Accretion in Astrophysics: Theory and Applications

Problem Set

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1 Luminosity of a Shakura-Sunyaev (SS) Disk

In lecture we derived the following expression for the effective temperature, $T_{\rm eff}$ as a function of radial distance from the central compact star:

$$T_{\rm eff} = \left[\frac{3GM\dot{M}}{8\pi\sigma r^3}\right]^{1/4} \left(1 - \sqrt{r_0/r}\right)^{1/4}$$

where σ is the Stefan-Boltzmann constant.

a.) Integrate the total power radiated from the disk (including both sides) and show that it equals

$$L = \frac{1}{2}\frac{GM\dot{M}}{r_0}$$

where r_0 is the radius of the inner edge of the accretion disk.

b.) Define the power radiated in an SS disk for all radii greater than r to be $L(>r)$. Find an analytic expression for the ratio:

$$\text{ratio} = \frac{L(>r)}{\frac{1}{2}\frac{GM\dot{M}}{r}}$$

Sketch the ratio as a function of r. This result demonstrates that the gravitational potential energy, released as the matter migrates inward, does not emerge from the disk locally, but rather is redistributed by the viscous stresses.

2 Temperature of an SS-Accretion Disk

a.) Use the above expression for $T_{\rm eff}$ of an SS-disk to find the location (i.e., the radial distance from the central star) where the temperature is a maximum. Express your answer in terms of r_0, the radius of the inner edge of the disk. If the central star is a non-rotating black hole, then $r_0 = 6R_g$. In this case, express your answer for the location of the maximum temperature in terms of R_g.

b.) Compute $T_{\rm max}$ for the following types of accreting sources:

accretor	mass	$\dot{M}$	r_0	source type
white dwarf	$1\ M_\odot$	10^{17} gm/sec	9×10^8 cm	"CV"
neutron star	$1.4\ M_\odot$	10^{18} gm/sec	1.2×10^6 cm	"LMXB"
black hole	$10^6\ M_\odot$	10^{24} gm/sec	9×10^{11} cm	"AGN"
black hole	$10^9\ M_\odot$	10^{27} gm/sec	9×10^{14} cm	"AGN"

3 Mass Stored in an Accretion Disk

In lecture we derived expressions for the midplane pressure, temperature, and density of an SS-disk, as well as for the thickness, H, all as functions of the radial distance r. In the handout, the dependence of these quantities on α and $\dot{M}$ were specified, but the leading dimensioned quantities were not given. These are provided below for the case of an accreting central neutron star with a mass of $1.4\ M_\odot$.

Use these results to compute the amount of mass stored in the accretion disk at a particular instant in time. Formally, you will find that this mass is infinite; however, if you restrict yourself to plausible integration limits for r, e.g., $r_0 < r < 10^4\ r_0$, you will find a sensible answer.

$$P \simeq 2 \times 10^5 \alpha^{-9/10} \dot{M}_{16}^{17/20} r_{10}^{-21/8} f^{17/20} \quad \text{dynes cm}^{-2}$$

$$H \simeq 1 \times 10^8 \alpha^{-1/10} \dot{M}_{16}^{3/20} r_{10}^{9/8} f^{3/20} \quad \text{cm}$$

$$T \simeq 2 \times 10^4 \alpha^{-1/5} \dot{M}_{16}^{3/10} r_{10}^{-3/4} f^{3/10} \quad \text{K}$$

$$\rho \simeq 7 \times 10^{-8} \alpha^{-7/10} \dot{M}_{16}^{11/20} r_{10}^{-15/8} f^{11/20} \quad \text{g cm}^{-3}$$

where $\dot{M}_{16}$ is the mass accretion rate in units of 10^{16} gm sec^{-1}, and r_{10} is the radial distance in units of 10^{10} cm. The function f is defined to be $f = (1 - \sqrt{r_c/r})^{1/4}$.

To make the integration easier, but with no significant loss of accuracy, you can safely set $f = 1$ in the above expressions. Take the inner edge of the accretion disk to be located at $r_0 = 10^7$ cm, and the accretion rate to be $\dot{M} = 10^{18}$ grams sec^{-1}. A plausible value to use for the α parameter is 0.1.

Given the amount of mass stored in such a disk and the accretion rate, estimate a timescale for "filling" the disk if it were initially empty.

4 Radial Velocity in an SS Accretion Disk

Use the expressions for $\rho(r)$ and $H(r)$ given in the previous problem to compute an expression for v_r, the radial in-spiral speed of the disk material. Show that for all choices of parameters α and $\dot{M}$, the radial speed $v_r \ll v_{\text{kepler}}$, as long as one considers radial distances significantly greater than r_0.

5 Spectrum of an SS Accretion Disk

Write out an integral expression for L_ν of an SS accretion disk, where L_ν is the spectral luminosity (units of power per unit frequency interval). Treat each annulus in the disk as a black body of temperature $T_{\text{eff}}(r)$ as defined in problem 2 above. Do not try to integrate the expression since it can't be done analytically.

For reference, the Planck function is:

$$P(\nu) = \frac{2\pi h \nu^3 c^{-2}}{\left[e^{(h\nu/kT)} - 1\right]}$$

Optional

If you make the following approximations, the spectrum (i.e., L_ν) can be obtained analytically:
• Approximate the Planck function by

$$P(\nu) = 2\pi h \nu^3 c^{-2} e^{-h\nu/kT}$$

• Take the factor $\left(1 - \sqrt{r_0/r}\right)^{1/4}$ in the expression for $T(r)$ to be approximately unity.
• Carry out the integration from $r = 0$ to $r = \infty$, even though a real disk obviously has limits at both ends.

Show that

$$L_\nu \propto \nu^{1/3}$$

6 The Last Stable Circular Orbit

In General Relativity, the equation for the radial coordinate r of a test particle orbiting a non-rotating black hole of mass M can be written as

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{r^2} + c^2\right) = \frac{1}{2}\frac{E^2}{c^2}, \tag{1}$$

where $\dot{r} = dr/dt$ and L and E are the angular momentum per unit rest mass and the energy per unit rest mass of the particle, respectively (the particle is assumed to have non-zero rest mass). This equation resembles the energy conservation equation in Newtonian dynamics, $E_{\mathrm{N}} = 1/2\,\dot{r}^2 + V_{\mathrm{eff}}(r)$, except for the additional term $-GML^2/c^2\,r^3$ in the effective potential V_{eff} that becomes dominant at small radii.

a) Treating the problem like a Newtonian one, sketch the effective potential for a particle near a black hole as a function of radius, both for a small and a large value of L. Characterize the possible types of trajectories/orbits in both cases.

b) Show that for each value of L there are two possible circular orbits

$$r_{\pm} = \frac{L^2 \pm [L^4 - 12G^2M^2L^2/c^2]^{1/2}}{2GM}, \tag{2}$$

provided that $L^2 > 12G^2M^2/c^2$.

c) Show that the r_+ solution has a minimum value of $r_+^{\mathrm{min}} = 6GM/c^2$ and argue that this is a stable orbit (i.e. corresponds to a minimum of the effective potential). What does this imply for the r_- solution?

d) Calculate the energy E of a particle at this innermost stable circular orbit and show that it's binding energy per unit rest mass E_{B} is

$$E_{\mathrm{B}} = (1 - (8/9)^{1/2})\,c^2 \simeq 0.06\,c^2.$$

e) Discuss briefly what happens as matter orbiting a black hole in an accretion disc approaches the innermost stable orbit. Compare this case to accretion onto a non-magnetic neutron star.