Problem sheet 2a

1. Let $\mathcal{H}$ be a bifurcate Killing horizon associated to Killing vector k_a and let the surface gravity κ be defined by $k^a\nabla_a k_b = \kappa k_b$. Show that $\nabla_a k^b k_b = -2\kappa k_a$. Use the fact that on $\mathcal{H}$, k_a is hypersurface-orthogonal, i.e., $k_{[a}\nabla_b k_{c]} = 0$, and Killing to show that

$$\kappa^2 = -\tfrac{1}{2}(\nabla_a k_b)(\nabla^a k^b) \tag{233}$$

Show that if the Killing horizon is bifurcate, then κ is constant. [*Hint: κ is constant up the generators, so work on the bifurcation surface where $k_a = 0$. First prove that for any Killing vector $\nabla_a\nabla_b k_c = -R_{bcad}k^d$.*]

2. Compute the surface gravity for the Reissner-Nordstrom metric

$$ds^2 = \frac{\Delta}{r^2}dt^2 - \frac{r^2}{\Delta}dr^2 - r^2 d\Omega^2\,, \qquad \Delta = (r-r_+)(r-r_-) = r^2 - 2mr + Q^2,$$

at the horizon $r = r_+ > r_-$.

[*Note that the original coordinates are singular at the horizon, whereas Eddington-Finkelstein coordinates make the horizon smooth with $dv = dt + \frac{r^2 dr}{\Delta}$ and $\partial/\partial t$ is $\partial/\partial v$ in these (v, r, θ, ϕ) coordinates.*]

3. Show that if a Killing vector k^a is thought of as a 1-form, $k_a dx^a$, then the 2-form ${}^*dk$ satisfies the identity

$$d^*dk = R_{ab}k^{a*}dx^b\,. \tag{234}$$

Assuming Einstein's equations, deduce that

$$\nabla^b J_b = 0\,, \qquad J_a = T_{ab}k^b - \tfrac{1}{2}T k_a\,, \quad T = T^a_a\,.$$

Evaluate the integral of ${}^*dk$ on a sphere of constant r in Reissner-Nordstrom. Interpret the answer.

4. Prove lemma 6.1 in the lecture notes.