STRING THEORY II

LECTURE X

OXFORD UNIVERSITY
MMATH PHYS TT 2020

TORUS-COMPACTIFICATIONS

$T^N = \mathbb{R}^N / \Lambda$ $\quad \Lambda \cong \mathbb{Z}^N$ Rank N LATTICE.

$X^\mu \equiv X^\mu + \lambda^\mu \qquad \lambda^\mu \in \Lambda$

MOMENTUM & WINDING LATTICE (NARAIN LATTICE):

$$\underline{l} = (\underbrace{l_L}_{N\text{ vector}}, \underbrace{l_R}_{N\text{ vector}}) \in \Gamma \cong \mathbb{Z}^{2N}. \leftarrow$$

$$l_L^i = \sqrt{\frac{\alpha'}{2}}\, K_L^i = \sqrt{\frac{\alpha'}{2}} \left(\frac{n^i}{R_i} + \frac{w^i R_i}{\alpha'} \right)$$

$$l_R^i = \sqrt{\frac{\alpha'}{2}}\, K_R^i = \sqrt{\frac{\alpha'}{2}} \left(\frac{n^i}{R_i} - \frac{w^i R_i}{\alpha'} \right).$$

(n_i, w_i) MOMENTA/WINDING #s IN i^{th} DIRECTION

R_i RADIUS OF THE i^{th} DIRECTION.

VERTEX OPERATOR $V_{\underline{l}} = \exp\left(i k_L \cdot X_L + i k_R \cdot X_R \right)$

$$V_l |\Omega\rangle = |k_L, k_R\rangle$$

OPE: $V_{\underline{\ell}}(z,\bar{z})\, V_{\underline{\ell}'}(0,0)$

$$= \exp(ik_L X_L + ik_R X_R)(z,\bar{z})\, \exp(ik_L'\cdot X_L + ik_R'\cdot X_R)(0,0)$$

$$\sim \underbrace{z^{\ell_L\cdot\ell_L'}\,\bar{z}^{\ell_R\cdot\ell_R'}}\; V_{\underline{\ell}+\underline{\ell}'}(0).$$

MONODROMY: THIS PICKS UP A PHASE

$$e^{2\pi i(\underbrace{\ell_L\cdot\ell_L' - \ell_R\cdot\ell_R'}_{=\langle\underline{\ell},\underline{\ell}'\rangle})} \stackrel{!}{=} 1$$

NEEDS TO BE TRIVIAL FOR OPE TO BE LOCAL.

CONDITION 1:

$$\boxed{\langle\underline{\ell},\underline{\ell}'\rangle \in \mathbb{Z}}$$

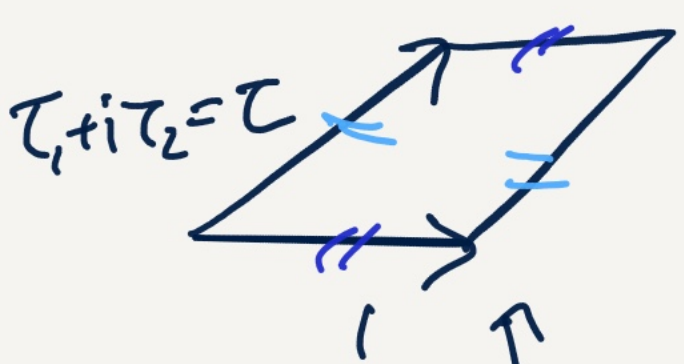
$\langle\underline{\ell},\underline{\ell}'\rangle = \ell_L\cdot\ell_L' - \ell_R'\,\ell_R'$ IS INNER PRODUCT ON THE NARAIN LATTICE Γ OF SIGNATURE (N,N)
$\cong \mathbb{Z}^{2N}$

FOR A LATTICE Γ, WE DEFINE THE DUAL LATTICE

Γ^* : $\underline{v}$ s.t. $\langle \underline{w}, \underline{v} \rangle \in \mathbb{Z} \quad \forall\, \underline{w} \in \Gamma$.

