Factorisation and Discrete-Logarithm Algorithms

Federico Pintore [1]

[1] Mathematical Institute,
University of Oxford

Outline

1. Factorization algorithms
2. Generic discrete logarithm algorithms
3. Discrete logarithms over finite fields

Outline

Integer factorization

Problem: Given a composite number N, which is the product of two n-bit primes, compute one of its factors.

Trial Divison: try every prime number up to $\sqrt{N}$. Running time is, at worst, $O(\sqrt{N} \cdot \text{polylog}(\text{N}))$.

Can we do better?

Pollard's rho

- It can be used to factor any arbitrary integer $N = pq$.
- Idea: find a **good** pair (x, y) s.t. $x = y \pmod{p}$ but $x \neq y \pmod{N}$.
- This implies that $\gcd(x - y, N) = p$, and therefore a non-trivial factor of N is obtained.
- Define some "pseudorandom" iteration function f (a standard choice would be $f(x) = x^2 + 1 \mod N$. It has the property that, if $x = x' \pmod{p}$, then $f(x) = f(x') \pmod{p}$.)
- At step i-th, compute x_i, x_{2i} and $\gcd(x_i - x_{2i}, N)$.
- By birthday's bound, a pair (x_i, x_{2i}) s.t. $x_i = x_{2i} \pmod{p}$ is expected to be found after $O(p^{1/2})$ trials on average.

Pollard's Rho

```
Input: integer N (a product of two n-bit primes)
a := b ← Z*_N
for i ∈ {2, ..., 2^{n/2}} do
    a := f(a)
    b := f(f(b))
    p := gcd(a − b, N)
    if p ∉ {1, N} then
        return p.
    end if
end for
```

Pollard's $p-1$ and Elliptic curve factorization methods

- Pollard's $p-1$ is an effective method if $p-1$ has only "small" prime factors.
- Elliptic-curve factorisation method generalises it when neither $p-1$ nor $q-1$ are smooth.
- The group order $\#E(\mathbb{F}_p)$ of an elliptic curve E can be smooth even when $p-1$ is not!
- Choosing *strong primes* for RSA, i.e. $p-1$ and $q-1$ both have large prime factors, can help against Pollard's $p-1$, but not against Elliptic-curve factorisation method or Number Field Sieve.

Quadratic Sieve Algorithm

- It runs in sub-exponential time in the length of N. Good choice for numbers up to about 300 bits long.
- Try to factor 8051. $8051 = 90^2 - 7^2 = (90-7)(90+7) = 83 \times 97$.
- **Idea**: find a, b s.t. $a^2 = b^2 \pmod N$ but $a \neq \pm b \pmod N$. Hence $\gcd(a-b, N)$ gives one non trivial factor of N.

Quadratic Sieve Algorithm

- Fix some bound $B \in \mathbb{N}$, and let $\mathcal{F} = \{p_1, \ldots, p_k\}$ the set of primes less than or equal to B.
- Search for integers x_i, where $x_1 = \left\lceil \sqrt{N} \right\rceil, x_2 = \left\lceil \sqrt{N} \right\rceil + 1, \ldots$, s.t. $q_i := x_i^2 \pmod N$ is B-smooth, and factor them.
- Find a subset S of $\{q_i\}_i$ such that the product of its elements is a square, i.e.

$$\prod_{j \in S} q_j = \prod_{\ell=1}^{k} p_\ell^{\sum_{j \in S} e_{j,\ell}} \quad s.t. \quad \sum_{j \in S} e_{j,\ell} = 0 \pmod 2 \quad \forall \ell \in \{1, \ldots, k\}$$

- S can be found using linear algebra.

Quadratic Sieve Algorithm

- Define the matrix of exponents (modulo 2) as follows:

$$\begin{pmatrix} e_{1,1} \pmod 2 & e_{1,2} \pmod 2 & \dots & e_{1,k} \pmod 2 \\ \vdots & \vdots & \ddots & \vdots \\ e_{m,1} \pmod 2 & e_{m,2} \pmod 2 & \dots & e_{m,k} \pmod 2 \end{pmatrix}$$

- If $m = k + 1$, then there exists a nonempty subset S of rows that sum to the zero vector modulo 2.

