STRING THEORY II

Lecture II

OXFORD UNIVERSITY
MMATHPHYS TT 2020.

Mode EXPANSION

CLOSED STRING.

$\Rightarrow$ PERIODIC B.C.

$$S_F = \frac{i}{2\pi}\int d^2\sigma \; (\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-)$$

$$\delta S_F = \int d\tau \; (\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\Big|_{\sigma=0}^{\sigma=\ell} \quad \circledast \quad \sigma\in[0,\ell]$$

+ TERMS THAT VANISH ON-SHELL (EOM).

PERIODIC B.C. : $(\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\Big|_{\sigma=0}^{\ell} = 0.$

IMPOSE B.C. ON $\psi_\pm$ THAT ARE COMPATIBLE W/ SPACETIME & WS POINCARÉ INV:

$$\left.\begin{aligned} \psi_+(\sigma+\ell) &= \boxed{\pm}\, \psi_+(\sigma) \\ \psi_-(\sigma+\ell) &= \boxed{\pm}\, \psi_-(\sigma) \end{aligned}\right\} \text{ 4 CHOICES OF B.C.}$$

$\boxed{\pm} = e^{2\pi i \phi_\pm}$

$+ \searrow \; \swarrow -$

(R, R)

(R, NS)

(NS, R)

(NS, NS)

$\phi_\pm = 0$ RAMOND STRING (R)

$\phi_\pm = \frac{1}{2}$ NEVEAU-SCHWARZ STRING (NS)

$$\phi = \begin{cases} 0 & R \\ 1/2 & NS \end{cases}$$

⇝ OSCILLATORS

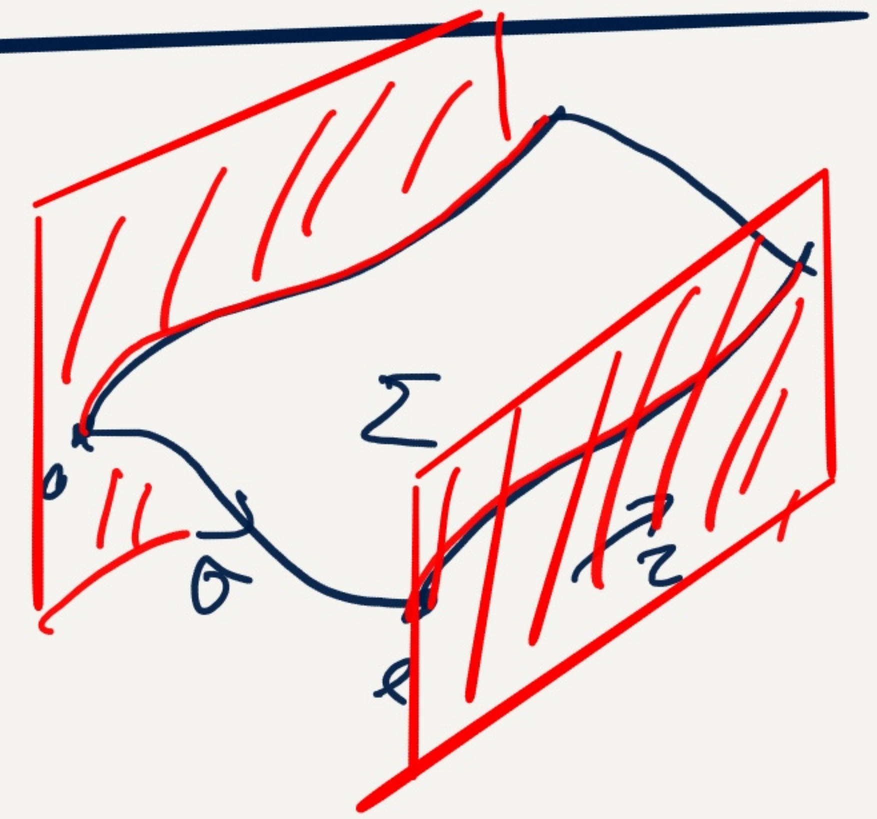
$$\psi_+^\mu(\sigma_+) = \sqrt{\frac{2\pi}{\ell}} \sum_{r \in \mathbb{Z}+\phi} \tilde{b}_r^\mu \, e^{-\frac{2\pi i r}{\ell}(\sigma_+)}$$

$$\sigma_\pm = \tau \pm \sigma.$$

$$\psi_-^\mu(\sigma_-) = \sqrt{\frac{2\pi}{\ell}} \sum_{r \in \mathbb{Z}+\phi} b_r^\mu \, e^{-\frac{2\pi i r}{\ell}\sigma_-}$$

⇒ CLOSED STRING.

