Further Partial Differential Equations
Problem Sheet 1

1. **Flow on a vertical substrate**
   Note: This is non-examinable material but may be of interest to those wanting to know how to derive the governing equations covered in the course.
   In lectures we used the following equation to find solutions for the spreading of liquid on a vertical wall as shown in figure 1:
   $$\frac{\partial h}{\partial t} + \frac{\rho g}{3\mu}\frac{\partial}{\partial z}\left(h^3\right) = 0. \tag{1}$$
   Here, h denotes the liquid thickness, z the vertical position, t time, g acceleration due to gravity and ρ and μ are respectively the density and viscosity of the fluid. All quantities are dimensional. In this question we will derive this equation.

   (a) By starting with the Stokes equations and assuming that the liquid film is thin, employ a lubrication scaling to show that the system is governed by the equations
   $$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2a}$$
   $$\frac{\partial p}{\partial y} = 0, \tag{2b}$$
   $$\frac{\partial p}{\partial z} - \rho g = \mu \frac{\partial^2 w}{\partial y^2}, \tag{2c}$$
   where y and z are the coordinates normal and tangential to the surface and v and w are the respective velocities, p is the fluid pressure and g denotes acceleration due to gravity (see figure 1).

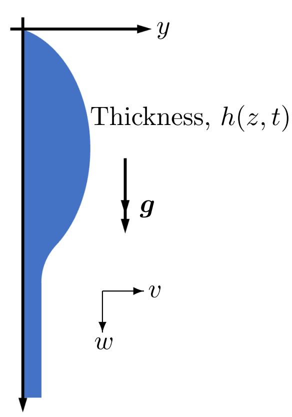

Figure 1: Schematic of liquid draining down a wall. The liquid profile is given by $h(z,t)$ at time t and vertical position z.

(b) Explain the physical significance of each of the following boundary conditions:

$$w = 0 \quad \text{on} \quad y = 0, \tag{3a}$$

$$v = 0 \quad \text{on} \quad y = 0, \tag{3b}$$

$$\frac{\partial h}{\partial t} + w\frac{\partial h}{\partial z} = v \quad \text{on} \quad y = h, \tag{3c}$$

$$p = 0 \quad \text{on} \quad y = h, \tag{3d}$$

$$\frac{\partial w}{\partial y} = 0 \quad \text{on} \quad y = h. \tag{3e}$$

(c) Integrate (2a) over the thickness of the liquid and use (3) to show that the liquid thickness satisfies the following equation for mass conservation:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial z}(\overline{w}h) = 0, \tag{4}$$

where $\overline{w}$ is the average velocity parallel to the wall, defined by

$$\overline{w} = \frac{1}{h}\int_0^h w \, \mathrm{d}y. \tag{5}$$

(d) Use the remaining equations and boundary conditions to show that

$$w = -\frac{\rho g}{2\mu}(y^2 - 2yh) \tag{6}$$

and hence show that (1) governs the flow of liquid on a vertical substrate

2. **An analytical solution for the flow on a vertical substrate**

(a) Use the method of characteristics to show that the solution to the dimensionless equation for the flow on a vertical substrate,

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) = 0. \tag{7}$$

subject to the initial condition $h(z,0) = h_0(z)$, is given by $h(z,t) = h_0(\xi(z,t))$, where $\xi(z,t)$ satisfies the implicit relation

$$h_0(\xi)^2 t + \xi = z. \tag{8}$$

(b) By expanding for small t by setting $t = O(\epsilon$ where $\epsilon \ll 1$ show that, for an initial profile of the form $h_0(z,0) = \tanh(\alpha z)$ for $z > 0$, the early time behaviour is

$$h \sim \tanh(\alpha z - \alpha \tanh^2(\alpha z) t). \tag{9}$$

(c) By expanding for large time by setting $t = O(1/\epsilon)$ where $\epsilon \ll 1$ show that the long time behaviour for large $z = O(1/\epsilon)$ is

$$h \sim \sqrt{\frac{z}{t}} \tag{10}$$

if we assume that $\xi = O(1)$.

(d) Comment on how the result (10) compares with the similarity solution found in lectures for the flow of liquid on a vertical surface and the implications of this result on the use of the similarity solution.