$\Longrightarrow$ CONDITION 1 : $\boxed{\Gamma \subseteq \Gamma^*.}$

RECALL: LAST LECTURE WE STUDIED MODULAR INVARIANCE OF $Z^{S^1}_{T^2}$ OF THE S^1-REDUCED BOSONIC STRING.



DEFINE FOR A LATTICE Γ:

$$Z^{\Gamma}_{T^2}(\tau) = \underbrace{\frac{1}{|\eta(q)|^{2N}}}_{\text{oscillators.}} \sum_{\underline{\ell} \in \Gamma} e^{\pi i \underline{\tau} \ell_L^2 - \pi i \underline{\bar{\tau}} \ell_R^2}$$

worldsheet

$q = e^{2\pi i \tau}$

REQUIRE: $Z^{\Gamma}_{T^2}$ IS MODULAR INVARIANT.

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}.$$

I.e. $\mathcal{T} : \tau \to \tau + 1 \qquad \mathcal{S} : \tau \to -1/\tau.$

1) $\mathcal{T}: \tau \to \tau+1$:

$$\boxed{l_L^2 - l_R^2 \in 2\mathbb{Z}}$$

$$\boxed{\langle \underline{l}, \underline{l} \rangle \in 2\mathbb{Z}}$$ CONDITION 2.

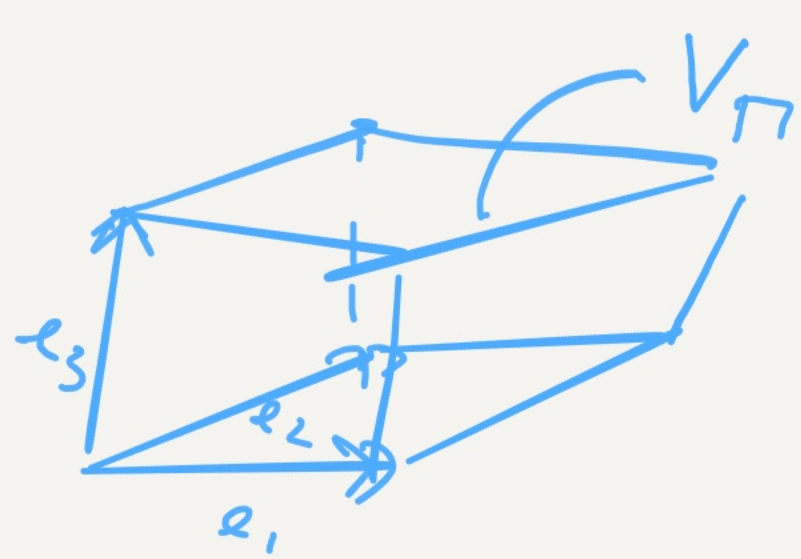
$\Rightarrow \Gamma$ IS AN EVEN LATTICE.

2) $\mathcal{S}: \tau \to -1/\tau$: POISSON RESUMMATION GENERALIZED TO A LATTICE Γ:

$$\sum_{\underline{l}' \in \Gamma} \delta(\underline{l} - \underline{l}') = \frac{1}{V_\Gamma} \sum_{\underline{l}'' \in \Gamma^*} e^{2\pi i \langle \underline{l}, \underline{l}'' \rangle}$$

Γ^* ← dual lattice.

V_Γ ← VOLUME OF FUNDAMENTAL CELL OF Γ



$$Z^{\Gamma}_{T^2}(\tau) = \frac{1}{V_\Gamma} \frac{1}{|\eta(q)|^{2N}} \times$$

$$\times \sum_{\underline{\ell}'' \in \Gamma^*} \int d^{2N}\ell \; e^{2\pi i \langle \underline{\ell}'', \underline{\ell} \rangle + \pi i \tau \ell_L^2 - i\pi \bar{\tau} \ell_R^2}.$$