Quadratic Sieve Algorithm

- Take $N = 377753$. We can compute the following:

$$620^2 \bmod N = 17^2 \cdot 23$$
$$621^2 \bmod N = 2^4 \cdot 17 \cdot 29$$
$$645^2 \bmod N = 2^7 \cdot 13 \cdot 23$$
$$655^2 \bmod N = 2^3 \cdot 13 \cdot 17 \cdot 29$$

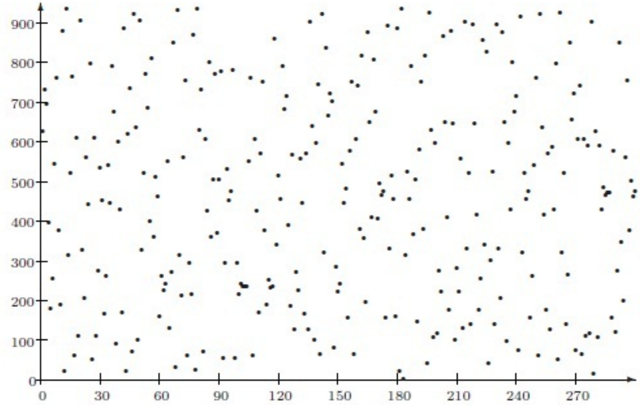
$$(620 \cdot 621 \cdot 645 \cdot 655 \pmod N)^2 = (2^7 \cdot 13 \cdot 17^2 \cdot 23 \cdot 29)^2 \pmod N$$
$$\Rightarrow 127194^2 = 45335^2 \pmod N$$

Since $127194 \neq \pm 45335 \pmod N$,
$\gcd(127194 - 45335, 377753) = 751$ gives a non trivial factor of N.

Outline

Why Discrete Logarithm?

A graph of $f(x) = 627^x \pmod{941}$ for $x = 1, 2, 3, \ldots$



Discrete logarithms

- Trivial if $(G, \circ) = (\mathbb{F}_p, +)$. Why?
- Recently broken if $(G, \circ) = (\mathbb{F}_{2^n}^*, *)$
  (more generally if characteristic is small)
- Believed to be hard for $G = \mathbb{F}_p^*$
  and harder for (well-chosen) elliptic curve groups

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Generic group model

- Algorithms do not exploit any special properties of the encodings of the group elements, other than the fact that each group element is encoded as a unique binary string.
- For instance, the attacker just receives bitstrings instead of $\mathbb{Z}_n$ elements (n itself is often hidden but the size of n cannot be hidden).
- Operations on group elements are performed using an oracle that provides access to the group operations.
- Some attacks are generic: they work for any group.
- This includes exhaustive search, BSGS, Pollard's rho
- There exist much better attacks for finite fields.
- Still no better attack for (well-chosen) elliptic curves.

Exhaustive search

Given $g, h \in G$ do the following:

```
1: k ← 1; h' ← g
2: if h' = h then
3:     return k
4: else
5:     k ← k + 1; h' ← h'g
6:     Go to Step 2
7: end if
```

- Generic algorithm
- Time complexity $|G|$ in the worst case
- Can we do better?

Pohlig-Hellman

- They observed that Dlog in a group $\mathbb{G}$ is as hard as the Dlog in the largest subgroup of prime order in $\mathbb{G}$.
- This applies in any arbitrary finite abelian group.
- Assume $|\mathbb{G}| = N = n_1 n_2$ and let g a generator of G.
- $h = g^k$ implies $h^{n_1} = (g^{n_1})^k$
  where g^{n_1} generates a subgroup of order n_2.
- Assuming that we can solve DLP in that subgroup, this would give us $k \bmod n_2$.
- Repeating the same thing for each factor of N and using CRT would give us k.

