STRING THEORY II

LECTURE XIV

OXFORD UNIVERSITY
MMATH PHYS TT 2020

LAST LECTURE: $\quad M_{10} = \mathbb{R}^{1,3} \times M_6$

M_6 ← COMPACT.

WHAT ARE THE CONDITIONS TO PRESERVE SUSY IN 4d?

SOLVE $\boxed{\delta_\epsilon \Psi_M = 0}$ (Ψ_M IS THE GRAVITINO; + DILATINO; HET: GAUGINO).

$$\langle \delta_\epsilon \Psi_M \rangle = \langle \nabla_M \epsilon + \cancel{H}_M \epsilon + e^{\Phi} \sum_n \cancel{F}_n \Gamma_M \epsilon \rangle = 0.$$

KILLING SPINOR EQ: (KSE) $\qquad \nabla_M = \partial_M + \frac{1}{2} \omega_M{}^{ab} \Gamma_{ab}$

$\underbrace{G, H, F, \Phi}_{\text{BOSONIC BACKGROUND FIELDS.}}, \epsilon$ ← SPINOR

LAST TIME: KEEP FLAT METRIC ALONG 4 directions ($\mathbb{R}^{1,3}$)

$\leadsto \epsilon \longrightarrow \sum \epsilon^{4d} \otimes \underbrace{\epsilon^{6d}}_{\text{KSE DETERMINE THESE COMPONENTS.}}$

START WITH: $H = F_n = 0$ ("NO FLUXES" TURNED ON).

$$\Rightarrow \quad \boxed{\nabla_M \epsilon = 0}$$

MORE GENERALLY WE CAN RETAIN FLUXES IN THE KSE. WE ALSO DO NOT HAVE TO CONSIDER MINKOWSKI BACKGROUNDS FOR SOME PART OF THE METRIC.

E.G. IIB: BACKGROUND $M_{10} = \underbrace{AdS_5}_{\text{5d Anti deSitter space } (\lambda=-1).} \times \underbrace{S^5}_{\text{5-sphere.}}$

THE KSE CAN BE SOLVED WITH THIS METRIC ANSATZ

$\Rightarrow$ IF $F_5 \neq 0$ ALONG BOTH AdS_5 & THE S^5.

KEY SOLUTION FOR HOLOGRAPHY.

WE ASK QUITE GENERALLY QUESTIONS LIKE: ($\to$ ACTIVE Research).

$M_{10} = AdS_d \times M_{10-d}$ WHAT PROPETIES DOES M_{10-d} HAVE TO SATISFY & FOR WHICH # SUSIES & FLUXES CAN ONE FIND SOLUTIONS TO THE KSE?

LETS RETURN TO $\nabla_M \epsilon = 0$ WITH $M_{10} = \mathbb{R}^{1,3} \times M_6$.

WE SAW: M_6 HAS TO HAVE REDUCED HOLONOMY $SU(3) \subsetneq SO(6)$

$\rightsquigarrow$ CALABI-YAU MANIFOLDS. (CY)

TODAY:

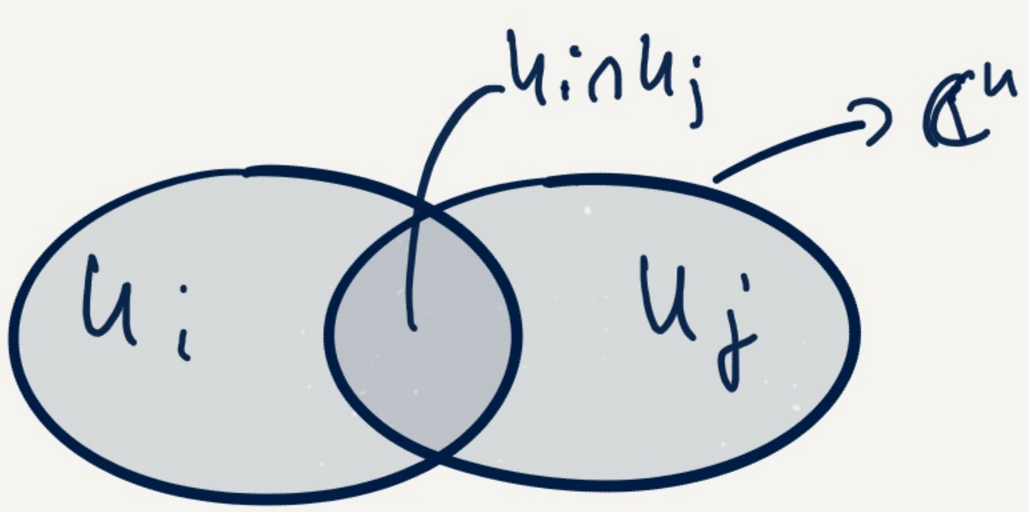
COMPLEX, KÄHLER & CY.