OPEN FERMIONIC STRING.



$$\partial\Sigma = \{\sigma = 0\} \cup \{\sigma = \ell\}.$$

$(*)$: $$\psi_+^\mu(\sigma)\Big|_{\substack{\sigma=0 \\ \sigma=\ell}} = \pm\, \psi_-^\mu(\sigma)\Big|_{\substack{\sigma=0 \\ \sigma=\ell}}.$$

⇒ 4 CHOICES

NEUMAN & DIRICHLET B.C.

Recap: BOSONIC STRING: $X^\mu(\sigma,\tau)$:

$$\partial_\sigma X^\mu \Big|_{\substack{\sigma=0\\ \sigma=\ell}} = 0 \qquad \text{(Neuman, Neuman)} \equiv (N,N)$$

$\Rightarrow$ FERMIONIC COUNTERPARTS:

$$\psi_+^\mu(0) = \psi_-^\mu(0)$$

$$\psi_+^\mu(\ell) = \pm\, \psi_-^\mu(\ell)$$

($\pm \leftarrow e^{2\pi i \phi}$)

$\phi = 0$: RAMOND SECTOR:

$$\psi_\pm^\mu(\sigma_\pm) = \sqrt{\frac{\pi}{\ell}} \sum_{r\in\mathbb{Z}} b_r^\mu \, e^{-\frac{\pi}{\ell} i r \sigma_\pm}$$

$\Rightarrow \psi_+ = \psi_-$ @ $\sigma = \ell$

$\phi = \frac{1}{2}$: NS SECTOR:

$$\psi_\pm^\mu(\sigma_\pm) = \sqrt{\frac{\pi}{\ell}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-\frac{\pi}{\ell} i r \sigma_\pm}.$$

$\Rightarrow \psi_+ = -\psi_-$ @ $\sigma = \ell$.

B.C. BREAK SUSY:

$$\partial_\sigma X^\mu \Big|_{\substack{\sigma=0 \\ \& \sigma=\ell}} = 0 \qquad (NN).$$

$$\psi_+^\mu(0) = \psi_-^\mu(0) \xrightarrow{\delta_\epsilon} \epsilon^+(0)\, \partial_+ X^\mu(0) \overset{!}{=} \epsilon^-(0)\, \partial_- X^\mu(0)$$

$$\psi_+^\mu(\ell) = \pm \psi_-^\mu(\ell)$$

$\xrightarrow{\delta_\epsilon}$ $\Rightarrow \epsilon^+(0) = \epsilon^-(0)$

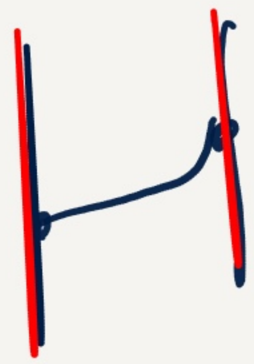
And likewise:

$$\epsilon^+(\ell) = \pm \epsilon^-(\ell).$$

$\Rightarrow$ SUSY PARAMETERS ϵI ARE RELATED

SO B.C. BREAK SUSY.

DIRICHLET - DIRICHLET B.C. (DD)



$$\partial_\tau X^\mu \Big|_{\substack{\sigma=0 \\ \sigma=\ell}} = 0 \qquad \begin{aligned} X^\mu(\tau,\sigma=0) &= x_0^\mu \\ X^\mu(\tau,\sigma=\ell) &= x_1^\mu \end{aligned}$$

$$\Rightarrow \quad X^\mu(\sigma,\tau) = x_0^\mu + \frac{1}{\ell}(x_1^\mu - x_0^\mu)\sigma + \sqrt{2\alpha'} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \frac{1}{n}\alpha_n^\mu e^{-i\frac{\pi}{\ell}n\tau} \sin\left(\frac{\pi n \sigma}{\ell}\right).$$

FERMIONS: EITHER USE SUSY OR

USE T-DUALITY: $N \Leftrightarrow D$

$$\left.\begin{aligned} X_+(\sigma^+) &\overset{T}{\longleftrightarrow} X_+(\sigma^+) \\ X_-(\sigma^-) &\overset{T}{\longleftrightarrow} -X_-(\sigma^-) \end{aligned}\right\}$$

