TOPOLOGICAL GROUPS, 2019–2020

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We begin with the course overview as described on `https://courses.maths.ox.ac.uk/node/46583`.

Course Overview: Groups like the integers, the torus, and GL_n share a number of properties naturally captured by the notion of a topological group. Providing a unified framework for these groups and properties was an important achievement of 20th century mathematics, and in this course we shall develop this framework.

Highlights will include the existence of Haar measure for (not necessarily Abelian) locally compact Hausdorff topological groups, Pontryagin duality, and the structure theorem for locally compact Hausdorff Abelian topological groups. Throughout, the course will use the tools of analysis to tie together the topology and algebra, getting at superficially more algebraic facts such as the structure theorem through analytic means.

References. There are some references in particular which may be of use: [Rud90], [Fol95], and [Kör08].

Teaching. The lectures and these notes will appear online as they are produced during the term. They will be supplemented by some tutorial-style teaching where we can discuss the course and also exercises scattered through the notes. Once I have a list of the MFoCS students attending I shall be in touch to arrange these.

Contact details and feedback. The current circumstances mean this course is appearing in a different way to normal. In particular, there will inevitably be less audience response so I encourage you to get in touch at tom.sanders@maths.ox.ac.uk if you have any questions or feedback.

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1. Introduction and recap

In this course we are interested in the interaction between group structure and a compatible topological structure. The results we use on groups are fairly basic and are covered in Prelims (see, for example, the notes [Ear14]). The topology we use is largely covered in the Part A: Topology course (notes are available at [DL18]).

We shall often describe topologies through a base. Given a set X, a **base** B is a collection of subsets of X such that B is a cover of X; and if $U, V \in B$ and $x \in U \cap V$ then there is some $W \in B$ with $x \in W \subset U \cap V$. Given a base B we write[1]

$$\tau(B) := \left\{\bigcup \mathcal{S} : \mathcal{S} \subset B\right\}$$

Exercise 1.1. Show that $\tau(B)$ is a topology on X.

In view of this exercise we call $\tau(B)$ the **topology generated by** B. Note that any topology is a base and it generates itself, but in general there may be multiple bases generating a given topology.

The **discrete topology** on a set X is $\mathcal{P}(X)$ – the topology in which every set is open – and has the singletons are an example of a base for this topology. If X has more than one element then the base of singletons is different to the base of all subsets of X.

Given two topological spaces (X, τ) and (X', τ') the **product topology** is the topology on $X \times X'$ generated by the base[2] $\{U \times V : U \in \tau, V \in \tau'\}$ on $X \times X'$.

With these basic topological definitions recalled we turn to the object of the course: suppose that G is both a group and a topological space. If the group multiplication map $G^2 \to G; (x, y) \mapsto xy$ (from G^2 with the product topology to G) and the group inversion map $G \to G; x \mapsto x^{-1}$ are both continuous then we say that G is a **topological group**.

Example 1.2 (Discrete groups)**.** Suppose that G is a group and also a discrete topological space. Then G is a topological group.

Proof. Since G is discrete so is G^2, and then since any map with a discrete domain is continuous we see that both multiplication and inversion are continuous as required. □

The reals under addition may be endowed with the discrete topology to make them into a topological group as in the above example. However, there are a number of other topologies on $\mathbb{R}$.

Example 1.3 (The real line)**.** The group $\mathbb{R}$ endowed with its usual topology is a topological group.

Proof. This example is quite instructive. The result is essentially just the algebra of limits (see, *e.g.* [Pri16, Theorem 8.3]): in particular if $x_n \to x_0$ then $-(x_n) = (-1)x_n \to (-1)x_0 = -x_0$; and if additionally $y_n \to y_0$, then $x_n + y_n \to x_0 + y_0$.

To connect this with the topological language we are using recall that the usual topology on $\mathbb{R}$ is generated by the base $\{(a, b) : a, b \in \mathbb{R}\}$ (where $(a, b) = \{x \in \mathbb{R} : a < x < b\}$) –

[1] Recall that $\bigcup \mathcal{S}$ is defined by $x \in \bigcup \mathcal{S}$ if $\exists S \in \mathcal{S}$ such that $x \in S$. In particular $\bigcup \varnothing = \varnothing$.

[2] We should check that this really is a base, but that is an easy exercise.

we could take this as our definition – and with this definition topological convergence and ϵ-N convergence coincide (almost tautologically).

From the definition of the product topology we also have that the (ordered pairs) $(x_n, y_n) \to (x_0, y_0)$ if and only if $x_n \to x_0$ and $y_n \to y_0$. Thus the results in the first paragraph exactly tell us that inversion and multiplication are sequentially continuous – a function f is **sequentially continuous** if $f(x_n) \to f(x_0)$ whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence with $x_n \to x_0$.

