Mathematical Methods I MT2019

Problem Sheet 3

Green's function

1. Infinite domains. Obtain the Green's function for each of the following boundary value problems and use it to express the solution for the given data.

   (a) $Ly \equiv y''(x) - \mu^2 y'(x) = f(x)$, $-\infty < x < 0$,

   with boundary conditions:

   $$y \to 0 \text{ as } x \to -\infty, \quad y(0) = 1. \tag{1}$$

   Here μ is a constant.

   (b) $Lu(x) \equiv u''(x) - (1 + x^2)u(x) = f(x), \quad -\infty < x < \infty,$

   with the regularity boundary conditions that the solution vanishes at $\pm\infty$.

   Hint: Show that $u_0 = e^{x^2/2}$ satisfies $Lu_0 = 0$. Then, seek a solution in the form $u = wu_0$. You may use without proof the fact that

   $$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}.$$

2. Green's function for Sturm-Liouville. Consider the Sturm-Liouville operator

   $$Ly \equiv -(py')' + qy, \quad a < x < b$$

   with boundary conditions

   $$B_l(y)|_a \equiv y(a) = 0$$
   $$B_r(y)|_b \equiv y(b) = 0.$$

   In lectures we gave the following formula for the Green's function, derived via variation of parameters:

   $$g(x,t) = \begin{cases} \frac{-y_l(x)y_r(t)}{W(t)p(t)} & a < x < t \\ \\ \frac{-y_l(t)y_r(x)}{W(t)p(t)} & t < x < b \end{cases} \tag{2}$$

   where $Ly_l = 0 = Ly_r$, $B_l(y_l) = 0$, $B_r(y_r) = 0$, and $W = y_l y_r' - y_l' y_r$ is the Wronskian.

   (a) Derive this formula in a different approach, by constructing the Green's function via the formulation $Lg(x,t) = \delta(x - t)$.

   (b) Give an alternative expression for the Green's function, through an eigenfunction expansion, and show that the two formulas agree.

   Hint: Expand the Green's function in Equation (2) in an eigenfunction expansion.