COMPLEX MANIFOLD M: OPEN COVER U_i

$\dim_{\mathbb{C}} M = n$

$\phi_i : U_i \to \mathbb{C}^n$ CHARTS W/

TRASITION FUNCTIONS THAT ARE HOLOMORPHIC:



$\phi_i \circ \phi_j^{-1} : \mathbb{C}^n \to \mathbb{C}^n$.

SIMPLEST (BEYOND $\mathbb{C}^n$): PROJECTIVE SPACE.
n-dim COMPLEX

$$\mathbb{CP}^n = \{ \underbrace{(z_1 \dots z_{n+1})}_{\equiv \underline{z}} \in \mathbb{C}^{n+1} : \underline{z} \sim \underline{w} \text{ IF } \exists \lambda \neq 0 : \underline{z} = \lambda \underline{w} \}.$$

$$\mathbb{CP}^1 = \{ (z_1, z_2) \in \mathbb{C}^2 : \dots \} \cong S^2$$

OPEN COVER:
$U_i = \{ z_i \neq 0 \}$.

DIFFENTIAL FORMS ON COMPLEX MANIFOLDS:

$\Omega^{p,q}$ – FORM:

$$\omega = \omega_{i_1 \dots i_p \bar{\imath}_1 \dots \bar{\imath}_q} dz^{i_1} \wedge \dots dz^{i_p} \wedge d\bar{z}^{\bar{\imath}_1} \wedge \dots d\bar{z}^{\bar{\imath}_q}$$

FROM ~~A~~ REAL PERSPECTIVE:

$$dz = dx + i\,dy.$$ (WORKS THE SAME ~~AS~~ BEFORE).

- [NAKAHARA] FOR A PHYSICS REF.
- [HUYBRECHTS: COMPLEX GEOMETRY: SPRINGER].

$$\partial : \Omega^{p,q} \to \Omega^{p+1,q}$$

$$\partial = \sum_{i=1}^{n} dz^i \frac{\partial}{\partial z^i}$$

$$\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$$

$$\bar{\partial} = \sum d\bar{z}^i \frac{\partial}{\partial \bar{z}^i}$$

COHOMOLOGY GROUPS:

$$H_{\partial}^{p,q} = \frac{\mathrm{KER}(\partial):\ \Omega^{p,q} \to \Omega^{p+1,q}}{\mathrm{Im}(\partial):\ \Omega^{p-1,q} \to \Omega^{p,q}}$$

closed forms

exact forms.

$$H_{\bar\partial}^{p,q} = \frac{\mathrm{Ker}(\bar\partial)}{\mathrm{Im}(\bar\partial)}$$

$$h^{p,q} = \dim H_{\partial}^{p,q} \qquad \text{Hodge NUMBERS OF } M.$$

$$\bar h^{p,q} = \dim H_{\bar\partial}^{p,q}.$$

ARRANGE INTO THE HODGE DIAMOND:

3 COMPLEX DIM MANIFOLD M_6

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{1,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

SIMILARLY: $\bar h^{p,q}$.

WE CONSTRAIN $h^{p,q}$ BY IMPOSING FURTHER PROPERTIES ON M_6.

