Recall from last time, we performed a worldsheet analysis of the S^1-compactified bosonic string. Found a string state space much like the non-compact case, but now with quantized "KK" momenta on the S^1 and with string winding.

24+1-dim'l momentum

$$p_{25} = \frac{m}{R}$$

$$(\text{osc}, \widetilde{\text{osc}})\,|0; K\rangle \longrightarrow (\text{osc}, \widetilde{\text{osc}})\,|\underline{K}; m, \omega\rangle$$

$$X^{25}(\tau, \sigma+\pi) = X^{25}(\tau,\sigma) + 2\pi R\omega$$

Virasoro operators as before, but with $\alpha_0^{25} = \alpha' p^{25} - R\omega$, $\tilde{\alpha}_0^{25} = \alpha' p^{25} + R\omega$. Gives modified $(24+1)$-dimensional mass-shell and level-matching conditions.

$$M_{25}^2 = -|\underline{K}|^2 = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2}R^2\omega^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

$$m\omega = N - \tilde{N}$$

Before proceeding with any study of small R, let's make an interesting observation about stringy physics. Consider the massless (at generic R) spectrum:

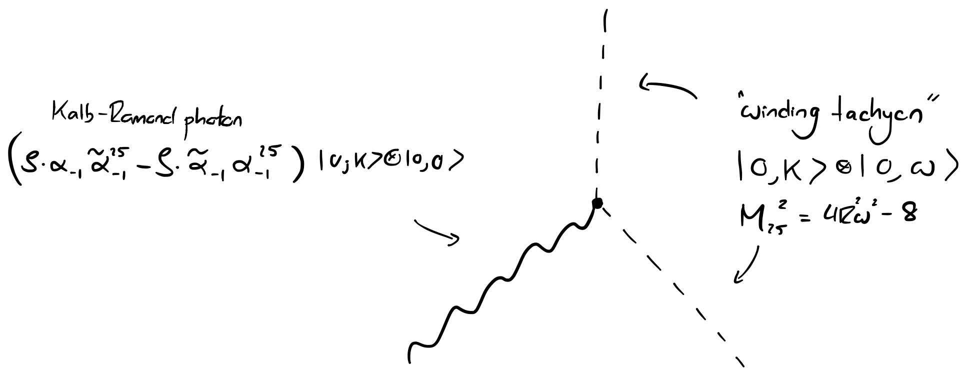
$$\gamma_{ij}\,\alpha_{-1}^i \tilde{\alpha}_{-1}^j\,|K^i; 0, 0\rangle \longleftarrow \text{25d Graviton}$$

$$B_{ij}\,\alpha_{-1}^i \tilde{\alpha}_{-1}^j\,|K^i; 0, 0\rangle \longleftarrow \text{25d B-field}$$

$$(\zeta\cdot\alpha_{-1}\tilde{\alpha}_{-1}^{25} \pm \zeta\cdot\tilde{\alpha}_{-1}\alpha_{-1}^{25})\,|K^i; 0, 0\rangle \longleftarrow \text{"Graviphoton" and "extra photon"}$$

$$\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}\,|K^i; 0, 0\rangle \longleftarrow \text{"Radion"}$$

Let's look at the couplings of the KR photon (remember in naive KK reduction, there were no charged particles with respect to the KR photon.) In particular let's look at the winding modes that otherwise appear as tachyons:



Can compute this vertex: $\mathcal{A} = \langle 0, -K_3; 0, \omega|\,(\zeta\cdot\partial_+X\,\partial_-X^{25} - \zeta\cdot\partial_-X\,\partial_+X^{25})\,e^{iK_2\cdot X}\,|0, K_1; 0, \omega\rangle$

$$= \langle 0, -K_3; 0, \omega|\,\zeta\cdot\tilde{\alpha}_0\alpha_0^{25} - \zeta\cdot\alpha_0\tilde{\alpha}_0^{25}\,|0, K_1+K_2; 0, \omega\rangle$$

$$= \zeta\cdot(K_1+K_2)\,\langle 0, -K_3; 0, \omega|\,\alpha_0^{25} - \tilde{\alpha}_0^{25}\,|0, K_1+K_2; 0, \omega\rangle = (2R\omega)\,\zeta\cdot K_3\,\delta(K_1+K_2+K_3)$$

This reproduces the vertex from a term $A_\mu \Phi \partial^\mu \Phi$ in the spacetime Lagrangian $\Rightarrow$ winding number is KR-photon charge! This is gratifying, strings ensure no spacetime gauge symmetry goes to waste.