In general topologies continuity is stronger than sequential continuity (see *e.g.* [DL18, Proposition 1.24]) however in the reals, and more generally in any first countable topological space, they are equivalent. A topological space X is said to be **first countable** if every $x \in X$ has a **countable local base**, meaning a countable sequence $(U_n)_{n\in\mathbb{N}}$ of neighbourhoods of x such that for any neighbourhood U of x there is some $n = n(U)$ such that $U_n \subset U$. In particular, $x \in \mathbb{R}$ has $\{(x-1/n, x+1/n) : n \in \mathbb{N}\}$ as a countable base for the neighbourhood $(x-1, x+1)$ of x so the usual topology is first countable. □

Exercise 1.4. Suppose that X is a first countable topological space. Show that f is a continuous function on X if (and only if) it is sequentially continuous. (Assuming the Axiom of Countable Choice, meaning that given a sequence $S_1, S_2, \ldots$ of sets then there is another sequence $x_1, x_2, \ldots$ with $x_n \in S_n$ for all $n \in \mathbb{N}$.)

We cannot relax the requirement that group inversion or multiplication are continuous. In the case of the group $\mathbb{R}$ group 'inversion' is negation *i.e.* the map $\mathbb{R} \to \mathbb{R}; x \mapsto -x$, and group 'multiplication' is addition *i.e.* the map $\mathbb{R}^2 \to \mathbb{R}; (x, y) \mapsto x + y$.

Example 1.5. Write τ for the usual topology on $\mathbb{R}$, and let τ_1 be the topology on $\mathbb{R}$ equal to the set of $U \in \tau$ such that there is some $a \in \mathbb{R}$ such that $U \supset (a, \infty)$. Then

(i) inversion is *not* continuous;
(ii) addition is continuous.

In particular $\mathbb{R}$ with the topology τ_1 is *not* a topological group.

Proof. For the first part $(0, \infty)$ is open but $-(0, \infty) = (-\infty, 0)$ is not open and so inversion is not continuous.

For continuity of addition first note that if $U \in \tau_1$ then there is some $a \in \mathbb{R}$ such that $(a, \infty) \subset U$. Since $\mathbb{R}$ is a topological group in τ and open intervals form a base for τ we see that there are sets $\mathcal{I}$ and $\mathcal{J}$ of open intervals such that

$$\begin{aligned}\{(x, y) : x + y \in U\} &= \bigcup\{I \times J : I \in \mathcal{I}, J \in \mathcal{J}\} \\ &= \bigcup_{I\in\mathcal{I}, J\in\mathcal{J}} (I \cup (a - \min J \cup \{a/2\}, \infty)) \times (J \cup (a - \min I \cup \{a/2\}, \infty)),\end{aligned}$$

which is a union of products of sets in τ_1 as required. To see the equality it is enough to check that if $x \in I \cup (a - \min J \cup \{a/2\}, \infty)$ and $y \in J \cup (a - \min I \cup \{a/2\}, \infty)$ then $x + y \in \{i + j : i \in I, j \in J\} \cup (a, \infty)$. This in turn follows by considering the cases:

(i) $x \in I$, $y \in J$ then $x + y \in \{i + j : i \in I, j \in J\}$;
(ii) $x \in I$, $y \in (a - \min I \cup \{a/2\}, \infty)$ then $x + y > a - \min I + x \geqslant a$ and so $x + y \in (a, \infty)$;

(iii) $x \in (a - \min J \cup \{a/2\}, \infty)$, $y \in J$ then $x+y > x+a-\min J \geqslant a$ and so $x+y \in (a,\infty)$;
(iv) $x \in (a - \min J \cup \{a/2\}, \infty)$, $y \in (a - \min I \cup \{a/2\}, \infty)$ then $x + y > (a - a/2) + (a - a/2) = a$ and so $x + y \in (a, \infty)$.

The result is proved. □

Example 1.6. Write τ for the usual topology on $\mathbb{R}$, and let τ_2 be the topology on $\mathbb{R}$ equal to the set of $U \in \tau$ such that there is some $a \in \mathbb{R}$ such that $U \supset (-\infty, -a) \cup (a, \infty)$. Then

(i) inversion is continuous;
(ii) addition is *not* continuous.

In particular $\mathbb{R}$ with the topology τ_2 is *not* a topological group.

Proof. The first part is immediate since $U \in \tau_2$ if and only if $-U \in \tau_2$.

If V is open (and non-empty) in the product then there is some $a > 0$ such that $V \supset ((-\infty, -a) \cup (a, \infty)) \times ((-\infty, -a) \cup (a, \infty))$ and hence V contains the ordered pair $(2a, -2a)$. However $2a + (-2a) = 0$ and so the preimage of $(-\infty, -1) \cup (1, \infty)$ is not open. □

Example 1.7 (Normed spaces)**.** Suppose that X is a normed space. Then the additive group of X with the topology induced by the norm is a topological group.

