Private-Key Encryption

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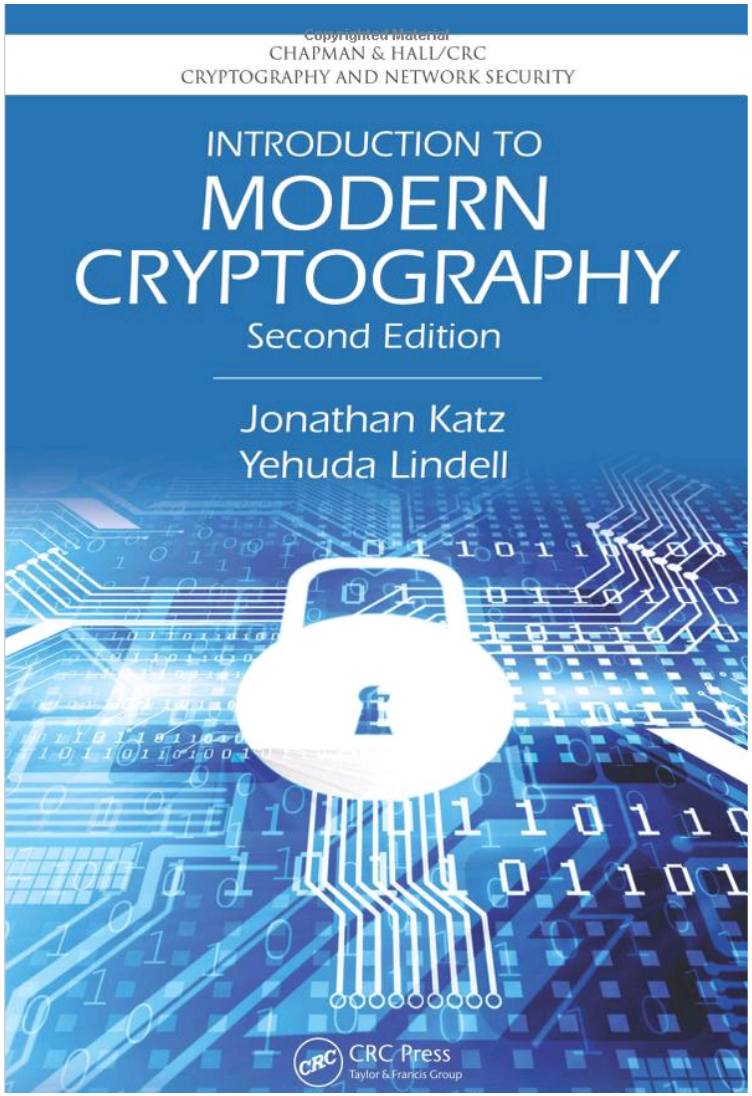
Outline

Course Main Reference



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The receiver decrypts the ciphertext, i.e. unhides it, using the same secret key. They recover the original message.

Syntax of Private-Key Encryption Schemes

A private-key encryption scheme consists of three algorithms $(\text{KeyGen}, \text{Enc}, \text{Dec})$:

- $k \leftarrow \text{KeyGen}(n)$: a randomised algorithm that takes the security parameter n and outputs the key k, chosen from the key space $\mathcal{K}$.
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Caesar Cipher (100-44BC)

Example

- Plaintext: ABCD $\cdots$ WXYZ.
- Shift: $+3 \pmod{26}$ *or* $+k \pmod{26}$ $(k \in \{0, \ldots, 25\})$
- Ciphertext: DEFG $\cdots$ ZABC.

Cryptanalysis:

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- Is it a sufficient condition?

Substitution Cipher (mono-alphabetic)

Example

- Plaintext: ABCZ
- Substitution: (A$\rightarrow$ T, B$\rightarrow$ N, C$\rightarrow$ L, . . . , Z$\rightarrow$ O) *or* any one-to-one map of the alphabet
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  - Frequency of pairs (or more) of letters, e.g. digrams, trigrams, etc.