Coming back to our mass formula (w/ α''s restored): $M^2 = \frac{m^2}{R^2} + \frac{\omega^2 R^2}{(\alpha')^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)$ & $N - \tilde{N} = m\omega$

Consider the limiting cases for R:

- $R \to \infty$, continuum of KK modes $(\omega = 0,\ m \text{ any})$ (sign of 25^{th} dimension)
- $R \to 0$, continuum of winding modes ?! $(m = 0,\ \omega \text{ any})$

A hint is to observe a curious symmetry of the physical state spectrum.

exchanges KK & KR photons.

- $m \leftrightarrow \omega$
- osc. $\leftrightarrow$ − osc. (b/c $\alpha_0^{25} = \frac{\alpha' m}{R} - R\omega$)
- $R \longleftrightarrow \alpha'/R$
- $\widetilde{\text{osc.}} \leftrightarrow \widetilde{\text{osc}}$

Claim: this is an exact symmetry of the CFT. Continuum of winding modes is related to opening up of dual circle!

$$\boxed{\text{"T-duality"}}$$

What is the invariant data of our theory?

- State space $\prod_i \alpha_{-n_i} |0, m, \omega\rangle$
- Virasoro operators $L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\alpha_n \qquad L_0 = \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-m}\alpha_m$

$$\tilde{L}_m = \frac{1}{2}\sum \tilde{\alpha}_{m-n}\tilde{\alpha}_n \qquad \tilde{L}_0 = \tilde{\alpha}_0^2 + \sum_{n=1}^{\infty}\tilde{\alpha}_{-m}\tilde{\alpha}_m$$

From this data we can re-construct the local field operator $X(\tau,\sigma) = x + 2\alpha'\tau p + 2\left(\frac{\alpha'}{R}\right)\omega\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\left(\alpha_n e^{-2in\sigma_-} + \tilde{\alpha}_n e^{-2in\sigma_+}\right)$

$= X_L(\sigma_+) + X_R(\sigma_-)$ where we split the field as:

$$X_L(\sigma_+) = x_L + 2\alpha'\sigma_+ p_L + i\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\frac{1}{n}\tilde{\alpha}_n e^{-2in\sigma_+}$$

$$X_R(\sigma_-) = x_R + 2\alpha'\sigma_- p_R + i\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\frac{1}{n}\alpha_n e^{-2in\sigma_-}$$

Defining $T_{\pm\pm} = \partial_\pm X \partial_\pm X$, we recover the L_m & $\tilde{L}_m$ as Fourier modes. The zero modes here are defined as follows:

$$p_L = \frac{1}{2}\left(p + \frac{R}{\alpha'}\omega\right) = \frac{1}{2}(p + \tilde{p}) \qquad \text{w/ } p|\ldots; m,\omega\rangle = \frac{m}{R}|\ldots; m,\omega\rangle$$

$$p_R = \frac{1}{2}\left(p - \frac{R}{\alpha'}\omega\right) = \frac{1}{2}(p - \tilde{p}) \qquad \text{w/ } \tilde{p}|\ldots; m,\omega\rangle = \frac{R\omega}{\alpha'}|\ldots; m,\omega\rangle$$

$$x_L = \frac{1}{2}(x + \tilde{x}) \qquad \text{w/ } e^{\frac{ix}{R}}|\ldots; m,\omega\rangle \to |\ldots; m+1,\omega\rangle \qquad [p,x] = -i$$

$$x_R = \frac{1}{2}(x - \tilde{x}) \qquad \text{w/ } e^{\frac{i\tilde{x}R}{\alpha'}}|\ldots; m,\omega\rangle = |\ldots; m,\omega+1\rangle \qquad [\tilde{p},\tilde{x}] = -i$$