Proof. The topology induced by the norm is the weakest[3] topology such that $x \mapsto \|x\|$ is continuous. For each $x \in X$, $(\{y \in X : \|x - y\| < 1/n\})_{n\in\mathbb{N}}$ is a countable local base so X is first countable, and the product of two first countable spaces is first countable. Hence by Exercise 1.4 it is enough to note from homogeneity that if $x_n \to x$ then $-x_n \to -x_0$; and from the triangle inequality that if $x_n \to x_0$ and $y_n \to y_0$ then $x_n + y_n \to x_0 + y_0$. □

In particular $\mathbb{R}^n$ and $\mathbb{C}^n$ are topological groups under addition.

Given a normed space X we write $\mathrm{GL}(X)$ for the set of linear homeomorphisms $X \to X$. Then $\mathrm{GL}(X)$ is a group under composition and it supports a number of natural topologies which it inherits from the larger set $B(X)$, of continuous linear maps $X \to X$; we shall mention two:

Example 1.8 ($\mathrm{GL}(X)$ with the operator norm topology)**.** $\mathrm{GL}(X)$ may be endowed with the subspace topology inherited from $B(X)$ with the operator norm topology. With this topology $\mathrm{GL}(X)$ is a topological group.

Proof. If $S_n \to S_0$ and $T_n \to T_0$ then $\|T_n\| \leqslant 2\|T_0\|$ for all sufficiently large n and hence

$$\|S_nT_n - S_0T_0\| \leqslant \|S_n - S_0\|\|T_n\| + \|S_0\|\|T_n - T_0\| \to 0$$

since the operator norm is sub-multiplicative; hence $S_nT_n \to S_0T_0$. $B(X)$ is a normed space so as in Example 1.7 the topology is first countable, whence so is the topology on $\mathrm{GL}(X)$ and on $\mathrm{GL}(X) \times \mathrm{GL}(X)$. Hence by Exercise 1.4 multiplication is continuous.

Similarly, for inversion suppose that $T_n \to T_0$. Then

$$\|T_n^{-1} - T_0^{-1}\| = \|T_n^{-1}(T_0 - T_n)T_0^{-1}\| \leqslant \|T_n - T_0\|\|T_n^{-1}\|\|T_0^{-1}\|.$$

[3]Recall that weakest here means fewest open sets.

There is $N \in \mathbb{N}$ such that for $n \geqslant N$ we have $\|T_n - T_0\| \leqslant 1/2\|T_0^{-1}\|$ which can be inserted in the above and rearranged to give $\|T_n^{-1}\| \leqslant 2\|T_0^{-1}\|$. Hence for $n \geqslant N$ we have

$$\|T_n^{-1} - T_0^{-1}\| \leqslant \|T_n - T_0\| 2\|T_0^{-1}\|^2,$$

and $T_n^{-1} \to T_0^{-1}$. Again inversion is continuous by Exercise 1.4. □

In particular $\mathbb{C}^*$ (which we may identify with $\mathrm{GL}(\mathbb{C})$) is a topological group under multiplication.

The second topology we shall consider is the **strong operator topology** on $B(X)$: this is the weakest topology on $B(X)$ such that the maps $T \mapsto Tx$ are continuous for all $x \in X$, and is sometimes called the **topology of point-wise convergence**. The sets

$$U(S; T_0, \epsilon) := \{T \in B(X) : \|Tx - T_0x\| < \epsilon\|x\| \text{ for } x \in S\}$$

for $T_0 \in B(X)$, $S \subset X$ *finite*, and $\epsilon > 0$ are all open in the strong operator topology. They also form a base and so the topology this base generates is contained in the strong operator topology. However, it also includes the sets $U(\{x\}; 0, \epsilon)$ (where 0 here is the 0 operator in $B(X)$) and so all the maps $T \mapsto Tx$ (for $x \in X$) are continuous and hence it is exactly the strong operator topology.

If $\dim X < \infty$ then the strong operator topology is the same topology as the operator norm topology, but if $\dim X = \infty$ then it is not. We shall consider the example $X = \ell_1$ in what follows, the space of (complex) sequences indexed by the naturals with the norm $\|x\| = \sum_i |x_i|$.