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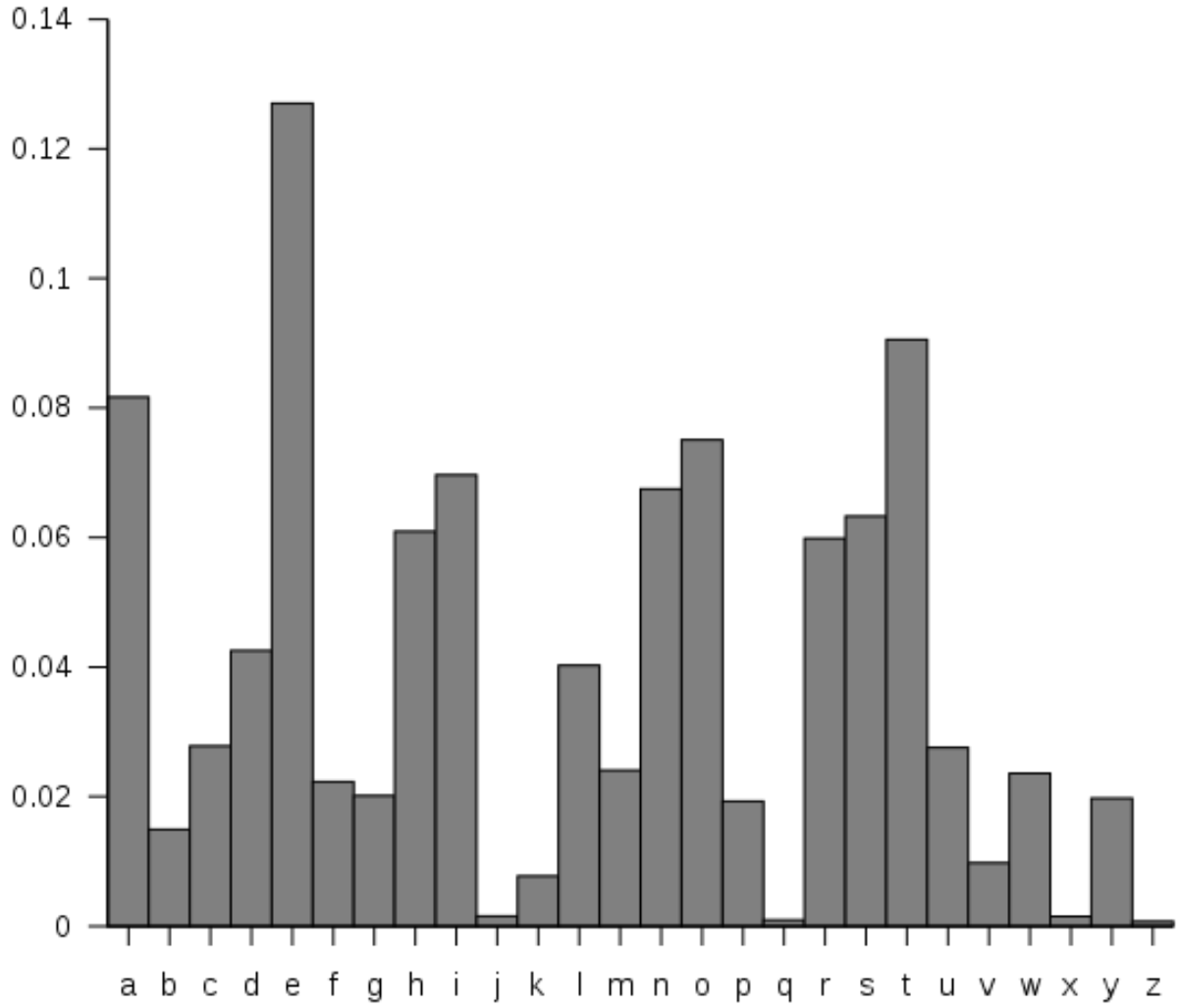

Vigenere Cipher (1553)

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- Poly-alphabetic shift:

Plaintext m:	TOBEORNOTTOBE
key k:(+ mod 26)	CRYPTOCRYPTOC
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  - If n is not known, use Kasiski method (Kasiski 1863) or *index of coincidence method* to find n, and do the rest as in the first case. (What if $n = |c| = |k|?$)

Kerckhoff's Principle (1883):

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Modern Cryptography:

- The encryption scheme's algorithms should be public. (Standardized, etc.)