- HERMITIAN METRIC.

$$g = \cancel{g_{ij}}\, dz^i \otimes dz^j + \cancel{g_{\bar{i}\bar{j}}}\, d\bar{z}^{\bar{i}} \otimes d\bar{z}^{\bar{j}}$$
$$+ g_{i\bar{j}}\, dz^i \otimes dz^{\bar{j}} + g_{\bar{i}j}\, d\bar{z}^{\bar{i}} \otimes dz^j$$

HERMITIAN: $g_{ij} = g_{\bar{i}\bar{j}} = 0.$

- HODGE $*$ OPERATOR:

$$\Omega^{p,q} \longrightarrow \Omega^{n-p,\,n-q}$$

$v, w \in \Omega^{p,q}$: $\quad v \wedge * w = (v, w)\ \mathrm{VOL}(M)$

$$(v,w) = \bar{v}_{\bar{i}_1 \dots \bar{i}_p\, i_1 \dots i_q}\, w_{j_1 \dots j_p \bar{j}_1 \dots \bar{j}_q}\, g^{\bar{i}_1 j_1} \cdots g^{i_q \bar{j}_q}.$$

$$\partial^\dagger = -*\partial*: \ \Omega^{p,q} \xrightarrow{*} \Omega^{n-p,\,n-q} \xrightarrow{\partial} \Omega^{n-p+1,\,n-q}$$
$$\xrightarrow{*} \Omega^{p-1,q}.$$

$$\Delta = \partial\partial^{\dagger} + \partial^{\dagger}\partial \quad : \Omega^{p,q} \to \Omega^{p,q}.$$

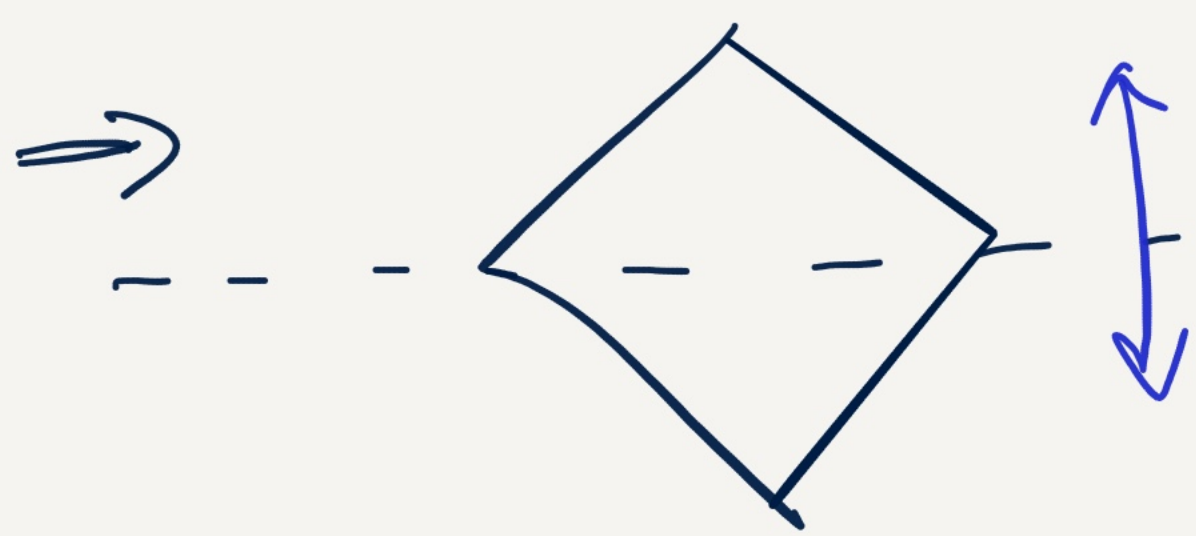
DEFINE: $\omega \in \Omega^{p,q}$ WITH $\Delta \omega = 0$ ARE CALLED <u>HARMONIC FORMS</u>.

HODGE THEOREM STATES:

$$h^{p,q} = \dim \{\omega \in \Omega^{p,q} : \Delta \omega = 0\}.$$

[Huybrechts].