$$= \frac{1}{V_\Gamma} \frac{1}{|\eta(q)|^{2N}} \frac{1}{(\tau\bar{\tau})^{N/2}} \sum_{\ell'' \in \Gamma^*} e^{-\pi i (\ell_R'')^2/\tau + \pi i \frac{(\ell_R'')^2}{\bar{\tau}}}.$$

$$= \frac{1}{V_\Gamma} Z^{\Gamma^*}_{T^2}\left(-\frac{1}{\tau}\right).$$

MODULAR INV $\Rightarrow$ $V_\Gamma = 1$

$$\boxed{\Gamma^* = \Gamma}$$

SELF-DUAL LATTICE

$\left(\Rightarrow V_\Gamma = \frac{1}{V_{\Gamma^*}} = 1\right)$.

$\Rightarrow$ Γ EVEN, SELF-DUAL LATTICE, UNIMODULAR OF SIGNATURE (N, N).

ALL LATTICES OF THIS TYPE ARE MAPPED INTO EACH OTHER BY $O(N,N,\mathbb{R})$ (PRESERVES THE $\langle \ell, \ell \rangle \in 2\mathbb{Z}$).

"signature (N,N) orthogonal gp".

NOTE: THIS IS NOT A SYMMETRY OF THE CFT.
(THIS DEPENDS SEPARATELY ON ℓ_L, ℓ_R).

THE SYMMETRY GROUP — THAT GENERALIZES THE T-DUALITY OF THE S^1- REDUCTION:

$$O(N,N,\mathbb{Z}) \quad \text{(w/ integer matrices).}$$

$\Rightarrow$ T-DUALITY GROUP
$\Rightarrow$ SYMMETRY OF THE CFT.

INEQUIVALENT T^N COMPACTIFICATIONS:

$$\mathcal{M}_{T^N} = O(N,N,\mathbb{Z}) \backslash O(N,N,\mathbb{R}) \big/ \underset{\substack{\uparrow \\ \text{preserve } \ell_L^2}}{O(N,\mathbb{R})} \times \underset{\substack{\uparrow \\ \ell_R^2}}{O(N,\mathbb{R})}$$

EACH POINT IN $\mathcal{M}_{T^N}$ CORRESPONDS TO A DISTINCT CFT.

$\Rightarrow$ $\mathcal{M}_{T^N}$ "MODULI/PARAMETER SPACE" OF T^N-COMPACTIFICATIONS.

THE $O(N,N,\mathbb{Z})$ IS THE T-DUALITY: CONTAINS e.g.

$$R \to \alpha'/R$$

$$SL_N\mathbb{Z}$$

$$B_{mn} \to B_{mn} + N_{mn} \leftarrow \text{antisymmetric integer matrix.}$$

GOOD EXERCISE: T^2-COMPACTIFICATION & WORK OUT THE ACTION OF $O(2,2,\mathbb{Z})$ on G_{mn} & B_{mn}.

ENHANCED SYMMETRY IN T^N- COMPACTIFICATIONS:

RECAP: $(S^1_R)^9$: $R \longrightarrow R' = \frac{\alpha'}{R}$

$\exists$ SPECIAL POINT: $R = R' = \frac{\alpha'}{R} \Rightarrow R = \sqrt{\alpha'}$

"SELF-DUAL POINT": IT ENJOYS SYMMETRY ENHANCEMENT:

$$\boxed{\begin{matrix} \partial X_L^9 \\ e^{\pm 2/\sqrt{\alpha'}\, X_L^9} \end{matrix}}$$

$U(1)$ CURRENT. $\partial X \partial X \sim \frac{1}{z^2}$

$$\Big\downarrow + e^{\pm \frac{2}{\sqrt{\alpha'}} X_L^9}.$$

GENERATES: $SU(2)$ CURRENT ALGEBRA (LEVEL 1)

THIS IS A GENERAL PRINCIPLE FOR T^N- COMPACTIFICATIONS:

$\rightsquigarrow$ $\mathfrak{g} = ADE$ $(SU(N),\ SO(2N),\ E_6,\ E_7,\ E_8)$

"SIMPLY-LACED" $\alpha^2 = 2$ (ROOTS), WE CAN CONSTRUCT A

$\mathfrak{g}$- CURRENT ALGEBRA AS FOLLOWS:

LET: $X^m(z)$ $m=1\ldots$ Rank($\mathfrak{g}$) FREE COMPACT BOSONS. ||

$$X^m(z)X^n(0) \sim -\delta_{mn}\ln(z).$$

LET: $H^m(z) = i\partial X^m(z)$ ← U(1) CURRENTS.

$$E^{\pm\alpha}(z) = e^{\pm i\alpha\cdot X(z)}$$

↖ MOMENTUM VERTEX OPS.

$\alpha = (\alpha_1 \ldots \alpha_r)$ ROOT of $\mathfrak{g}$

$\alpha^2 = 2.$

THE OPE OF THESE:

$$H^m(z)\,E^\alpha(0) \sim \frac{\alpha^m E^\alpha(0)}{z}$$

$$E^\alpha(z)\,E^\beta(0) \sim \begin{cases} \alpha=-\beta: & \frac{\overset{\to k}{1}}{z^2} + \frac{\alpha\cdot H}{z} \\ (\alpha,\beta)=-1: & \frac{E^{\alpha+\beta}}{z} \end{cases}$$

SU(N):

$$(\alpha,\beta) = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \\ & & & \ddots \end{pmatrix}$$

H^m = CARTANS

$E^{\pm\alpha}$ = "RAISING/LOWERING OPS".

$\Rightarrow$ LEVEL 1 CURRENT ALGEBRA FOR $\mathfrak{g}$.

$\hat{\mathfrak{g}}_{1=k}$ KAC MOODY Algebra. (AFFINE Lie Algebra)

GIVEN SUCH A CURRENT ALGEBRA WE CAN CONSTRUCT A VIRASORO ALGEBRA (SUGAWARA CONSTRUCTION): $\hat{\mathfrak{g}}_k$

$$T(z) = \frac{1}{2(k+g^\vee)} \sum_a J^a J^a$$

$J^a = \{E^\alpha, H\}$.

$g^\vee =$ Dual Coxeter # of $\mathfrak{g}$

W/ CENTRAL CHARGE

$$c = \frac{k \dim \mathfrak{g}}{k+g^\vee}.$$

ADE : $g^\vee = \frac{\dim \mathfrak{g}}{\operatorname{rank} \mathfrak{g}} - 1$.

MODE EXPANDING:

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n$$

OPE $\Rightarrow$ ALGEBRA OF J^a_n:

$$[J^a_n, J^b_m] = \sum_c i f_{abc} J^c_{n+m} + k n \delta_{ab} \delta_{n,-m}$$

$n, m \in \mathbb{Z}$
$a = 1 \dots \dim \mathfrak{g}$.

STRUCTURE CONSTANTS OF THE (FIN. DIM) LIE ALGEBRA $\mathfrak{g}$.

$\{J^a_0\} \cong \mathfrak{g}$ "HORIZONTAL Algebra". : $[J^a_0, J^b_0] = i f_{abc} J^c_0$

MORAL: $X^m(z)$ rank of MANY FREE BOSONS
WE CAN STRUCT A LEVEL 1 ADE affine LIE ALGEBRA $\hat{g}_1$ WHICH HAS A VIRASORO w/ $c = \frac{\dim g}{1 + g^\vee}$.

EG: RANK 1: SU(2) dim 3 $g^\vee = 2$: $c = 1$ ✓.

LITERATURE:

FINITE DIM LIE ALGEBRAS.

- REPRESENTATION THEORY: FULTON, HARRIS (SPRINGER).
- CFT YELLOW BOOK: DI FRANCESCO, MATHIEU, SENECHAL, (CURRENT ALG. etc).
- KAC: INFINITE DIMENSIONAL LIE ALGEBRAS. (CUP).