Unofficial principle

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It is composed by:

- a security guarantee
- a threat model

Security Definitions

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- Chosen-ciphertext attack (CCA): now, they additionally get the decryption of ciphertexts of its choice.

Outline

Discrete Probability

Let Ω be a (finite) set of outcomes, known as sample space.

The event space $\mathcal{A}$ is a subset of $\mathcal{P}(\Omega)$ s.t.

- $\Omega \in \mathcal{A}$
- if $A \in \mathcal{A}$, then $\bar{A} \in \mathcal{A}$ ($\bar{A}$ denotes the complementary in Ω)
- if $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$

Usually, $\mathcal{A} = \mathcal{P}(\Omega)$ when Ω is finite.

Discrete Probability

A probability distribution $\Pr$ is a map from $\mathcal{A}$ to $[0,1]$ s.t.

- $\Pr[\Omega] = 1$
- if $A_1, \ldots, A_t$ are pairwise disjoint events, then

  $$\Pr[\cup_{i=1}^{t} A_i] = \sum_{i=1}^{t} \Pr[A_i]$$

Assuming Ω finite we have:

- Let $A \subseteq \Omega$, $\Pr[A] = \sum_{\omega \in A} \Pr[w]$.
- Union Formula: $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$.
- Union Bound: $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$.

Discrete Probability

- Conditional Probability: $\Pr[A|B] = \Pr[A \cap B]/\Pr[B]$ (if $\Pr[B] > 0$).
- Bayes' Theorem: $\Pr[A|B] = \dfrac{\Pr[A] \cdot \Pr[B|A]}{\Pr[B]}$ (if $\Pr[A], \Pr[B] > 0$)
- A and B are independent $\Leftrightarrow \Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.

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Notation

From

- $(\mathcal{K}, \mathcal{P}(\mathcal{K}), \Pr_{\mathcal{K}})$
- $(\mathcal{M}, \mathcal{P}(\mathcal{M}), \Pr_{\mathcal{M}})$

a probability distribution $\Pr_{\mathcal{K}\times\mathcal{M}}$ over the sample space $\mathcal{K} \times \mathcal{M}$ is deduced. It is defined as:

$$\Pr_{\mathcal{K}\times\mathcal{M}}(A) = \sum_{(k,m)\in A} \Pr_{\mathcal{K}}[k] \Pr_{\mathcal{M}}[m]$$

- $\Pr[K = k] = \Pr_{\mathcal{K}\times\mathcal{M}}[\{k\} \times \mathcal{M}] = \Pr_{\mathcal{K}}[k]$
- $\Pr[M = m] = \Pr_{\mathcal{K}\times\mathcal{M}}[\mathcal{K} \times \{m\}] = \Pr_{\mathcal{M}}[m]$
- $\Pr[C = c] = \Pr_{\mathcal{K}\times\mathcal{M}}[A]$ with $A = \{(k, m) \mid \mathsf{Enc}(k, m) = c\}$

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Definition (Perfect Secrecy)

For every probability distribution $\Pr_{\mathcal{M}}$ *over the message space* $\mathcal{M}$*, we have*

$$\Pr[M = m | C = c] = \Pr[M = m]$$

$\forall m \in \mathcal{M}, \forall c \in \mathcal{C} \;\; s.t. \;\; \Pr[C = c] > 0.$

Equivalently,

$$\Pr[C = c | M = m] = \Pr[C = c]$$

Perfect Indistinguishability

Perfect Indistinguishability Experiment $\text{PrivK}^{\text{perfect}-\text{ind}}_{\mathcal{A},E}$