Example 1.9 ($\mathrm{GL}(\ell_1)$ with the strong operator topology)**.** $\mathrm{GL}(\ell_1)$ may be endowed with the strong operator topology inherited from $B(\ell_1)$. Then $\mathrm{GL}(\ell_1)$ is not a topological group

Proof. For $\eta > 0$ and $n \in \mathbb{N}$ define the linear map $T_{\eta,n}$ on the standard basis $(e_i)_{i\in\mathbb{N}}$ of ℓ_1 by letting

$$T_{\eta,n}e_i := e_i + \eta e_{n+i} \text{ and } T_{\eta,n}e_{n+i} := -\eta^{-1}T_{\eta,n}e_i,$$

for $1 \leqslant i \leqslant n$, and $T_{\eta,n}e_i = 0$ for all $i > n$. Then the image of $T_{\eta,n}$ is finite dimensional and so it is a bounded linear operator and an element of $B(\ell_1)$, and by design $T_{\eta,n}^2 = 0$ so $(I + T_{\eta,n})^{-1} = I - T_{\eta,n}$. Note that the existence of these operators for arbitrarily large n is where we use that ℓ_1 is infinite dimensional.

Now, suppose $x \in \ell_1$ is non-zero, $\epsilon > 0$, $\eta \in (0, \epsilon/4]$ and $\delta := \min\{\eta^2, \eta\}$. Then there is some $n_0 = n_0(\eta, x)$ such that $\sum_{i>n_0} |x_i| < \delta\|x\|$ where $x = \sum_i x_ie_i$. Hence for $n \geqslant n_0$ we have

$$\begin{aligned}\|x - T_{\eta,n}x\| &= \left\|\sum_{i=1}^{\infty} x_ie_i - \sum_{i=1}^{n} x_ie_i - \sum_{i=1}^{n} \eta x_ie_{n+i} + \sum_{i=1}^{n} \eta^{-1}x_{n+i}e_i + \sum_{i=1}^{n} x_{n+i}e_{n+i}\right\| \\ &= \sum_{i=1}^{n} |\eta^{-1}x_{n+i}| + \sum_{i=1}^{n} |2x_{n+i} - \eta x_i| + \sum_{i>2n} |x_i| \leqslant \eta\|x\| + (2 + \eta^{-1})\delta\|x\| < \epsilon\|x\|.\end{aligned}$$

Thus $\|T_{\eta,n}x - x\| < \epsilon\|x\|$ for all $n \geqslant n_0$. But then if $S \subset \ell_1$ is a set of (non-zero elements) it follows that for $n_1 = \max\{n_0(\eta, x) : x \in S\}$ we have $T_{\eta,n} \in U(S; I, \epsilon)$ for all $n \geqslant n_1$.

Now, let x be a non-zero element of ℓ_1 and consider the open set $U := U(\{x\};(1/2)I,1/4)$. Then $(1/2)I \in U$, and so $2I \in U^{-1}$ and if U^{-1} is open then it contains a non-empty set $U(S;2I,\epsilon)$ for $S \subset \ell_1$ finite. Since $U(S;2I,\epsilon)$ is non-empty, S contains only non-zero elements, and so the preceding paragraph gives for all $\eta \in (0,\min\{\epsilon/4,1/4\}]$ some $n_1 = n_1(\eta, S \cup \{x\})$ such that $I + T_{\eta,n} \in U(S;2I,\epsilon) \subset U^{-1}$ and $T_{\eta,n} \in U(\{x\};I,1/4)$ for all $n \geqslant n_1$. Finally, $(I+T_{\eta,n})^{-1} = I - T_{\eta,n} \in U$ by construction and so

$$\|(I-T_{\eta,n})x - x/2\| < \|x\|/4 \text{ and } \|T_{\eta,n}x - x\| < \|x\|/4$$

The triangle inequality gives a contradiction. □

It can be shown similarly that multiplication is not continuous.

⚠The same argument works to show that $\mathrm{GL}(\ell_2)$ with the strong operator topology is not a topological group. However, here it is more natural to consider the subgroup $U(\ell_2)$ of unitary maps $\ell_2 \to \ell_2$. This group *is* a topological group in the strong operator topology.

⚠If G is a group endowed with a topology then we say that multiplication is **separately continuous** if the maps $x \mapsto xz$ and $x \mapsto zx$ are continuous for all $z \in G$. If more clarity is needed when referring to the continuity of multiplication we shall say that multiplication is **jointly continuous** to mean it is continuous as a map $(x,y) \mapsto xy$.

2. Basics of the topology

Suppose that G is a group written multiplicatively. We shall write 1 or 1_G for its identity. For $S,T \subset G$ and $x \in G$ we write

$$ST := \{st : s \in S, t \in T\}, xS := \{xs : s \in S\} \text{ and } Sx := \{sx : s \in S\}.$$

We also write powers in a natural way: specifically for $n \in \mathbb{N}$

$$S^0 := \{1_G\} \text{ and } S^{n+1} := S^n S.$$

⚠Note that $SS^{-1} \neq S^0$ and $S^2 \neq \{s^2 : s \in S\}$ (in general).

We call a set S with $S = S^{-1}$ **symmetric**.

