Practical Numerical Analysis: Sheet 7

1. On Sheet 2 we saw that

$$\int_0^1 4\pi x \sin(20\pi x)\cos(2\pi x)\mathrm{d}x \;=\; -\frac{20}{99}\;.$$

   Show that the trapezium rule and the standard Monte Carlo approach give the expected convergence rates. Use stratified sampling with $L = 1, 5, 50$ samples per stratum and comment on the results.

2. Define the function $f(x, y, z)$ by

$$f(x, y, z) \;=\; \begin{cases} 1 & 0 \le x^2 + y^2 + z^2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   so that

$$I(f) \;=\; \int_0^1 \int_0^1 \int_0^1 f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \;=\; \frac{\pi}{6}\;.$$

   First calculate approximations to $I(f)$ using the trapezium rule in 3D and show how your results converge to the exact solution.

   Next use the standard Monte Carlo approach to approximating $I(f)$. How do these results converge to the exact solution? How many sample points should you use if you want to approximate $I(f)$ to within a tolerance of 0.01 with probability 0.999?