$\star \Delta = \Delta \star$: $\underline{h^{p,q} = h^{n-p,\, n-q}}$



symmetry of the Hodge Diamond.

KÄHLER MANIFOLDS

(CY ARE COMPLEX AND KÄHLER ... WE NEED STUDY THIS)

M HERMITIAN MANIFOLD, $g_{ij} = g_{\bar{\jmath}\bar{\jmath}} = 0$.

$$\omega \equiv J \equiv i\, g_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}} \in \Omega^{1,1} \quad (*)$$

MATH LITERATURE (ω)   PHYSICS (J).

IF $d\omega = 0$ M KÄHLER W/ KÄHER FORM ω.

$d\omega$ FROM THE DEF $(*)$:

$$\partial_k g_{i\bar{\jmath}} = \partial_i g_{k\bar{\jmath}}$$

$$\partial_{\bar{k}} g_{i\bar{\jmath}} = \partial_{\bar{\jmath}} g_{i\bar{k}}.$$

$d\omega = 0$ IS LIKE AN INTEGRABILITY CONDITION:

$$\Rightarrow \qquad g_{i\bar{\jmath}} = \frac{\partial^2 K(z,\bar{z})}{\partial z_i\, \partial \bar{z}_{\bar{\jmath}}}$$

$K(z,\bar{z})$ = KÄHLER POTENTIAL (YOU MIGHT ENCOUNTERED THIS IN SUSY COURSE).

PS4: COMPUTE KÄHLER POT. FOR $\mathbb{CP}^n$.

THEOREM (See HUYBRECHTS).

$\Delta = \partial\partial^\dagger + \partial^\dagger\partial$ $\qquad$ $\bar{\Delta} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$

$\rightsquigarrow h^{p,q}$ $\qquad$ $\rightsquigarrow \bar{h}^{p,q}$.