Challenger Ch **Adversary** A

$\xleftarrow{m_0, m_1, |m_0| = |m_1|}$

$b \leftarrow\!\!\$ \{0,1\}$

$\xrightarrow{c = \text{Enc}(k, m_b)}$ Outputs his guess b'

Definition

An encryption scheme is perfectly indistinguishable if for every adversary $\mathcal{A}$ the following holds:

$$\Pr[\text{PrivK}^{\text{perfect}-\text{IND}}_{\mathcal{A},E} = 1] = 1/2$$

Where $\text{PrivK}^{\text{perfect}-\text{IND}}_{\mathcal{A},E} = 1$ if $b' = b$, and 0 otherwise.

Perfect Indistinguishability

Theorem (Perfect indistinguishability)

*An encryption scheme (*KeyGen, Enc, Dec*) has perfect secrecy iff for every probability distribution* $\Pr_{\mathcal{M}}$ *over* $\mathcal{M}$ *we have*

$$\Pr[C = c | M = m_0] = \Pr[C = c | M = m_1]$$

$\forall m_0, m_1 \in \mathcal{M}\ s.t.\ |m_0| = |m_1|, \forall c \in \mathcal{C}.$

Proof.

$(\Rightarrow) : \Pr[C = c|M = m_0] = \Pr[C = c] = \Pr[C = c|M = m_1]$

$(\Leftarrow) :$

$$\begin{aligned}
\Pr[C = c] &= \sum_m \Pr[C = c|M = m] \cdot \Pr[M = m] \\
&= \sum_m \Pr[C = c|M = m_0] \cdot \Pr[M = m] \\
&= \Pr[C = c|M = m_0] \cdot \sum_m \Pr[M = m] \\
&= \Pr[C = c|M = m_0]
\end{aligned}$$

which is correct for any m_0 □

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One Time Pad (Vernam 1917 or some 35 years earlier!)

Fix an integer $n > 0$. Let $\mathcal{M} = \mathcal{C} = \mathcal{K} = \{0,1\}^n$.

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It was used between the White House and the Kremlin during the Cold War!

Security of OTP

Theorem

The one time pad (OTP) encryption scheme is perfectly secret.

Proof.

$$\begin{aligned}\Pr[C = c | M = m] &= \Pr[M \oplus K = c | M = m] \\ &= \Pr[m \oplus K = c] \\ &= \Pr[K = m \oplus c] \\ &= \frac{1}{2^n}\end{aligned}$$

because the key k is a uniform n-bit string. Therefore, for any m_0, m_1, we have $\Pr[C = c | M = m_0] = \frac{1}{2^n} = \Pr[C = c | M = m_1]$

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If an encryption scheme E is perfectly secret, then $|\mathcal{K}| \geq |\mathcal{M}|$.

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Proof.

Assume that $|\mathcal{K}| < |\mathcal{M}|$, we will show that E is not perfectly secure. We first fix a uniform distribution $\Pr_{\mathcal{M}}$ over $\mathcal{M}$, and let

$$\mathcal{M}(c) = \{m \mid m = \mathsf{Dec}(k, c) \text{ for some } k \in \mathcal{K}\}$$

but $|\mathcal{M}(c)| \leq |\mathcal{K}|$, then there exists $m' \in \mathcal{M}$ s.t. $m' \notin \mathcal{M}(c)$. Therefore, $\Pr[M = m' | C = c] = 0 \neq \Pr[M = m']$ □

Is there a way to make OTP practical?

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- Perfect secrecy: No leakage of information about an encrypted message even to an eavesdropper with *unlimited computational power*.
- Computational secrecy: an encryption scheme is still considered to be secure even if it leaks some information with a very small probability to eavesdroppers with *limited power*.
- Real-world application: happy with a scheme that leaks information with probability at most 2^{-60} over 200 years using fastest supercomputers!

Further Reading (1)


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