BY SUSY:

$$\left.\begin{aligned} \psi_-(\sigma^-) &\longleftrightarrow -\psi_-(\sigma^-) \\ \psi_+(\sigma^+) &\longleftrightarrow \psi_+(\sigma^+). \end{aligned}\right\}$$

$$\Rightarrow (DD) \quad \psi_+^\mu(0) = -\psi_-^\mu(0)$$

$$\psi_+^\mu(\ell) = \mp \psi_-^\mu(\ell) \equiv -e^{2\pi i\phi}\psi_-^\mu(\ell).$$

$$\Rightarrow \quad \psi_\pm(\sigma_\pm) = \pm\sqrt{\frac{\pi}{\ell}} \sum_{r\in\mathbb{Z}+\phi} \overset{(\sim)}{b}{}_r^\mu \, e^{-\frac{i\pi r\sigma_\pm}{\ell}} \qquad (DD).$$

MODE EXPANSION OF THE EM TENSOR & THE SUPERCURRENT:

$$T_{\pm\pm} = -\frac{1}{\alpha'}\partial_\pm X\cdot\partial_\pm X - \frac{i}{2}\psi_\pm\partial_\pm\psi_\pm .$$

$$J_\pm = -\frac{1}{\sqrt{2\alpha'}}\psi_\pm\cdot\partial_\pm X$$

$\Rightarrow$ MODE EXPANSION FOR THE CLOSED STRING:

$$\left\{\begin{matrix}\tilde{L}_m \\ L_m\end{matrix}\right. = -\frac{\ell}{2\pi^2}\int_0^\ell d\sigma \left(e^{\pm i\frac{\pi}{\ell}m\sigma}\, T_{\pm\pm}\right).$$

Virasoro generators.

e.g. $$\boxed{L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_{-n}\cdot\alpha_{n+m} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\left(r+\frac{m}{2}\right)b_{-r}\cdot b_{m+r}}$$

MODES FOR $J_\pm$:

$$\boxed{G_r^+ = \sum_{n\in\mathbb{Z}}\alpha_{-n}\cdot b_{r+n} \quad . \; r\in\mathbb{Z}+\phi}.$$

Super generators for SUPER-VIRASORO Algebra.

CHAPTER 2: QUANTIZATION OF THE RNS STRING

RECAP: BOSONIC STRING CANONICAL QUANTIZATION.

$\alpha_m^\mu,\ \tilde{\alpha}_m^\mu,\ x^\mu,\ p^\mu \qquad \alpha_m^\dagger = \alpha_{-m}.$

POISSON BRACKET $\longrightarrow \frac{1}{i}[\,,\,]$: $\quad [x^\mu, p^\nu] = i\eta^{\mu\nu}$

$$[\overset{(\sim)}{\alpha}{}_m^\mu, \overset{(\sim)}{\alpha}{}_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}.$$

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0.$$

NORMAL ORDERING: (N.O.)

$$:\alpha_m^\mu \alpha_n^\nu: \; = \begin{cases} \alpha_m^\mu\, \alpha_n^\nu & m \leq n \\ \alpha_n^\nu\, \alpha_m^\mu & n < m. \end{cases}$$

$$\Rightarrow L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n:$$

$$\tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n:$$

N.O. IMPORTANT FOR $L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot\alpha_n: = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n$

$L_0 \longrightarrow L_0 - a$

← N.O. constant.

CENTRAL CHARGE.

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}.$$

E.G. FOR X^μ: $\mu = 0 \dots d-1$: $c = d$.

PHYSICAL STATES ARE DETERMINED AS THE SUBSPACE OF THE FOCK SPACE OF α^μ_m, $\tilde{\alpha}^\nu_n$ SUBJECT TO PHYSICAL STATE CONDITIONS:

$$\begin{cases} L_m |\phi\rangle = \tilde{L}_m |\phi\rangle = 0 & m > 0 \\ (L_0 - a) |\phi\rangle = 0. \end{cases}$$

$|\phi\rangle \in$ FOCK SPACE.

CANONICAL QUANTIZATION OF THE RNS-STRING

$\psi_\pm^\mu$: DIRAC BRACKETS $\{ , \}_{DB} \longrightarrow \frac{1}{i}\{ , \}$

$$\Pi_\pm = \frac{i}{2\pi}\psi_\pm$$

↑ anticommutators.