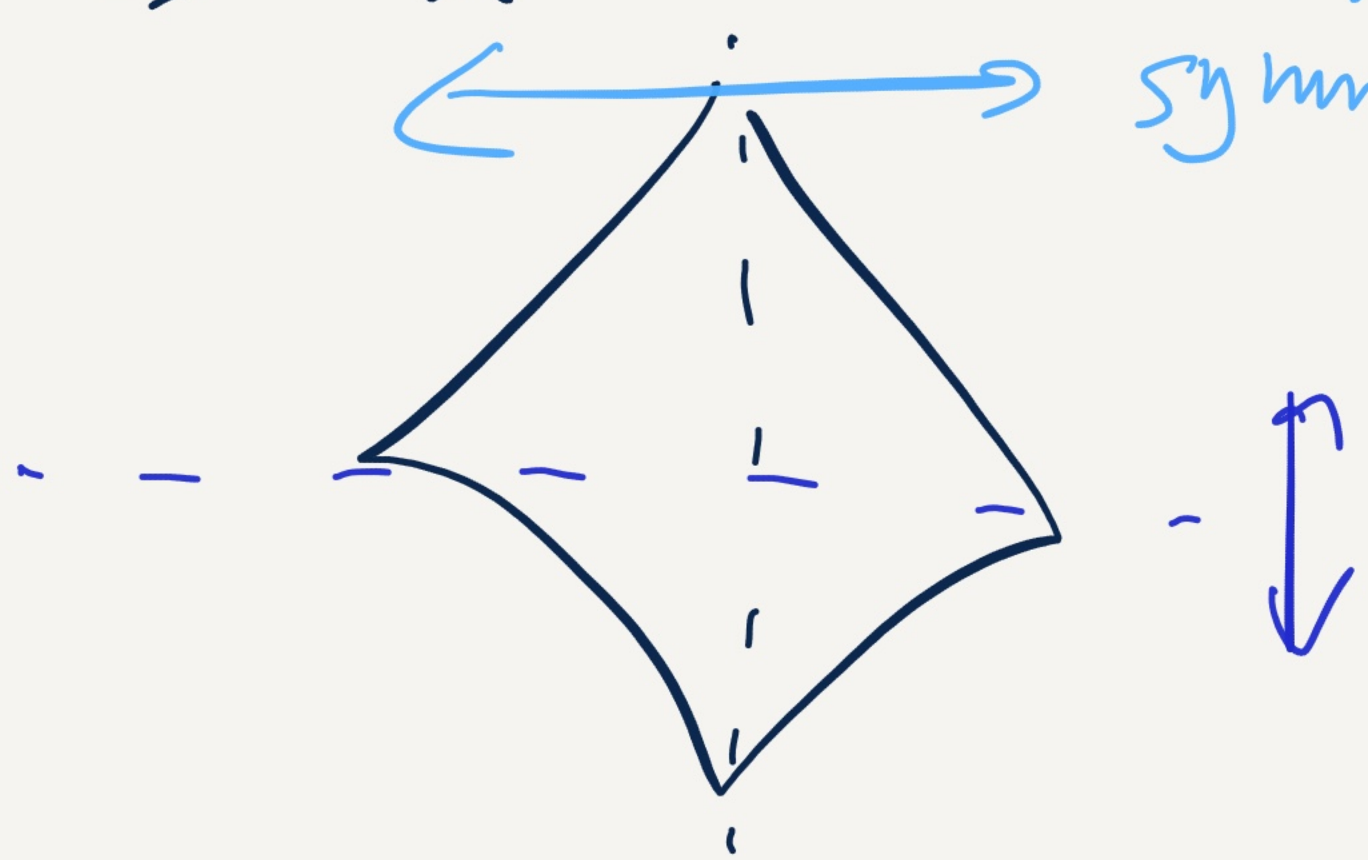
M IS KÄHLER: $\Delta = \bar{\Delta}$

& so $h^{p,q} = \bar{h}^{p,q}$

ALSO USE COMPLEX CONJUGATION: $\bar{h}^{p,q} = h^{q,p}$.

$\Rightarrow h^{p,q} = h^{q,p}$.

symmetry.



symmetry. (*).

CALABI-YAU MANIFOLD: (CY)

M_{2n} n COMPLEX DIM. MANIFOLD, KÄHLER & RICCI FLAT.

RICCI FOR KÄHLER: $R_{\bar{i}j} = -\dfrac{\partial \Gamma^{\bar{k}}_{\bar{i}\bar{k}}}{\partial z_j}$.

CY 3-FOLD "M_6":

HODGE DIAMOND: INDIPENDENT COMPONENTS ARE

$h^{0,0}, h^{3,3}, h^{3,0}, h^{0,3} = 1$

$h^{1,2}, h^{1,1}$ ARE DIFFERENT FOR EACH CY 3-FOLD.

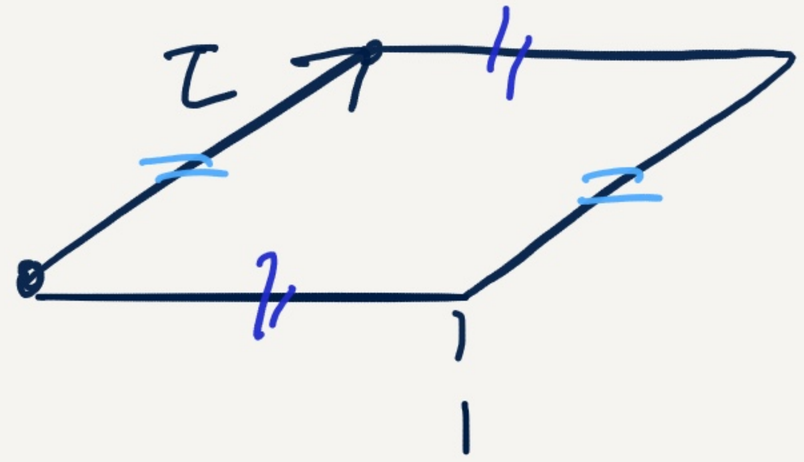
$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

$h^{1,2} \equiv$ COMPLEX STRUCTURE DEFORMATIONS.

$h^{1,1} \equiv$ KÄHLER STRUCTURE DEFORMATIONS.

