M Sc in Mathematical and Computational Finance

Quantitative Risk Management: Summary

The topics addressed in the course were:

- Risk in perspective (Ch 1)
- Basic concepts in risk management (Ch 2)
- Market risk (Ch 9)
- Extreme value theory (Ch 5)
- Multivariate models (Ch 6)
- Copulas and dependence (Ch 7)

The problem sheets also considered two examples of financial crises.
The exam questions will consist of a combination of paragraph answer and calculation questions.
An example follows:

Let X be a random variable with a Pareto distribution, that is, with the density

$$f(x) = \begin{cases} \theta x^{-(1+\theta)} & x > 1 \\ 0 & x \leq 1 \end{cases}$$

where $\theta > 0$ is a real parameter. The expected value of X is $E[X] = \theta/(1-\theta)$.

a) Show that the Value at Risk of X at level α (where X is to be treated as losses) is given by

$$\mathrm{V@R}_\alpha(X) = (1-\alpha)^{-1/\theta}.$$

   i. How does this change as $\theta \to 0$?

   Suppose a trader must hold cash deposits with an exchange equal to their value at risk at a fixed level α.

   ii. Explain how small increases in θ can lead to liquidity problems for a company in this setting.

   iii. If θ is estimated based on historical losses, explain how this can lead to a liquidity crisis even when historical losses are relatively small.

b) The expected shortfall is defined by

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{V@R}_\gamma d\gamma$$

   Show that, for the Pareto distribution, this has the equivalent representation

$$\mathrm{ES}_\alpha = E[X | X \geq V@R_\alpha]$$

   and calculate its value, assuming an appropriate value of θ.

c) Suppose θ is estimated, and a confidence interval $[\theta_*, \theta^*]$ is obtained for its value, with $\theta_* > 0$. Calculate the corresponding confidence interval for the Value at Risk at level α. Comment on the width of this interval as $\alpha \to 1$, and how this affects risk estimation.

d) i. Show that there exists a decomposition $X = Y + Z$, for some random variables Y, Z, such that $\mathrm{V@R}_{0.95}(X) > \mathrm{V@R}_{0.95}(Y) + \mathrm{V@R}_{0.95}(Z)$.

   ii. Is it possible to have the same result with $V@R$ replaced by ES?

   iii. Give a brief discussion of the consequences of this for risk management on multiple trading desks.

Solutions:

-a) i. The CDF is given by

$$F(x) = \int_1^x \theta x^{-(1+\theta)} dx = 1 - x^{-\theta}$$

Hence the quantile is $q(y) = (1-y)^{1/\theta}$. Evaluating this at $y\alpha$, we get

$$\mathrm{V@R}_\alpha = (1-\alpha)^{-1/\theta}$$

As $\theta \to 0$, the V@R goes to ∞, for any $\alpha \in (0,1)$.

ii. If θ changes only a little, this can have a big impact on the value at risk when $\alpha \approx 1$. For example, if $\alpha = 0.99$ and $\theta = 1$ then the value at risk is 100, but if $\theta = 2$ then the value at risk is 10. Therefore, if θ decreases by a small amount (and is already small) the value at risk (and hence the cash deposit) can increase substantially.

iii. If θ is estimated by historical losses, then small random variations are to be expected. If these cause the value at risk to rise substantially (as discussed in part ii) then this will lead to a substantial increase in the cash which a trader must maintain in their account. This may cause them to be in significant difficulty, even if the realized losses are relatively moderate.

-b) We can compute (assuming $\theta > 1$)

$$\begin{aligned}
\mathrm{ES}_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 (1-y)^{-1/\theta} dy = \frac{-1}{(1-1/\theta)(1-\alpha)}(1-y)^{1-1/\theta}\Big|_\alpha^1 \\
&= \frac{(1-\alpha)^{1-1/\theta}}{(1-1/\theta)(1-\alpha)} = \frac{(1-\alpha)^{-1/\theta}}{(1-1/\theta)}
\end{aligned}$$

and also

$$\begin{aligned}
E[X|X > (1-\alpha)^{-1/\theta}] &= \frac{1}{1-\alpha}\int_{(1-\alpha)^{-1/\theta}}^\infty x\theta x^{-(1+\theta)} dx = \frac{\theta}{1-\alpha}\int_{(1-\alpha)^{-1/\theta}}^\infty x^{-\theta} dx \\
&= \frac{\theta}{(1-\theta)(1-\alpha)}(x^{1-\theta})\Big|_{(1-\alpha)^{-1/\theta}}^\infty = \frac{-\theta}{(1-\theta)(1-\alpha)}(1-\alpha)^{(-1/\theta)(1-\theta)} \\
&= \frac{(1-\alpha)^{-1/\theta}}{(1-1/\theta)}
\end{aligned}$$

as desired. Hence the two formulae agree.

-c) The interval for the value at risk is $[(1-\alpha)^{-1/\theta^*}, (1-\alpha)^{-1/\theta_*}]$. As $\theta_* > 0$, this is a bounded interval, but grows quickly as $\alpha \to 1$, as our uncertainty has a larger impact the further in the tail we seek to estimate.

-d) i. A construction similar to that considered in lectures will be enough. Let $v = \mathrm{V@R}_{0.95}(X) + \epsilon$ for some small $\epsilon > 0$, and consider the decomposition $Y = XI_{X>v}$, $Z = XI_{X\leq v}$. Then $\mathrm{V@R}_{0.95}(Y) = 0$, but $\mathrm{V@R}_{0.95}(Z) \approx \mathrm{V@R}_{0.9}(X) < \mathrm{V@R}_{0.95}(X)$ for $\epsilon \approx 0$. This gives the desired inequality.

ii. As ES is subadditive, this inequality would not be possible.

iii. This shows that using V@R over multiple desks means that, depending on how it is divided between desks, the same position may result in different total levels of risk. This means that different accounting rules can have a significant bearing on how risk is managed, even when the positions considered are essentially the same.