Exercises: Algorithmic Trading

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March 1, 2020

1 Exercises

1. The agent wishes to liquidate $\mathfrak{N}$ shares between t and T using MOs. The value function is

$$H(t,S,q) = \sup_{\nu \in \mathcal{A}_{t,T}} \mathbb{E}_{t,S,q}\left[\int_t^T (S_u - k\,\nu_u)\,\nu_u\,du + Q_T^{\nu}\,(S_T - \alpha\,Q_T^{\nu})\right],$$

where $k > 0$ is the temporary market impact, ν_t is the speed of trading, $\alpha \geq 0$ is the liquidation penalty, and $dS_t = \sigma\,dW_t$.

(a) Show that the value function H satisfies

$$0 \quad = \quad -\left(\partial_q H - S\right)^2 - 4\,k\,\partial_t H - 2\,k\,\sigma^2\,\partial_{SS} H\,.$$

(b) Make the ansatz

$$H(t,S,q) = h_2(t)\,q^2 + h_1(t)\,q + h_0(t) + q\,S \tag{1}$$

and show that the optimal liquidation rate is

$$\nu_t^* = \frac{Q_t^{\nu^*}}{T - t + \frac{k}{\alpha}}\,. \tag{2}$$

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(c) Let $\alpha \to \infty$ and discuss the intuition of the strategy.

2. This exercise is similar to that above but with a slightly different setup. The agent wishes to liquidate $\mathfrak{N}$ shares and her objective is to maximise expected terminal wealth which is denoted by X_T^ν (in the exercise above we wrote terminal wealth as $\int_t^T (S_u - k\,\nu_u)\,\nu_u\,du$). The value function is

$$H(t,x,S,q) = \sup_\nu \mathbb{E}_{t,x,S,q}\left[X_T^\nu + Q_T^\nu\,(S_T - \alpha\,Q_T^\nu)\right]\,, \tag{3}$$

where

$$dX_t^\nu = (S_t - k\,\nu_t)\;\nu_t\,dt\,. \tag{4}$$

(a) Show that the HJB satisfied by the value function $H(t,S,q,x)$ is

$$0 = \left(\partial_t + \tfrac{1}{2}\,\sigma^2\,\partial_{SS}\right) H + \sup_\nu \left\{(-\nu\,\partial_q + (S - k\,\nu)\;\nu\partial_x)\;H\right\}\,, \tag{5}$$

and the optimal liquidation rate in feedback form is

$$\nu_t^* = \frac{\partial_q H - S\,\partial_x H}{-2\,k\,\partial_x H}\,. \tag{6}$$

(b) To solve (5), use the terminal condition $H(T,x,S,q) = x + q\,S - \alpha\,q^2$ to propose the ansatz

$$H(t,S,x,q) = x + h(t)\,q^2 + q\,S\,, \tag{7}$$

where $h(t)$ is a deterministic function of time. Show that

$$h(t) = -\frac{k}{T - t + \frac{k}{\alpha}}\,, \tag{8}$$

and

$$\nu_t^* = \frac{Q_t^{\nu^*}}{T - t + \frac{k}{\alpha}}\,.$$

3. Let the stock price dynamics satisfy

$$dS_t = \mu\,dt + \sigma\,dW_t\,,$$

where $\sigma > 0$, μ is a constant and W_t is a standard Brownian motion. The agent wishes to liquidate $\mathfrak{N}$ shares and her trades create a temporary adverse move in prices so the price at which she transacts is

$$\hat{S}_t^\nu = S_t - k\,\nu_t\,,$$

with $k > 0$ and the inventory satisfies

$$dQ_t^\nu = -\nu_t \, dt \,,$$

where ν_t is the liquidation rate. Any outstanding inventory at time T is liquidated at the midprice and picks up a penalty of $\alpha \, Q_T^2$ where $\alpha \geq 0$ is a constant. The agent's value function is

$$H(t, S, q) = \sup_\nu \mathbb{E}_{t,S,q} \left[\int_t^T (S_u - k \, \nu_u) \, \nu_u \, du + Q_T^\nu \, (S_T - \alpha Q_T^\nu) \right] . \tag{9}$$

(a) Show that the optimal liquidation rate in feedback form is

$$\nu^* = \frac{\partial_q H - S}{-2k} \, . \tag{10}$$

(b) Use the ansatz $H(t, S, q) = q \, S + h(t, S, q)$ to show that the optimal liquidation rate is given by

$$\nu_t^* = \frac{Q_t^{\nu^*}}{(T-t) + \frac{k}{\alpha}} - \frac{1}{4k} \, \mu \, (T-t) \, \frac{(T-t) + 2\frac{k}{\alpha}}{(T-t) + \frac{k}{\alpha}} \, .$$

Comment on the magnitude of μ and the sign of the liquidation rate.