EX: $T^2 = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$

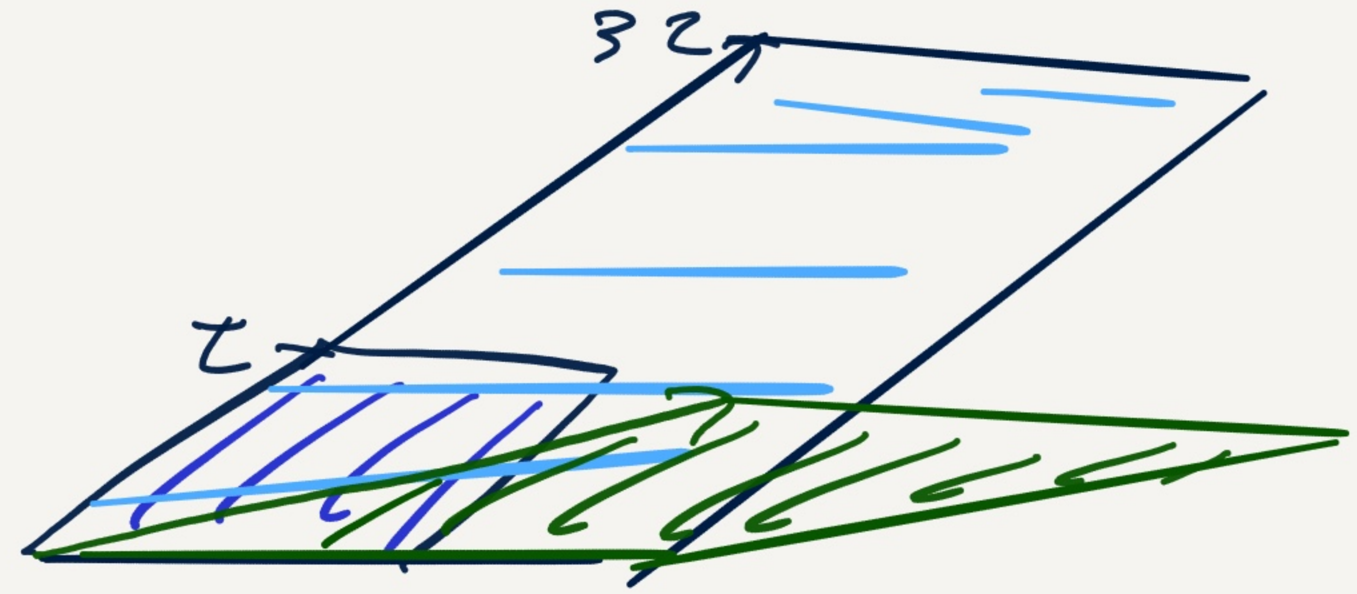
$T^2 = \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$

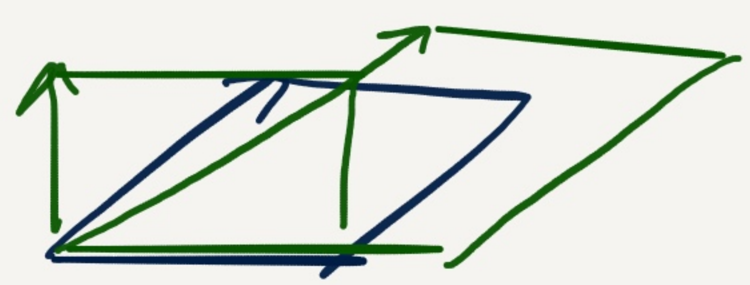


KÄHLER STRUCTURE: CHANGING THE VOLUME BUT KEEPING THE SHAPE THE.

COMPLEX STRUCTURE: τ

SHAPE PARAMETER.





FOR A HIGHER DIM CY: EG. CY 3-FOLD:
MANY COMPLEX & KÄHLER DEFORMATIONS.
[$T^2_{\tau_1} \times T^2_{\tau_2} \times T^2_{\tau_3}$ IS THE SIMPLEST EXAMPLE].