k^i\rangle\otimes|0,0\rangle \longleftarrow \text{25d Graviton}$$

$$B_{ij}\,\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0;k^i\rangle\otimes|0,0\rangle \longleftarrow \text{25d B-field}$$

$$(\zeta\cdot\alpha_{-1}\tilde{\alpha}_{-1}^{25}\pm\zeta\cdot\tilde{\alpha}_{-1}\alpha_{-1}^{25})|0;k^i\rangle\otimes|0,0\rangle \longleftarrow \text{"Graviphoton" and "extra photon"}$$

$$\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}|0;k^i\rangle\otimes|0,0\rangle \longleftarrow \text{"Radion"}$$

Energy in winding & KK modes quantized in R-dependent units, so generically cannot give massless states. (however, special cases allow it — see problem sheets) might come back to this. However, there is an interesting new effect we can observe. The winding modes provide charged states for the Kalb-Ramond photon!



Can compute this vertex: $\mathcal{A} = \langle 0,-k_3;0,\omega|(\zeta\cdot\partial_+X\,\partial_-X^{25}-\zeta\cdot\partial_-X\,\partial_+X^{25})e^{ik_2\cdot X}|0,k_1;0,\omega\rangle$

$$= \langle 0,-k_3;0,\omega|\zeta\cdot\tilde{\alpha}_0\alpha_0^{25}-\zeta\cdot\alpha_0\tilde{\alpha}_0^{25}|0,k_1+k_2;0,\omega\rangle$$

$$= \zeta\cdot(k_1+k_2)\langle 0,-k_3;0,\omega|\alpha_0^{25}-\tilde{\alpha}_0^{25}|0,k_1+k_2;0,\omega\rangle = (2R\omega)\,\zeta\cdot k_3\,\delta(k_1+k_2+k_3)$$

This reproduces the vertex from a term $A_\mu\Phi\partial^\mu\Phi$ in the spacetime Lagrangian $\Rightarrow$ winding number is KR-photon charge! This is gratifying, strings ensure no spacetime gauge symmetry goes to waste.

Coming back to our mass formula (α''s restored): $M^2 = \frac{m^2}{R^2}+\frac{\omega^2R^2}{(\alpha')^2}+\frac{2}{\alpha'}(N+\tilde{N}-2)$ & $N-\tilde{N}=m\omega$

Consider the limiting cases for R:

- $R\to\infty$, continuum of KK modes ($\omega=0$, m any) (sign of 25th dimension)
- $R\to 0$, continuum of winding modes?! ($m=0$, ω any)

A hint is to observe a curious symmetry of the physical state spectrum.

- $m\leftrightarrow\omega$
- osc. $\leftrightarrow$ − osc. (b/c $\alpha_0^{25}=\frac{\alpha' m}{R}-R\omega$)
- $R\longleftrightarrow \alpha'/R$
- $\tilde{\text{osc}}.\leftrightarrow\tilde{\text{osc}}$

Claim: this is an exact symmetry of the CFT. Continuum of winding modes is related to opening up of dual circle!

"T-duality"

What is the invariant data of our theory?

- State space $\prod_i \alpha_{-n_i} |0, m, \omega\rangle$
- Virasoro operators $L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\alpha_n \qquad L_0 = \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-m}\alpha_m$

(includes Hamiltonian $H = L_0 + \tilde{L}_0$) $\qquad \tilde{L}_m = \frac{1}{2}\sum \tilde{\alpha}_{m-n}\tilde{\alpha}_n \qquad \tilde{L}_0 = \tilde{\alpha}_0^2 + \sum_{n=1}^{\infty}\tilde{\alpha}_{-m}\tilde{\alpha}_m$

From this data we can re-construct the local field operator $X(\tau,\sigma) = x + 2\alpha'\tau p + 2\left(\frac{\alpha'}{R}\right)\omega\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\left(\alpha_n e^{-2in\sigma_-} + \tilde{\alpha}_n e^{-2in\sigma_+}\right)$

$= X_L(\sigma_+) + X_R(\sigma_-)$ where we split the field as:

$$X_L(\sigma_+) = x_L + 2\alpha'\sigma_+ p_L + i\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\frac{1}{n}\tilde{\alpha}_n e^{-2in\sigma_+}$$

$$X_R(\sigma_-) = x_R + 2\alpha'\sigma_- p_R + i\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\frac{1}{n}\alpha_n e^{-2in\sigma_-}$$

Defining $T_{\pm\pm} = \partial_\pm X \partial_\pm X$, we recover the L_m & $\tilde{L}_m$ as Fourier modes. The zero modes here are defined as follows:

$$p_L = \frac{1}{2}\left(p + \frac{R}{\alpha'}\omega\right) = \frac{1}{2}(p + \tilde{p}) \qquad \text{w/ } p|\ldots;m,\omega\rangle = \frac{m}{R}|\ldots;m,\omega\rangle$$

$$p_R = \frac{1}{2}\left(p - \frac{R}{\alpha'}\omega\right) = \frac{1}{2}(p - \tilde{p}) \qquad \text{w/ } \tilde{p}|\ldots;m,\omega\rangle = \frac{R\omega}{\alpha'}|\ldots;m,\omega\rangle$$

$$x_L = \frac{1}{2}(x + \tilde{x}) \qquad \text{w/ } e^{\frac{ix}{R}}|\ldots;m,\omega\rangle \to |\ldots;m+1,\omega\rangle \qquad [p,x] = -i$$

$$x_R = \frac{1}{2}(x - \tilde{x}) \qquad \text{w/ } e^{\frac{i\tilde{x}R}{\alpha'}}|\ldots;m,\omega\rangle = |\ldots;m,\omega+1\rangle \qquad [\tilde{p},\tilde{x}] = -i$$

Now with the exact same data, we define $\tilde{X}(\tau,\sigma) = X_L(\tau,\sigma) - X_R(\tau,\sigma)$

$$= \tilde{x} + 2\alpha'\tau\tilde{p} + 2\alpha' p\sigma + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n=-\infty}^{\infty}\frac{1}{n}\left\{-\alpha_n e^{-2in\sigma_-} + \tilde{\alpha}_n e^{-2in\sigma_+}\right\}$$

$$= \tilde{x} + 2\alpha'\tau\tilde{p} + 2\tilde{R}\tilde{\omega}\sigma + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n=-\infty}^{\infty}\frac{1}{n}\left\{(-\alpha_n)e^{-2in\sigma_-} + \tilde{\alpha}_n e^{-2in\sigma_+}\right\}$$ where $\tilde{R} = \frac{\alpha'}{R}$, $\tilde{\omega} = Rp$ acts as m.

This is exactly the expression for a compact boson of radius $\tilde{R} = \alpha'/R$! Writing $\hat{T}_{\pm\pm} = \partial\tilde{X}_\pm\partial\tilde{X}_\pm = T_{\pm\pm}$, we see that some L_m's & $\tilde{L}_m$'s will be derived. An exact equivalence of string backgrounds!

We have found that in the "stringy" regime $R \ll (\alpha')^{1/2}$, there is a new, hidden geometry with $\tilde{R} \gg (\alpha')^{1/2}$. "Strings see spacetime differently." So the most intrinsically stringy point should be $R = \sqrt{\alpha'} = \tilde{R}$, the "minimum length" available if we permit changes in duality frame. This is the "self-dual radius", and there are interesting phenomena there (cf., PS4).