String Theory II: Assignment 2

(1) Torus-Partition Function and Modular Invariance: Free Boson

1. Fundamental Domain D of the torus:
   Let $\tau \in \mathbb{C}$ be the modular parameter (modulus) of the torus $T^2_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. Show that the torus T^2_τ and $T^2_{\gamma\tau}$, where $\gamma \in SL_2\mathbb{Z}$, i.e.

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}\,, \qquad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = 1 \tag{1}$$

   describe the same space, i.e. the same identifications in $\mathbb{C}$. Using these $SL_2\mathbb{Z}$ transformations one can restrict τ to the fundamental domain

$$D: \qquad -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2}\,, \qquad |\tau| \geq 1\,. \tag{2}$$

   Sketch D and determine the images of D under

$$\begin{aligned} T: &\qquad \tau \to \tau + 1 \\ S: &\qquad \tau \to -\frac{1}{\tau}\,. \end{aligned} \tag{3}$$

2. For a single free boson $X^\mu(z, \bar{z})$ in d dimensions, the torus partition function is

$$Z(\tau) = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \tag{4}$$

   where $q = e^{2\pi i\tau}$. By performing the integral over momenta k and performing the sum over oscillator modes evaluate $Z(\tau)$ and show that it is modular invariant, i.e. $Z(\tau) = Z(\gamma\tau)$, with $\gamma \in SL_2\mathbb{Z}$. You may use that $\eta(\tau + 1) = e^{i\pi/12}\eta(\tau)$ and $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$. What is the relevance of this result?

(2) Torus-Partition Function: RNS String

The one-loop or torus partition function for the closed RNS string is

$$Z_{T^2} = V_{10} \int_D \frac{d^2\tau}{2\tau_2} \int \frac{d^{10}k}{(2\pi)^{10}}\, \mathrm{Tr}_{\mathcal{H}_k} (-1)^{\mathcal{F}} q^{\alpha'(k^2+M^2)/4} \bar{q}^{\alpha'(k^2+\tilde{M}^2)/4}\,, \tag{5}$$

where $\mathcal{H}_k$ is the physical state (including GSO-projection) space with momentum k ground state, and $q = e^{2\pi i\tau}$, where τ is again the modular parameter (modulus) of the torus $T^2_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ and D the fundamental domain. The spacetime Fermion number operator is $\mathcal{F}$ (not to be confused with the worldsheet one F).

1. Evaluate the NS-sector and R-sector partition functions.
2. Using the results on theta-functions from the lecture, show that this partition function vanishes.

(3) Ghosts!

Let b and c be anticommuting fields, β and γ commuting fields, with action

$$S = \frac{1}{2\pi} \int d^2 z (b\bar{\partial} c + \beta \bar{\partial} \gamma) \,. \tag{6}$$

The OPE algebra is

$$b(z)c(0) \sim \frac{1}{z} \,, \qquad c(z)b(0) \sim \frac{1}{z} \,, \qquad \beta(z)\gamma(0) \sim -\frac{1}{z} \,, \qquad \gamma(z)\beta(0) \sim \frac{1}{z} \,. \tag{7}$$

Define

$$\begin{aligned} T_{\text{ghost}}(z) =& (\partial b)c - \lambda \partial(bc) + (\partial \beta)\gamma - \frac{1}{2}(2\lambda - 1)\partial(\beta\gamma) \\ J_{\text{ghost}}(z) =& -\frac{1}{2}(\partial \beta)c + \frac{2\lambda - 1}{2}\partial(\beta c) - 2b\gamma \end{aligned} \tag{8}$$

1. Compute the conformal weights of b and c, β and γ.
2. Furthermore compute the OPE of TT and determine the central charge.
3. This system of ghosts can be used in the Faddeev-Popov gauge-fixing for $\lambda = 2$. Check that for this value the total central charge of T_{ghost} and T_{RNS} vanishes for $d = 10$.