(e) Show that if we also assume that $\xi = O(1/\epsilon)$ in (c) then the solutions are travelling waves of the form $h_0(\alpha(z-t))$.

3. **Similarity solutions in higher dimensions**

Consider the equation for the thickness of liquid on a vertical substrate derived in question 1 with the addition of surface tension smoothing in the transverse direction:

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu}\frac{\partial}{\partial \hat{z}}\left(\hat{h}^3\right) + \frac{\gamma}{3\mu}\frac{\partial}{\partial \hat{x}}\left(\hat{h}^3\frac{\partial^3 \hat{h}}{\partial \hat{x}^3}\right) = 0, \tag{11}$$

where γ is the coefficient of surface tension.

(a) Non-dimensionalize the system using

$$\hat{h} = Hh, \qquad \hat{z} = Lz, \qquad \hat{x} = Lx, \qquad \hat{t} = Tt,$$

to obtain the dimensionless version of (11),

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) + \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial^3 h}{\partial x^3}\right) = 0, \tag{12}$$

for suitably chosen H, L and T, which you should find.

(b) Assume first that the thickness is independent of transverse direction, x. Seek a similarity solution of the form $h = f(\eta)$ where $\eta = z/t^\alpha$ and find the equation that is satisfied by f and the required value of the parameter α.

(c) By solving the differential equation found in (b), show that $f = (z/t)^{1/2}$.

(d) Now assume that the thickness depends on x, z and t. Seek a solution of the form $h = f(\eta)g(\nu)$ where f is the function given in (c) and $\nu = xt^\beta z^\delta$. By substituting this ansatz into (12) show that g satisfies the equation

$$-12g + 12g^3 + 3\nu g' - 9\nu g^2 g' + 24g^2 g' g''' + 8g^3 g'''' = 0$$

where primes denote differentiation, for suitably chosen β and δ that you should find.

4. **Spreading of oil in a frying pan: a radial gravity current**
In lectures we looked at the two-dimensional spreading of a liquid. In this question we will consider radial spreading. The height of the liquid, $\hat{h}$ in terms of the radial coordinate $\hat{r}$ and time $\hat{t}$ is given by the equation

$$\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{\Delta\rho g}{3\mu\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\hat{h}^3\frac{\partial \hat{h}}{\partial \hat{r}}\right) = 0, \tag{13}$$

where $\Delta\rho$ is the difference in density between the liquid and the surrounding air, g denotes acceleration due to gravity and μ is the viscosity of the liquid.

(a) Explain the physical significance of the expression

$$2\pi \int_0^{\hat{r}_f(\hat{t})} \hat{r}\hat{h}(\hat{r},\hat{t})\,\mathrm{d}\hat{r} = \hat{V}, \tag{14}$$

and the quantity $\hat{V}$, where $\hat{r}_f$ is the position of the liquid front.

(b) Non-dimensionalize the system (13) and (14) using suitable scalings.

(c) Use a scaling argument to show that

$$r_f \sim t^{1/8} \qquad\qquad h \sim t^{-1/4}, \tag{15}$$

where the lack of hats denotes dimensionless quantities.

(d) By setting $\eta = r/t^{1/8}$ and $h = t^{-1/4}f(\eta)$ derive an ordinary differential equation for f.

(e) By defining the scaled coordinate $z = \eta/\eta_f$ and $f(z) = \alpha g(\eta_f z)$ for an appropriate choice in α that you should determine, show that g satisfies

$$\left(zg^3g'\right)' + \frac{1}{8}z^2g' + \frac{1}{4}zg = 0, \tag{16}$$

where primes denote differentiation, and the position of the moving front is given by

$$\eta_f = \left(\int_0^1 zg(z)\,\mathrm{d}z\right)^{-3/8}. \tag{17}$$

(f) Consider the behaviour near the propagating front by setting $z = 1 - \epsilon\xi$ and $g = \delta G$ where $\epsilon, \delta \ll 1$. Find an appropriate relationship between ϵ and δ that provides a leading-order balance and use this to show that the behaviour near the front is given by

$$g \sim \left(\frac{3}{8}\right)^{1/3}(1-z)^{1/3}. \tag{18}$$