Pohlig-Hellman

- Let $\mathbb{G} = \langle g \rangle$ of order $N = \#\mathbb{G} = \prod_{i=1}^{\ell} p_i^{e_i}$
- Given $h = g^x$, we want to first find $x \mod p_i^{e_i}$ and then use CRT to recover it mod N.
- There is a group isomorphism $\phi : \mathbb{G} \to C_{p_1^{e_1}} \times \cdots \times C_{p_\ell^{e_\ell}}$.
- Define the projection map $\phi_{p_i} : \mathbb{G} \to C_{p_i^{e_i}}$ where $\phi_{p_i}(g) = g^{N/p_i^{e_i}}$. ϕ_{p_i} is a group homomorphism, i.e., if $h = g^x$ in $\mathbb{G}$, then $\phi_{p_i}(h) = \phi_{p_i}(g)^x$ in $C_{p_i^{e_i}}$.
- Solving the discrete logarithm in $C_{p_i^{e_i}}$ reduces to solving e_i discrete logarithm in the group C_{p_i} following an inductive procedure.
- Given $h' = g^{x'} \in C_{p_i^{e_i}}$, we write $x' = x_0 + x_1 p_i + \cdots + x_{e_i - 1} p_i^{e_i - 1}$ and then find $x_0, x_1, \ldots, x_{e_i - 1}$ in turn.

Baby step, giant step (BSGS)

- Given a public cyclic group $\mathbb{G} = \langle g \rangle$, now we can assume that $\mathbb{G}$ has a prime order p.
- Given $h \in \mathbb{G}$, find the value of k s.t. $h = g^k$.
- Let $N' = \lceil \sqrt{|\mathbb{G}|} \rceil$
- There exist $0 \leq i, j < N'$ such that $k = jN' + i$

$$h = g^{jN'+i} \Leftrightarrow hg^{-jN'} = g^i$$

- Compute $L_B := \{g^i | i = 0, \ldots, N' - 1\}$
- Compute $L_G := \{hg^{-jN'} | j = 0, \ldots, N' - 1\}$
- Attack requires time and memory each $\mathcal{O}\left(|\mathbb{G}|^{1/2}\right)$
- Can we do better in terms of space requirement and still obtain a time complexity of $\mathcal{O}\left(\sqrt{|\mathbb{G}|}\right)$

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Pollard's Algorithms

- John Pollard, a famous name in factoring/Dlog algorithms in the 20th century.
- Known for $(P-1)$ method, Rho-method, Number Field Sieve.
- The idea in the Rho method is to find a collision in a random mapping.
- Using the birthday paradox *naively* is no better than Baby-Step/Giant-Step method in terms of space/time requirements.
- Similar to the improved birthday paradox attack on hash functions, we can use Floyd's cycle finding algorithm, i.e. given (x_i, x_{2i}), we compute

$$(x_{i+1}, x_{2i+2}) = (f(x_i), f(f(x_{2i})))$$

- We stop when $x_\ell = x_{2\ell}$

Pollard's rho

- Define the sets G_1, G_2, G_3 of about the same size such that $G = G_1 \cup G_2 \cup G_3$ and $G_i \cap G_j = \{\}$, assuming that $1 \notin G_2$.
- Over $\mathbb{Z}_p^*$, one can choose
  $G_1 = \{0, \ldots, \lfloor p/3 \rfloor\}$,
  $G_2 = \{\lfloor p/3 \rfloor + 1, \ldots, \lfloor 2p/3 \rfloor\}$, $G_3 = \{\lfloor 2p/3 \rfloor + 1, \ldots, p - 2\}$
- Define a random walk $f : G \to G$ such that

$$x_{i+1} = f(x_i) = \begin{cases} hx_i & x_i \in G_1 \\ x_i^2 & x_i \in G_2 \\ gx_i & x_i \in G_3 \end{cases}$$