When a group is written additively we write 0 or 0_G for the group identity. Additively written groups will always be commutative, and we shall write $S+T$ instead of ST *etc.* above.

This notation interacts well with the topology of a topological group.

Lemma 2.1. *Suppose that G is a topological group. Then U is open (resp. closed) if and only if xU is open (resp. closed), and similarly for Ux. In particular, if U is open and V is any set then UV is open.*

Proof. This is just the separate continuity of multiplication, in particular that the maps $G \to G; u \mapsto x^{-1}u$ and $G \to G; u \mapsto xu$ are continuous. □

The next lemma is more important making use of *joint* continuity of multiplication.

Lemma 2.2. *Suppose that G is a topological group and U is a neighbourhood of 1_G. Then there is an open symmetric neighbourhood V of 1_G such that $V^2 \subset U$.*

Proof. We may suppose that U is open by shrinking it if necessary. The map $(x, y) \mapsto xy^{-1}$ is continuous and so $\{(x, y) : xy^{-1} \in U\}$ is an open subset of $G \times G$, and hence there are sets $\mathcal{S}$ and $\mathcal{T}$ of open subsets of G such that

$$\{(x, y) : xy^{-1} \in U\} = \bigcup \{S \times T : S \in \mathcal{S}, T \in \mathcal{T}\}.$$

Since $1_G 1_G^{-1} = 1_G \in U$, there is some $S \in \mathcal{S}$ and $T \in \mathcal{T}$ such that $(1_G, 1_G) \in S \times T$. Thus S and T are open neighbourhoods of 1_G. It follows that $S \cap T$ is an open neighbourhood of 1_G, and since $1_G^{-1} = 1_G$ the set $V := (S \cap T) \cap (S \cap T)^{-1}$ is an open neighbourhood of the identity. Moreover, $V^{-1} = V$ and $V^2 = VV^{-1} \subset ST^{-1} \subset U$ as required. □

This lemma can be applied repeatedly as follows.

Corollary 2.3. *Suppose that G is a topological group and U is a neighbourhood of 1_G. Then there are open symmetric neighbourhoods $(V_n)_{n \in \mathbb{N}_0}$ of 1_G such that $V_{n+1}^2 \subset V_n$ for all $n \in \mathbb{N}_0$, and $V_0 \subset U$.*

Proof. Apply Lemma 2.2 iteratively (using the Axiom of Dependent Choice), beginning with the set U to get a set U_0, and then to the set U_0 to get U_1 *etc.* □

⚠ The neighbourhoods $(V_n)_{n \in \mathbb{N}}$ are not necessarily a local base for the identity in the topology. (Indeed, not all topological groups are first countable.)

3. Separation axioms

The taxonomy of separation in topological spaces has a somewhat involved history with a range of different naming convention (see *e.g.* [nLa20]) so some caution is advised when consulting references. In topological groups we shall see that much of the hierarchy collapses because in some sense ‘every point looks the same’.

A topological space X is **Kolmogorov** if for any distinct $x, y \in X$, either there is an open set containing x and not y, or there is an open set containing y and not x. If we can replace the ‘or’ by an ‘and’ then the space is said to be **Fréchet**. Equivalently a space is Fréchet if every singleton in X is closed [DL18, Proposition 1.47].

Lemma 3.1. *Suppose that G is a topological group. Then G is Kolmogorov if and only if G is Fréchet.*

Proof. Suppose $x, y \in G$ are distinct, and U is an open set containing x and not y. Since inversion is continuous and multiplication is separately continuous, the set $y(x^{-1}U \cap U^{-1}x)$ is open and contains y but not x. □

We only used separate continuity of multiplication in the above, but joint continuity through Lemma 2.2 can be used to show more collapse in the separation hierarchy.

Similarly, a topological space X is said to be **Hausdorff** if for any $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

Proposition 3.2. *Suppose that G is a topological group. Then G is Hausdorff if and only if $\{1_G\}$ is closed (equivalently if and only if G is Fréchet).*

Proof. First, if G is Hausdorff then for each $x \neq 1_G$ there is an open set U_x containing x and not containing 1_G. Hence $G\backslash\{1_G\} = \bigcup_{x\in G} U_x$ is open as required.

Conversely, suppose that $x, y \in G$ are distinct. Then $G\backslash\{x^{-1}y\}$ is open and so by Lemma 2.2 there is an open neighbourhood of the identity V such that $VV^{-1} \subset G\backslash\{x^{-1}y\}$. It follows that $xV \cap yV = \emptyset$, but of course these are both open sets and since $1_G \in V$ we have $x \in xV$ and $y \in yV$. The claim is proved. □

Note the topology τ_1 on $\mathbb{R}$ in Example 1.5 is Fréchet since if $x, y \in \mathbb{R}$ are distinct then $\mathbb{R}\backslash\{x\}$ is open in τ_1, contains y and does not contain x. On the other hand $(\mathbb{R}, \tau_1)$ is *not* Hausdorff since any two non-empty open sets in τ_1 have a non-empty intersection. Since $\{1_G\}$ is closed in τ_1 the preceding lemma gives another proof that $(\mathbb{R}, \tau_1)$ is *not* a topological group.

