Further Partial Differential Equations
Problem Sheet 3

NB: Updated on 28/2/2020. Problem 1 has been developed; Problem 3 has now been reduced to help coincide with the lectures. The rest of problem 3 that originally featured on this sheet will now be moved to Sheet 4 so don't worry if you had already attempted part of that question.

1. **A solid–liquid interface with a density change**
Consider the one-dimensional Stefan problem for melting of a solid considered in lectures. The full system behaviour may be described by equations expressing conservation of mass, momentum and total energy, which are given respectively by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho v^2 + p\right) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\rho h + \frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left(\rho v - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right) = 0, \tag{3}$$

where ρ is the density, v the velocity, p the pressure, T the temperature and

$$h = \begin{cases} c(T - T_\mathrm{m}) + L & T > T_\mathrm{m} \\ c(T - T_\mathrm{m}) & T < T_\mathrm{m}. \end{cases}$$

is the *enthalpy* of the system, which is the total energy per unit mass, including heat. Here, c is the specific heat and L the latent heat.

Suppose that liquid occupies a region $0 \le x \le s(t)$ and solid occupies a region $s(t) \le x \le 1$.

(a) Show that when the density of the fluid and the solid are the same then $v = 0$ and the temperature in the liquid and the solid is described by the one-dimensional heat equation

$$\frac{\partial}{\partial t}(\rho c T) - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = 0. \tag{4}$$

(b) Now suppose that the densities in the solid and the liquid phases are different. Integrate (1) over a domain $x_1 < x < x_2$ that contains the interface (so $x_1 < s(t)$ and $x_2 > s(t)$). Divide the integral into $x_1 \le x \le s(t)$ and $s(t) \le x \le x_2$ and take the limit as $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$ to show that the following jump condition is satisfied by the density:

$$[\rho]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = [\rho v]_-^+ . \tag{5}$$

(c) By performing an identical process for (2) and (3) obtain the jump conditions

$$[\rho v]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[\frac{1}{2}\rho v^2 + p\right]_-^+, \tag{6}$$

$$\left[\rho h + \frac{1}{2}\rho v^2\right]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[\rho v - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right]_-^+. \tag{7}$$

(d) Explain how these reduce to the Stefan condition presented in lectures when the solid and liquid densities are equal.

2. **Linear stability of a two-dimensional Stefan problem**
Consider the linear stability of the free boundary problem depicted in Figure 2.2 in the limit $\mathrm{St} \to 0$. Assume that the free boundary is moving at constant speed V under a constant temperature gradient $-\lambda_{1,2}$ in each phase before being perturbed, so the solutions take the form

$$u_1(x, y, t) = -\lambda_1(x - Vt) + \tilde{u}_1(x, y, t), \qquad u_2(x, y, t) = -\lambda_2(x - Vt) + \tilde{u}_2(x, y, t)$$

and the position of the free boundary is given by

$$x = Vt + \xi(y, t).$$

By linearising the problem with respect to $\tilde{u}_1$, $\tilde{u}_2$ and ξ, show that perturbations with wavenumber $k > 0$ and growth rate σ are possible provided

$$\frac{\sigma}{Vk} = -\frac{\lambda_1 + K\lambda_2}{\lambda_1 - K\lambda_2}.$$

3. **One-dimensional welding**

(a) Derive the dimensionless one-dimensional welding problem (2.31).

(b) Show that the normalised heating coefficient is given by

$$q = \frac{a^2 J^2}{\sigma k (T_{\mathrm{m}} - T_0)} = \frac{\sigma V^2}{k (T_{\mathrm{m}} - T_0)},$$

where V is the applied voltage. Assuming that we require $q = O(1)$ to melt the plate, roughly how high must the voltage be to achieve melting?