(c) Let $\alpha \to \infty$ and show that the inventory along the optimal strategy is given by

$$Q_t^{\nu^*} = (T-t) \left(\frac{\mathfrak{N}}{T} + \frac{\mu}{4k} t \right) \, .$$

4. Consider the framework developed in the market making lecture (see slides), where the MM posts only at-the-touch, but assume that when an MO arrives, and the agent is posted on the matching side of the LOB, her order is filled with probability $\rho < 1$. Derive the DPE and compute the optimal strategy in feedback form.

5. The agent wishes to maximise the profit from liquidating $\mathfrak{N}$ shares. The agent uses limit orders and/or market orders to trade in the market. The mid-price process follows

$$\mathrm{d}S_t = \sigma \, \mathrm{d}W_t \,, \tag{11}$$

and ξ is half of the spread (which we assume constant). When the agent sends a MO for a unit quantity, she receives $S_t - \xi$, whereas when she is posted on the ask side with depth δ_t and an incoming order fills the liquidity offered by the agent (which happens with probability $\mathrm{e}^{-\delta_t\,\kappa}$) the agent receives $S_t + \delta_t$. Let $\boldsymbol{\tau} = \{\tau_k : k = 1, 2, \dots\}$ be an increasing sequence of stopping times when the agent executes an MO.

We summarise the components of the optimisation problem we wish to solve:

• $\mathfrak{q} : [0, T] \mapsto \mathbb{R}_+$ is the target inventory or schedule the client has, where T is the terminal time.

• $\delta = (\delta_t)_{\{0 \le t \le T\}}$ is our control variable for the depth to post Limit Orders, i.e. we post an LO of $S_t + \delta_t$ at time t.

• $M = (M_t)_{\{0 \le t \le T\}}$ is the Poisson process with intensity λ modelling the MOs arriving to the market (this does not imply that the arriving MOs are going to fill the agent's LO, that, depends on the posted depth).

• $N^\delta = \left(N^\delta_t\right)_{\{0 \le t \le T\}}$ is the controlled counting process for the number of filled LOs.

• $P(\delta) = e^{-\kappa\,\delta}$ is the probability of an arriving MO to fill the posted LO δ-ticks away from the mid price.

• β denotes the rebate amount the agent receives when her posted LO is filled.

• ξ denotes the half-spread between the best-bid and best-ask price. We assume this is constant throughout the trading day.

• $M^{a,\boldsymbol{\tau}} = (M^{a,\boldsymbol{\tau}}_t)_{\{0 \le t \le T\}}$ is counting process for the MOs sent by the agent. From this definition, it follows that $M^{a,\boldsymbol{\tau}}_t = \sum \mathbb{1}_{\tau_k \le t}$.

• $X^{\boldsymbol{\tau},\delta} = \left(X^{\boldsymbol{\tau},\delta}_t\right)_{\{0 \le t \le T\}}$ is agent's cash process who is now affected by the control process δ and the stopping times $\boldsymbol{\tau}$. This process satisfies the SDE given by

$$\mathrm{d}X^{\boldsymbol{\tau},\delta}_t = (S_t + \beta + \delta_{t^-})\,\mathrm{d}N^\delta_t + (S_t - \xi)\,\mathrm{d}M^{a,\boldsymbol{\tau}}_t$$

where we see that the change in the wealth process is described by the filled LOs and posted MOs. On the right-hand side of the SDE, the first term is the amount by which our cash process changes when an LO is filled, we note that apart from the mid-price, we receive the depth at which we posted and a rebate for providing liquidity to the market. The second term is representing the executed MOs, where we see that instead of receiving more than the mid-price, we have to pay for crossing the spread an amount of ξ.

- $Q^{\boldsymbol{\tau},\delta} = \left(Q_t^{\boldsymbol{\tau},\delta}\right)_{\{0\leq t\leq T\}}$ is the inventory process, given by

$$Q_t^{\boldsymbol{\tau},\delta} = \mathfrak{N} - N_t^{\delta} - M_t^{a,\boldsymbol{\tau}}\,.$$

The agent wishes now find the optimal control δ and stopping times $\boldsymbol{\tau}$ that maximizes her performance criterion,

$$V(t,\,x,\,s,\,q) = \sup_{(\boldsymbol{\tau},\delta)\in\mathcal{A}_{t,T}} \mathbb{E}_{t,x,s,q}\left[X_T^{\boldsymbol{\tau},\delta} + Q_T^{\boldsymbol{\tau},\delta}\,S_T - \ell(Q_T^{\boldsymbol{\tau},\delta}) - \phi\int_t^T \left(Q_u^{\boldsymbol{\tau},\delta} - \mathfrak{q}(u)\right)^2 \mathrm{d}u\right], \tag{12}$$

with $\ell(q) = q\,(\xi + \alpha\,q)$, representing the terminal penalty when liquidating the remaining inventory[1]. Furthermore, $\mathcal{A}$ is now the set for all $\mathcal{F}$-predictable depths and also the $\mathcal{F}$-stopping times bounded by T.

(a) Derive the corresponding DPE for the associated value function, and solve for the optimal trading strategy in feedback form.

(b) Make a suitable ansatz to reduce the system to one of only two-dimensions.

[1]Note that you lose the half-spread for the whole value of q and also a final penalty as dictated by α.