A topological space X is said to be **regular** if for every closed set $S \subset X$ and any $x_0 \in X\backslash S$ there are disjoint open sets U and V with $S \subset U$ and $x_0 \in V$. ⚠Note that we do *not* require that X be Hausdorff.

Exercise 3.3. Suppose that G is a topological group. Show that G is regular.

We say that a topological space X is **completely regular** if for every closed set $S \subset X$ and any $x_0 \in X\backslash S$ there is a continuous function $f : X \to \mathbb{R}$ with $f(x_0) = 0$ and $f(x) = 1$ for all $x \in S$.

The next result is important because it starts to give us a supply of non-constant continuous functions on any topological group. This means that we can study the group through a function space with all of the attendant tools.

Theorem 3.4. *Suppose that G is a topological group. Then G is completely regular.*

Proof. Suppose that X is a closed set in G and $x_0 \in G\backslash X$; without loss of generality we may assume $x_0 = 1_G$. Apply Corollary 2.3 to $G\backslash X$ to get a sequence $(U_n)_{n\in\mathbb{N}_0}$ of symmetric open neighbourhoods of the identity with $U_{n+1}^2 \subset U_n$ and $U_0 \subset G\backslash X$.

The idea is to use the sets U_n to define a sort of metric between the set X and the element $x_0 = 1_G$. We use the sets U_n to do this, and think of them as playing the role of an interval of length 2^{-n} in the reals, so that if we were proving this result in the case $G = \mathbb{R}$ we could use those intervals to produce the usual notion of distance.

Since G need not be commutative we have to take a bit of care with the order in which we multiply the sets U_n. By induction (note the first inequality is weak, and all the others strict)

$$U_{n_1} \cdots U_{n_k} \subset U_{n_k - 1} \text{ whenever } n_1 \geqslant n_2 > \cdots > n_k > 0. \tag{3.1}$$

Given $n_1 > \cdots > n_k > 0$ and $\epsilon \in (0, 1]$, there is some j and $n_{j+1} < n_0 \leqslant \lfloor \log_2 \epsilon^{-1} \rfloor$ such that

$$U_{n_1} \cdots U_{n_k} \subset U_{n_0} U_{n_{j+1}} \cdots U_{n_k} \text{ and } 2^{-n_0} \leqslant 2\epsilon + 2^{-n_1} + \cdots + 2^{-n_j}. \tag{3.2}$$

Put $n_0^* := \lfloor \log_2 \epsilon^{-1} \rfloor$ and let i be maximal such that $U_{n_1} \cdots U_{n_i} \subset U_{n_0^*}$. Then by (3.1) we have $n_{i+1} \leqslant n_0^*$. If $n_{i+1} < n_0*$ then set $n_0 := n_0^*$ and $j := i$ and we are done; if not let j be maximal such that $n_{i+1},\dots,n_j$ are consecutive (counting down) and set $n_0 := n_j - 1$ and we are done since $2^{-n_0} = 2^{-n_j} + \cdots + 2^{-n_{i+1}} + 2^{-n_0^*}$.

With these preliminaries we can define our function. Put

$$S(x) := \left\{2^{-n_1} + \cdots + 2^{-n_k} : x \in U_{n_1}\cdots U_{n_k} \text{ where } k \in \mathbb{N}, n_1 > \cdots > n_k > 0\right\}$$

and let $f(x) := \inf S(x)$ when $S(x) \neq \emptyset$ and $f(x) = 1$ otherwise. f takes values in the range $[0,1]$ and $f(1_G) = 0$ since $1_G \in U_n$ for all $n \in \mathbb{N}$. We have $f(x) = 1$ if and only if $S(x) = \emptyset$, which in turn is true if and only if $x \notin \bigcup_k U_kU_{k-1}\cdots U_1$. Hence $f^{-1}(1)$ is closed and it contains X since $\bigcup_k U_kU_{k-1}\cdots U_1 \subset U_0 \subset G\setminus X$ by (3.1).

We have to show that f is continuous, and to do this it will be enough to show that for all $x \in G$ and $\epsilon > 0$ sufficiently small the preimage of $(f(x)-\epsilon, f(x)+\epsilon)$ contains a neighbourhood of x.

Let n_0 be large enough that $2^{-n_0} < \epsilon$. If $f(x) < 1$ then there is some $x \in U_{n_1}\cdots U_{n_k}$ with $n_1 > \cdots > n_k > 0$ and $2^{-n_1} + \cdots + 2^{-n_k} < f(x) + \epsilon$, and we may additionally assume that $n_0 > n_1$.

