String Theory II: Assignment 1

(1) Super-Virasoro Algebra

Consider the algebra of oscillators for the RNS-string

$$\begin{aligned} [\alpha_n^\mu, \alpha_m^\nu] &= n\eta^{\mu\nu}\delta_{n+m,0}\,, \qquad n,m \in \mathbb{Z} \\ \{b_r^\mu, b_s^\nu\} &= \eta^{\mu\nu}\delta_{r+s,0}\,, \qquad r,s \in \mathbb{Z}+\phi\,, \end{aligned} \tag{1}$$

where $\phi = 0, \frac{1}{2}$ for the R, NS-sector. Define the generators of the Super-Virasoro algebra by the normal ordered expressions

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_{-m}\cdot\alpha_{m+n}: + \sum_{r\in\mathbb{Z}+\phi}(r+n/2):b_{-r}b_{n+r}: \\ G_r &= \sum_{m\in\mathbb{Z}}\alpha_{-m}\cdot b_{r+m}\,, \end{aligned} \tag{2}$$

for $n \in \mathbb{Z}$ and $r \in \mathbb{Z}+\phi$. Determine the algebra that these operators generate, i.e. compute $[L_m, L_n]$, $[L_m, G_r]$ and $\{G_r, G_r\}$.

(2) Supersymmetry of the RNS-String

Consider the superconformal gauge-fixed RNS-string action

$$S = S_B + S_F = \frac{1}{2\pi}\int d^2\sigma\left(\frac{2}{\alpha'}\partial_+X\cdot\partial_-X + i\left(\psi_+\cdot\partial_-\psi_+ + \psi_-\cdot\partial_+\psi_-\right)\right)\,. \tag{3}$$

Show that this action is invariant under the global supersymmetry transformations

$$\begin{aligned} \sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu &= i\bar\epsilon\psi^\mu = i(\epsilon^+\psi_+^\mu + \epsilon^-\psi_{-\mu}) \\ \delta_\epsilon\psi_+^\mu &= -\sqrt{\frac{2}{\alpha'}}\epsilon^+\partial_+X^\mu \\ \delta_\epsilon\psi_-^\mu &= -\sqrt{\frac{2}{\alpha'}}\epsilon^-\partial_-X^\mu\,. \end{aligned} \tag{4}$$

Note: In the notes from lectures 1 and 2 I added a summary page including conventions for spinors in 2d.

(3) Gauge Fixing the RNS-String [Bonus]

The fully covariant, supersymmetric RNS-string is coupled to a world-sheet metric $h^{\alpha\beta}$ and gravitino superpartner χ_α (i.e. what would be called a 2d $N = 1$ supergravity multiplet). Let e_a^α be the zweibein, satisfying

$$e_\alpha^a e_b^\alpha = \delta_b^a \,, \qquad e_a^\alpha e_b^\beta h_{\alpha\beta} = \eta_{ab} \,. \tag{5}$$

The action

$$\begin{aligned} S^{\text{cov}} = S_B^{\text{cov}} + S_F^{\text{cov}} + S^\chi = &-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + i\bar{\psi}^\mu\gamma^\alpha\partial_\alpha\psi_\mu \\ &- \frac{i}{8\pi}\int d^2\sigma\sqrt{-h}\bar{\chi}_\alpha\gamma^\beta\gamma^\alpha\psi^\mu\left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta\psi_\mu\right) \end{aligned} \tag{6}$$

is then invariant under the following supersymmetry transformations

$$\begin{aligned} \sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu \\ \delta_\epsilon\psi^\mu &= \frac{1}{2}\gamma^\alpha\left(\sqrt{\frac{2}{\alpha'}}\partial_\alpha X^\mu - \frac{i}{2}\bar{\chi}_\alpha\psi^\mu\right)\epsilon \\ \delta_\epsilon e_\alpha^a &= \frac{i}{2}\bar{\epsilon}\gamma^a\chi_\alpha \\ \delta_\epsilon\chi_\alpha &= 2\nabla_a\epsilon \,, \end{aligned} \tag{7}$$

Weyl transformations

$$\delta_\Lambda X^\mu = 0 \,, \delta_\Lambda e_\alpha = \Lambda e_\alpha^a \,, \quad \delta\psi^\mu = -\frac{1}{2}\Lambda\psi^\mu \,, \quad \delta_\Lambda\chi_\alpha = \frac{1}{2}\Lambda\chi_\alpha \,, \tag{8}$$

Super-Weyl transformations

$$\delta_\eta\chi_\alpha = \gamma_\alpha\eta \tag{9}$$

with all others vanishing, 2d Lorentz transformations

$$\delta_l X^\mu = 0 \,, \quad \delta_l\psi^\mu = -\frac{1}{2}l\gamma\psi^\mu \,, \quad \delta_l e_\alpha^a = l\varepsilon^a{}_b e_\alpha^b \,, \quad \delta_-\chi_\alpha = -\frac{1}{2}l\gamma\chi_\alpha \,, \tag{10}$$

where $\gamma = \gamma^0\gamma^1$ is the chirality operator, and finally reparametrizations

$$\begin{aligned} \delta_\xi X^\mu &= -\xi^\beta\partial_\beta X^\mu \\ \delta_\xi\psi^\mu &= -\xi^\beta\partial_\beta\psi^\mu \\ \delta_\xi e_\alpha^a &= -\xi^\beta\partial_\beta e_\alpha^a - e_\beta^a\partial_\alpha\xi^\beta \\ \delta_\xi\chi_\alpha &= -\xi^\beta\partial_\beta\chi_\alpha - \chi_\beta\partial_\alpha\xi^\beta \,. \end{aligned} \tag{11}$$

1. Use the bosonic symmetries (two worldsheet reparametrizations ξ, one Lorentz l and one Weyl scaling Λ) to bring the zweibein into the form

$$e_\alpha^a = \delta_\alpha^a\,. \tag{12}$$

   This analysis is very much like in the bosonic string.

2. Use the two supersymmetries and two superconformal symmetries ($\epsilon_\pm$ and $\eta_\pm$) to gauge fix the gravitino to

$$\chi_\alpha = 0 \tag{13}$$

3. Using the equations of motion of e and χ evaluated in the gauged fixing (12) and (13) show that the resulting equations are precisely the Super-Virasoro constraints

$$T_{\pm\pm} = J_\pm = 0\,. \tag{14}$$