Practical Numerical Analysis: Sheet 6

1. We consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

on the interval $x \in [0,1]$ and $t \in [0, 0.25]$ with initial and boundary conditions given by

$$\begin{aligned} u(x,0) &= \sin\left(\frac{3\pi x}{2}\right) - \frac{3\pi}{2}\cos\left(\frac{3\pi x}{2}\right) , \\ u(0,t) &= -\frac{3\pi}{2}\mathrm{e}^{-(3\pi/2)^2 t} , \\ u(1,t) &= -\mathrm{e}^{-(3\pi/2)^2 t} . \end{aligned}$$

The exact solution to the problem is given by

$$u(x,t) = \mathrm{e}^{-(3\pi/2)^2 t}\left(\sin\left(\frac{3\pi x}{2}\right) - \frac{3\pi}{2}\cos\left(\frac{3\pi x}{2}\right)\right) .$$

Solve the problem using the θ-method and take in turn $\theta = 0$, 0.5, 1. Use $N = 32$, 64, 128 and 256 mesh spacings in the x-direction and choose Δt so that $\mu = \Delta t/\Delta x^2 = 1/2$. By looking at the maximum absolute difference between the exact solution and the numerical solution at time $t = 0.25$, confirm that you achieve the expected convergence rates.

Again solve the problem using the θ-method and take in turn $\theta = 0$, 0.5, 1. Now fix $N = 512$ and use $M = 32$, 64, 128 and 256 equally spaced timesteps in the interval $[0, 0.25]$. Again look at the error defined as the maximum absolute difference between the exact solution and the numerical solution at time $t = 0.25$, and explain your results.

Finally use the Crank Nicolson scheme with $N = 400$ mesh spacings and $M = 400$ timesteps. Record the maximum error at $t = 0.25$ and the time taken for the whole computation. Repeat with the explicit scheme with $N = 400$ mesh spacings and $\mu = 1/2$. Comment on your results.

Further Reading

1. K. W. Morton and D. F. Mayers, *Numerical Solution of Partial Differential Equations*, CUP, 1994 (1st Edition) or 2005 (2nd Edition). Chapter 2 contains an introduction to finite difference methods for 1D parabolic PDEs.