Suppose that $y \in U_{n_0}x$. If $f(x) < 1$ we see that $f(y) < f(x) + 2\epsilon$ by (3.1); the equality holds trivially if $f(x) = 1$.

On the other hand if $f(y) \leqslant f(x) - 3\epsilon$ then there are naturals $n_1' > \cdots > n_{k'}' > 0$ such that $y \in U_{n_1'}\cdots U_{n_{k'}'}$ and $2^{-n_1'} + \cdots + 2^{-n_{k'}'} < f(y) + \epsilon$. By (3.2) there is some $n_{j+1}' < n_0' \leqslant \lfloor \log_2 \epsilon^{-1}\rfloor$ such that

$$U_{n_1'}\cdots U_{n_{k'}'} \subset U_{n_0'}U_{n_{j+1}'}\cdots U_{n_{k'}'} \text{ and } 2^{-n_0'} \leqslant 2\epsilon + 2^{-n_1'} + \cdots + 2^{-n_j'}.$$

But $n_0 > n_0' > n_{j+1}'$ and since U_{n_0} is symmetric we have

$$x \in U_{n_0}U_{n_0'}U_{n_{j+1}'}\cdots U_{n_{k'}'} \text{ and } 2^{-n_0} + 2^{-n_0'} + 2^{-n_{j+1}'} + \cdots + 2^{-n_{k'}'} \leqslant f(y) + 3\epsilon.$$

We conclude that $f(x) < f(y) + 3\epsilon \leqslant f(x)$, a contradiction. Hence $f(y) > f(x) - 3\epsilon$, and the result is proved. □

Note that if G is Kolmogorov then this result gives (a long proof) that G is Hausdorff.

There is a final separation axiom we mention: we say that a topological space X is **normal** if for any two disjoint closed sets $S, T \subset X$ there are disjoint open sets U and V containing S and T respectively. We shall see later (in Example 5.3) that not all topological groups are normal, but also that there is a natural condition which makes them normal.

4. Subgroups, homomorphisms, and quotient groups

When considering subgroups of a topological group we should like them to interact with the topological structure. We begin with the following slightly surprising result.

Lemma 4.1. *Suppose that G is a topological group and $H \leqslant G$. Then H is a topological group when endowed with the subspace topology. Moreover, if H is a neighbourhood it is open; if H is open then it is closed; and if H is closed and of finite index then it is open.*

Proof. Suppose U is an open set in H, and let W be an open subset of G such that $U = W \cap H$. Then $U^{-1} = W^{-1} \cap H$ which is open since inversion is continuous. Then the set $V = \{(x,y) : xy \in W\}$ is open and so a union of products of the form $S \times T$ with S and T open in G. But then $V \cap (H \times H) = \{(x,y) \in H \times H : xy \in U\}$ and $(S \times T) \cap (H \times H) = (S \cap H) \times (T \cap H)$ so that the preimage of U under multiplication on H is open.

For the first part let U be a non-empty open set in H – this is exactly what it means to say that H is a neighbourhood. Then $H = HU = \bigcup_{x \in H} xU$ is a union of open sets and so open.

Since the left cosets of H partition G we have $H = G \setminus \bigcup((G/H)\setminus\{H\})$. If H is open then any left coset of H is open and so $\bigcup((G/H)\setminus\{H\})$ is a union of open sets and so open, whence H is closed. If H is closed then any left coset of H is closed and $\bigcup((G/H)\setminus\{H\})$ is a *finite* union of closed sets, and so closed and hence H is open. □

Lemma 4.2. *Suppose that G is a topological group. Then the connected component of the identity is a closed*[4] *normal subgroup of G.*

Proof. Let L be the[5] connected component of the identity. Then if $\overline{L} = A \sqcup B$ with A and B both closed in $\overline{L}$. Then $L = (L \cap A) \sqcup (L \cap B)$ and so without loss of generality $L = L \cap A$ and so $L \subset A$, but then A is closed and contains L so $A \supset \overline{L}$. We conclude that $\overline{L}$ is connected and so by maximality of L we have $\overline{L} = L$.

Since $1_G = 1_G^{-1}$ we have that $L \cap L^{-1}$ is a closed set containing the identity and hence $L \subset L \cap L^{-1}$ so that if $x \in L$ then $x^{-1} \in L$. Thus for $x \in L$ the set Lx is closed and contains the identity. Hence $L \subset Lx$, so $Lx^{-1} \subset L$ and L is a subgroup by the subgroup test (since it contains 1_G so is non-empty). Similarly, for $x \in G$ the set xLx^{-1} is closed and contains the identity. Hence $L \subset xLx^{-1}$, and so $x^{-1}Lx \subset L$ as required for normality. The result is proved. □

The closure operation also preserves some of the algebraic structure.