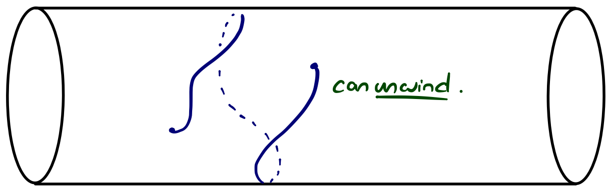
Now with the exact same data, we define $\tilde{X}(\tau,\sigma) = X_L(\tau,\sigma) - X_R(\tau,\sigma)$

$$= \tilde{x} + 2\alpha'\tau\tilde{p} + 2\alpha' p\sigma + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n=-\infty}^{\infty}\frac{1}{n}\left\{-\alpha_n e^{-2in\sigma_-} + \tilde{\alpha}_n e^{-2in\sigma_+}\right\}$$

$$= \tilde{x} + 2\alpha'\tau\tilde{p} + 2\tilde{R}\tilde{\omega}\sigma + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n=-\infty}^{\infty}\frac{1}{n}\left\{(-\alpha_n)e^{-2in\sigma_-} + \tilde{\alpha}_n e^{-2in\sigma_+}\right\}$$ where $\tilde{R} = \frac{\alpha'}{R}$, $\tilde{\omega} = Rp$ acts as m.

This is exactly the expression for a compact boson of radius $\tilde{R} = \alpha'/R$! Writing $\hat{T}_{\pm\pm} = \partial\tilde{X}_\pm\partial\tilde{X}_\pm = T_{\pm\pm}$, we see that same L_m's & $\tilde{L}_m$'s will be derived. An exact equivalence of string backgrounds!

We need to consider how open-strings behave under T-duality, if it is to be a true string duality. The puzzle is that for open strings, while KK momentum still makes sense as a conserved charge, there is no reasonable analogue of winding.

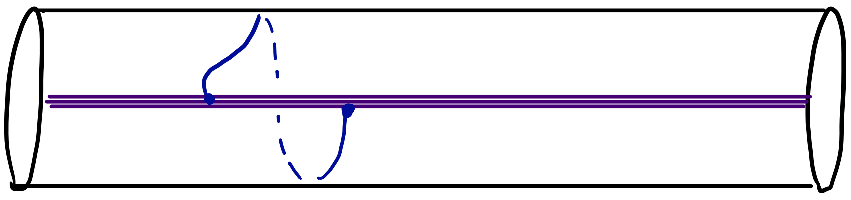
S^1_R 

After T-duality, we should have a kind of string for which winding is a good quantum number, but KK-momentum is not. So translation invariance in the S^1 direction should be broken!

How does it happen? Consider our boundary conditions for $X^{25}(\tau,\sigma)$ in the open string:

$$0 = \partial_\sigma X^{25} = \partial_+ X_L^{25} - \partial_- X_R^{25} = \partial_\tau \tilde{X}^{25} \quad \text{at } \sigma = 0, \pi$$

So $\tilde{X}^{25}(\tau,\sigma)$ obeys Dirichlet boundary conditions $\tilde{X}_{25}(\tau,\sigma)\Big|_{\sigma=0,\pi} = \text{constant}$.

$\tilde{S}^1_{\tilde{R}}$ 

Now have winding but no momentum (translations in compact direction not a symmetry). The subspace where the string endpoints live is called a D-brane. In particular, the convention is that a Dp-brane has p spatial directions of extent (plus time). Thus our open strings now end on a D24-brane.

This sheds a new perspective on our original open string. We say that our open strings end on a space-filling D25-brane, and T-duality along a circle filled by the brane sends $D_p \to D_{(p-1)}$. Similarly, T-duality along an unfilled direction sends $D_{(p)} \to D_{(p+1)}$.