Pollard's rho

- Given $g, h = g^x$, we start from $x_0 := 1$ and apply f recursively to get $\{x_i, x_{2i}\}_i$.
- By the way f is defined, we can keep track of (x_t, a_t, b_t) such that $x_t = g^{a_t} h^{b_t}$, where

$$a_{i+1} = \begin{cases} a_i \\ 2a_i \mod p \\ a_i + 1 \mod p \end{cases}, b_{i+1} = \begin{cases} b_i + 1 \mod p & x_i \in G_1 \\ 2b_i \mod p & x_i \in G_2 \\ b_i & x_i \in G_3 \end{cases}$$

- We stop when a collision is found, i.e. $x_\ell = x_{2\ell}$, therefore $x = \dfrac{a_{2\ell} - a_\ell}{b_\ell - b_{2\ell}} \mod p$.
- If f is "random enough", then we should find the Dlog in expected time $\mathcal{O}\left(\sqrt{|G|}\right)$.

Pollard's rho

$N \leftarrow \lceil \sqrt{|G|} \rceil$
$a_1 = 0; b_1 = 0; x_1 = 1$
$(x_2, a_2, b_2) = f(x_1, a_1, b_1)$
for $k \in \{2, \ldots, N\}$ **do**
  $(x_1, a_1, b_1) = f(x_1, a_1, b_1)$
  $(x_2, a_2, b_2) = f(f(x_2, a_2, b_2))$
  if $x_1 = x_2$ **break;**
end for
if $b_1 = b_2 \mod p$ **then**
  return $\perp$
else
  return$(a_2 - a_1)/(b_1 - b_2) \mod p$
end if

Pollard's Rho: example

Example (Smart's book)

Consider $\mathbb{G} = \langle g \rangle$, a subgroup of $\mathbb{F}_{607}^*$ of order $p = 101$, with $g = 64$. Given $h = 122 = 64^x$. Solve for x.
We split $\mathbb{G}$ into three sets S_1, S_2, S_3 as follows:

$$S_1 = \{x \in \mathbb{F}_{607}^* : x \leq 201\}$$

$$S_2 = \{x \in \mathbb{F}_{607}^* : 202 \leq x \leq 403\}$$

$$S_3 = \{x \in \mathbb{F}_{607}^* : 404 \leq x \leq 606\}$$

Pollard's Rho: example

Example

A collision is found when $i = 14$, this implies that $g^0 h^{12} = g^{64} h^6$, so $[12x = 64 + 6x \mod 101]$ and therefore $x = 78$.

More from Pollard

- Pollard's Lambda Method: similar to the Rho method in that it uses deterministic random walk, but it is particularly designed to the cases where we know that the Dlog lies in a particular interval.
- Parallel Pollard's Rho: designed to be able to use computing resources of different sites across the internet.

Outline

L notation

$$L_Q(\alpha; c) = \exp(c(\log Q)^{\alpha}(\log\log Q)^{1-\alpha})$$

- Q is the size of the field
- $\alpha = 0 \Rightarrow L_Q(\alpha; c) = (\log Q)^c$ polynomial
- $\alpha = 1 \Rightarrow L_Q(\alpha; c) = Q^c$ exponential

(simplified) Index Calculus for $\mathbb{F}_p^*$

- DLP: given $g, h \in \mathbb{F}_p^*$, find x such that $h = g^x$
- Factor basis made of **small primes**

$$\mathcal{F}_B := \{\text{primes } p_i \leq B\} = \{p_1, \ldots, p_k\}$$

- **Relation search**
  - Compute $g_i := g^{a_i}$ for random $a_i \in \{1, \ldots, p-1\}$
  - **If** all factors of g_i are $\leq B$, we have a relation

$$g^{a_i} = \prod_{p_j \in \mathcal{F}} p_j^{e_{i,j}} \qquad (1)$$

- **Linear algebra** Once we have $\ell \geq k$ linearly independent equations similar to equations (1), we solve $\bmod (p-1)$ for $\log_g p_i$, $i = 1, \ldots, k$.
- Search for t such that $[g^t \cdot h \mod p]$ is B-smooth. Once found, solve for $\log_g h \mod (p-1)$.

