Further Partial Differential Equations
Problem Sheet 2

1. **Outwardly radial spreading in a porous medium**

   Consider the radial spreading of a fixed volume of liquid in a porous medium. The height $\hat{h}$ of the liquid is governed by the equation

$$\phi \frac{\partial \hat{h}}{\partial \hat{t}} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{h} \hat{Q} \right) = 0, \qquad \hat{Q} = -\frac{\rho g k}{\mu} \frac{\partial \hat{h}}{\partial \hat{r}} \tag{1}$$

   where $\hat{r}$ and $\hat{t}$ denote respectively the radial coordinate and time and $\hat{Q}$ is the flux; ρ denotes the density of the fluid, g acceleration due to gravity, k the permeability, ϕ the porosity and μ the fluid viscosity.

   (a) Write down the equation that expresses conservation of mass.

   (b) By choosing suitable non-dimensionalization show that the system may be reduced to one that contains no physical parameters.

   (c) By finding the appropriate form of the similarity solution, show that the problem can be reduced to solving the following ordinary differential equation system,

$$(\eta f f')' + \frac{1}{4}\eta^2 f' + \frac{1}{2}\eta f = 0, \tag{2}$$

$$\int_0^{\eta_f} \eta f(\eta)\, \mathrm{d}\eta = 1, \tag{3}$$

$$f'(0) = 0, \tag{4}$$

$$f(\eta_f) = 0, \tag{5}$$

   where you should define the functions $\eta = \eta(r,t)$, $\eta_f = \eta_f(r,t)$ and $f = f(h,t)$.

   (d) By rescaling $s = \eta/\eta_f$ and $g = f/\eta_f^2$ find the ordinary differential equation that is satisfied by g and a condition for η_f in terms of g.

   (e) By performing a local analysis show that the conditions at the front are

$$g(1) = 0, \qquad g'(1) = -\frac{1}{4}. \tag{6}$$

   (f) Hence show that the solution is given by $g(s) = (1 - s^2)/8$, $\eta_f \approx 2$.

   (g) Based on the results of this analysis, is this a similarity solution of the first or second kind? What physical feature of the problem indicates that it is a similarity solution of this kind?

2. **Inwardly radial spreading in a porous medium**

Consider again the radial spreading of a fixed volume of liquid in a porous medium as described by equation (1). Suppose that the liquid is now confined in a cylindrical container of radius $\hat{r}_0$ and the liquid occupies a region $\hat{r}_f(\hat{t}) \leq \hat{r} \leq \hat{r}_0$ where $\hat{r}_f$ moves inwardly with time.

(a) Write down the equation that expresses conservation of mass in this case and comment on how it differs from that in question 1.

(b) By using the results of question 1, show that the system may be reduced to one that contains no physical parameters.

(c) Let t_c denote the time at which the central dry hole closes. Define

$$\tau = t_c - t, \qquad h = \frac{r^2}{\tau}\bar{h}(r,\tau), \qquad Q = \frac{r}{\tau}\bar{Q}(r,\tau) \tag{7}$$

and show that in terms of these new variables the system may be written as

$$2\bar{h} + \bar{Q} + r\frac{\partial \bar{h}}{\partial r} = 0, \tag{8}$$

$$\tau\frac{\partial \bar{h}}{\partial \tau} - \bar{h} - 4\bar{h}\bar{Q} - r\frac{\partial}{\partial r}\left(\bar{h}\bar{Q}\right) = 0. \tag{9}$$

(d) Now suppose that $\bar{h} = \bar{h}(\eta)$, $\bar{Q} = \bar{Q}(\eta)$ where $\eta = r/\tau^{\alpha}$ is a similarity variable, for some α. Find the equations that are satisfied by $\bar{h}$ and $\bar{Q}$.

(e) Show that the system can be written in the form

$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}\bar{h}} = \frac{\bar{h} + 4\bar{h}\bar{Q} - \alpha(\bar{Q} + 2\bar{h}) - \bar{Q}(\bar{Q} + 2\bar{h})}{\bar{h}(\bar{Q} + 2\bar{h})}. \tag{10}$$

(f) Based on the results of this analysis, is this solution a similarity solution of the first or second kind? What physical feature of this problem indicates that it is a similarity solution of this kind?

3. **Similarity solutions in the two-phase Stefan problem**

Consider the two-phase Stefan problem (2.15) in the limit $t \to 0$. Show that the leading-order behaviour is given by

$$u(x,t) \sim \begin{cases} f(\eta) & 0 < \eta < \beta, \\ g(\eta) & \beta < \eta < \infty, \end{cases} \qquad s(t) \sim \beta\sqrt{t}, \qquad \eta = \frac{x}{\sqrt{t}},$$

where

$$g(\eta) = \theta \left(\frac{\operatorname{erfc}\left(\eta\sqrt{St}/2\sqrt{\kappa}\right)}{\operatorname{erfc}\left(\beta\sqrt{St}/2\sqrt{\kappa}\right)} - 1 \right), \qquad f(\eta) = \theta \left(1 - \frac{\operatorname{erf}\left(\eta\sqrt{St}/2\right)}{\operatorname{erf}\left(\beta\sqrt{St}/2\right)} \right),$$

and β satisfies the transcendental equation

$$\frac{\beta\sqrt{\pi}}{2\sqrt{St}} = \frac{\mathrm{e}^{-\beta^2 St/4}}{\operatorname{erf}\left(\beta\sqrt{St}/2\right)} - \frac{K\theta\mathrm{e}^{-\beta^2 St/4\kappa}}{\sqrt{\kappa}\operatorname{erfc}\left(\beta\sqrt{St}/2\sqrt{\kappa}\right)}.$$

4. **Asymptotic analysis of Stefan problems**

(a) Show that the transcendental relation (2.12) between β and St may be parameterized as

$$\mathrm{St} = \sqrt{\pi}\xi\mathrm{e}^{\xi^2}\operatorname{erf}(\xi), \qquad \beta = \frac{2\sqrt{\xi}\mathrm{e}^{-\xi^2/2}}{\pi^{1/4}\sqrt{\operatorname{erf}(\xi)}}, \tag{11}$$

where $0 < \xi < \infty$. By taking the limits $\xi \to 0$ and $\xi \to \infty$, derive the asymptotic expressions (2.13).