[BLT].

$$\{\psi_\pm^\mu(\sigma,\tau), \psi_\pm^\nu(\sigma',\tau)\} = 2\pi\eta^{\mu\nu}\delta(\sigma-\sigma')$$

$$\{\psi_+ , \psi_-\} = 0.$$

$\Rightarrow$ MODE EXPANSION:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}$$

$$\{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}$$

$$\{b_r^\mu, \tilde{b}_s^\nu\} = 0$$

$r, s \in \mathbb{Z} + \phi$

$$\phi = \begin{cases} 0 & R \\ 1/2 & NS. \end{cases}$$

KEY OBSERVATION: R : $r \in \mathbb{Z}$: $r = 0$: $\boxed{\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}}$

μ = SPACE-TIME VECTOR INDICES.

$\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$ IS THE CLIFFORD ALGEBRA OF $SO(1,d-1)$!

ψ_1^ν : $\mathbb{R}^{1,d-1}$ VECTORS.

STARTED w/ WS FERMIONS $\nearrow$ R / $\searrow$ NS B.C.

$\xrightarrow{\text{QUANTIZE}}$ R SECTOR: SPACETIME SPINORS AS THE PHYSICAL STATE SPACE!

SUPER-VIRASORO ALGEBRA: T_{--} & J_- MODES.

0 $\quad L_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot\alpha_{m+n}: + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} (r+\frac{m}{2}) :b_r\cdot b_{m+r}:$

1 $\quad G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n}$

$\Rightarrow [L_m, L_n] = (m-n) L_{n+m} + \frac{c}{12} m(m^2-2\phi)\,\delta_{m+n,0}.$

$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}$

$\{G_r, G_s\} = 2 L_{r+s} + \frac{c}{12}(4r^2 - 2\phi)\,\delta_{r+s,0}.$

SUPER-VIRASORO ALGEBRA: $\mathbb{Z}_2$-GRADED ALGEBRA.

$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ $\qquad \mathfrak{g}^{(i)}$ $i \hat{=}$ $\mathbb{Z}_2$-grading.

$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}^{i+j \bmod 2}$

$[\,,\,] \hat{=}$ COMMUTATOR EXCEPT FOR $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \equiv \{\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\}$.

- C = CENTRAL CHARGE GETS CONTRIBUTIONS FROM

  BOSONS: $\Delta c = 1$

  FERMION: $\Delta c = 1/2$.

  $\Big\}$ $c = \frac{3}{2} d$ $\qquad d$ = dim of spacetime.

- ϕ IN THE ALGEBRA CAN BE SHIFTED AWAY:

  $L_0^R \to \left(L_0^R + \frac{d}{16}\right) \Rightarrow \phi = \frac{1}{2}$ FOR BOTH SECTORS

  ↑ R- SECTOR.

PHYSICAL STATE CONDITIONS: (PSC) SUPER-VIR CONSTRAINTS

FOCK: $\mathcal{F} = \left\{ \prod \alpha_{-m_i}^{\mu_i} \, b_{-r}^{\nu_1} \cdots b_{-r_I}^{\nu_I} |0\rangle \right\}$.

NOTE: $b_r^2 = 0$.

VACCUM: $b_r |0\rangle = \alpha_m |0\rangle = 0 \qquad r > 0, \; m > 0$

PSC:

$$(L_0 - a_{R/NS}) |\phi\rangle = 0$$

$$L_m |\phi\rangle = 0 \quad m > 0 \qquad (\text{sim. for } \tilde{L}_m, \tilde{a}_r).$$

$$G_r |\phi\rangle = 0 \quad r > 0$$

N.O.: BOSONIC STRING: $\zeta(-1) = -\frac{1}{12} \quad a = \frac{1}{24}$.

N.O. RAMOND SECTOR:

$$L_0^{(R)} = \frac{1}{2}\sum_{r=1}^{\infty} r\, b_{-r} b_r + \underbrace{\frac{1}{2}\sum_{-\infty}^{-1} r\, b_{-r} b_r}$$

$$= \frac{1}{2}\sum_{r=1}^{\infty} -r\, b_r \cdot b_{-r}$$

$$= \frac{1}{2}\sum_{r=1}^{\infty} r\, b_{-r}\cdot b_r - \underbrace{\frac{1}{2}\sum r}_{= -a_R}$$

$\{b_r, b_{-r}\} = +1$

SAME SUM AS IN BOSONIC STRING: $a_R = -\frac{1}{24}$.

NS SECTOR: 1/2 - INTEGER:

$$a_{NS} = \frac{1}{2}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) = \frac{1}{2}\,\zeta(-1,q)\Big|_{q=\frac{1}{2}}$$

$$= \frac{1}{48}$$

$$a_{NS} = \frac{1}{48}$$

$$\Rightarrow \quad a^B + a^F_R = 0$$

$$a^B + a^F_{NS} = \frac{1}{16}.$$