Size of the factor basis

- By the prime number theorem,

$$|\{\text{primes } p_i \leq B\}| \approx \frac{B}{\log B}$$

- Fact: 30% of all numbers have no prime factors above their square root. Surprisingly, a large proportion of numbers can be built out of so few primes!

Complexity Analysis

- How to choose an optimal B: If B is large, then it is more likely that the generated elements are B-smooth, but then testing that they are B-smooth is more difficult now. Therefore, we need to balance the cost!
- In order to choose an optimal B, we also need to know the probability that a random integer that is smaller than N is B-smooth.
- We will assume that the cost of generating relations dominates the overall complexity of Algorithm, i.e. assume that the linear algebra is negligible in terms of time complexity.
- We will simply use the trial-division to factor over $\mathcal{F}_B$.

Complexity Analysis

- A number is B-smooth if all its prime factors are smaller than B.
- Define $\Psi(N, B) = \#\{B\text{-smooth numbers} \leq N\}$.
- The probability that a positive integer $m \leq N$ is B-smooth is approximately equal to $\frac{1}{N} \cdot \Psi(N, B)$.
- The Canfield-Erdos-Pomerance Theorem: Let $u = \frac{\log N}{\log B}$, we have $\frac{1}{N} \cdot \Psi(N, B) = u^{-u+o(u)}$. This is the *Dickman-de Bruijn* function ρ, i.e. $\rho(u) \approx u^{-u}$.
- The expected number of random trials of choosing numbers in $[1; N]$ to find one that is B-smooth is $\approx u^u$

Complexity Analysis

- Let $|\mathcal{F}_B| = k$, the expected running time of the algorithm is

$$\approx \underbrace{(k+1)}_{\text{nb of relations}} \cdot \underbrace{u^u}_{\text{expected nb of trials}} \cdot \underbrace{k}_{\text{nb of trial divisions}} \cdot \underbrace{M(\log N)}_{\text{time for a trial division}} \quad (2)$$

$$\approx B^2 \cdot u^u \quad \textit{drop the logarithmic factors, where } k \approx \frac{B}{\log B} \quad (3)$$

$$= N^{2/u} \cdot u^u \quad (4)$$

- We want to minimize $f(u) = N^{2/u} \cdot u^u$. If we set $f'(u) = 0$, we need a u s.t. $u^2 \log u \approx 2 \log N$.
- Let $u = 2\sqrt{\frac{\log N}{\log \log N}}$, we then get $u^2 \log u = 2 \log N + o(\log N)$

Complexity Analysis

- Back to our bound B:

$$
\begin{aligned}
B &= N^{1/u} \\
&= exp(\frac{1}{u}\log N) \\
&= exp(\frac{1}{2}\sqrt{\log N \log\log N)} \\
&= L_N(1/2, 1/2)
\end{aligned}
$$

- Note that $u^u = L_N(1/2, 1)$, therefore $B^2 u^u = L_N(1/2, 2)$.
- The cost of the linear algebra step is bounded by $\tilde{O}(B^3)$, i.e. $L_N(1/2, 3/2)$.

Further Reading (1)

- Andrew Granville.
  Smooth numbers: computational number theory and beyond.
  Algorithmic number theory: lattices, number fields, curves and cryptography, 44:267–323, 2008.

- Antoine Joux, Andrew Odlyzko, and Cécile Pierrot.
  The past, evolving present, and future of the discrete logarithm.
  In *Open Problems in Mathematics and Computational Science*, pages 5–36. Springer, 2014.

- Hendrik W Lenstra Jr.
  Factoring integers with elliptic curves.
  Annals of mathematics, pages 649–673, 1987.

Further Reading (2)

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A tale of two sieves.
Biscuits of Number Theory, 85, 2008.

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Lower bounds for discrete logarithms and related problems.
In *Advances in Cryptology—EUROCRYPT'97*, pages 256–266. Springer, 1997.

Further Reading (3)

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