Lemma 4.3. *Suppose that G is a topological group and $H \leqslant G$. Then $\overline{H}$, the topological closure of H, is a subgroup of G. If H is normal then so is $\overline{H}$.*

Proof. Suppose that $(x,y) \in G^2$ is such that $xy^{-1} \notin \overline{H}$. Then since $(x,y) \mapsto xy^{-1}$ is continuous, there are open sets $S, T \subset G$ such that $x \in S$, $y \in T$ and $ST^{-1} \cap \overline{H} = \varnothing$. Since $\overline{H} \supset H$, and H is a subgroup, if $S \cap H \neq \varnothing$ then $T \cap H = \varnothing$, and hence $\overline{H} \subset G \setminus T$ so that $T \cap \overline{H} = \varnothing$. On the other hand, if $S \cap H = \varnothing$ then $S \cap \overline{H} = \varnothing$. It follows that $x \notin \overline{H}$ or $y \notin \overline{H}$ and so $\overline{H}$ is a group.

Conjugation is continuous and hence $a^{-1}\overline{H}a$ is closed for all $a \in G$, and also contains H if H is normal. Hence $\bigcap_{a \in G} a^{-1}\overline{H}a$ is a closed normal subgroup of G containing H. It follows that it contains $\overline{H}$, but it is visibly also contained (take $a = 1_G$) and so the result is proved. □

[4]There was an error in the lecture here suggesting that this group was open. As we shall see later this need not be the case.

[5]It may be worth recalling that we define the connected component of x to be the union of all connected components containing x, and that this union is itself connected.

In particular this lemma tells us that even if $\{1_G\}$ is not closed, its closure is a closed normal subgroup. With this in mind we are led naturally to want to be able to take quotients.

For topological groups G and H a map $\theta : G \to H$ is a **homomorphism of topological groups** if it is a continuous group homomorphism. Topological groups G and H are **isomorphic as topological groups** if there are continuous homomorphisms $\theta : G \to H$ and $\psi : H \to G$ such that $\theta \circ \psi = \iota_H$ and $\psi \circ \theta = \iota_G$.

It is well known that a continuous bijection of topological spaces need not be a homeomorphism, while a homomorphism that is a bijection is necessarily an isomorphism. The group structure of a topological group does not mitigate the topological problem as the following easy example shows.

Example 4.4. Given a group G, the identity map $G \to G$ is a group homomorphism. If the domain is endowed with the discrete topology then this is a continuous group homomorphism, but unless the codomain has the same topology (and it needn't, for example if G is not trivial and it is indiscrete) then this map is not a homeomorphism and so not an isomorphism of topological groups.

Given a topological group G and a subgroup H the quotient map $q : G \to G/H; x \mapsto xH$ naturally induces a topology on the quotient space: $U \subset G/H$ is open if and only if $\bigcup U$ is open in G. If H is normal then G/H also has a group structure and it turns out that this is compatible with the topology even without any topological restrictions on H.

Proposition 4.5. *Suppose that G is a topological group and H is a normal subgroup of G. Then G/H is a topological group when endowed with the quotient topology and the quotient map $q : G \to G/H$ is open.*

Proof. To show the quotient map is open it suffices to note that if U is open in G then $UH = \bigcup\{Uh : h \in H\}$ is a union of open sets and $q(U) = \{uH : u \in U\}$ so that $\bigcup q(U) = UH$. Thus $\bigcup q(U)$ is open and hence $q(U)$ is open by definition.

Suppose that $U \subset G/H$ is open. Then

$$\bigcup U^{-1} = \bigcup\{(xH)^{-1} : xH \in U\} = \left\{x^{-1} : x \in \bigcup U\right\} = \left(\bigcup U\right)^{-1}$$

and so U^{-1} is open in G/H by definition since $\bigcup U$ is open in G and inversion is continuous on G.

Finally, define

$$W := \left\{(zH, wH) \in (G/H)^2 : (zH)(wH) \in U\right\} \text{ and } V := \left\{(z, w) \in G^2 : zw \in \bigcup U\right\}.$$

Suppose that $(xH, yH) \in W$. Then since V is open and contains (x, y), there are open sets $S, T \subset G$ such that $x \in S$, $y \in T$, and $S \times T \subset V$. If $h, k \in H$ then $(xh)(yk) \in xyH \subset \bigcup U$ and so $SH \times TH \subset V$.

On the other hand SH and TH are unions of open sets and so they are themselves open in G, and so the sets $S' := \{sH : s \in S\}$ and $T' := \{tH : t \in T\}$ are open in G/H; $xH \in S'$ and $yH \in T'$; and $S' \times T' \subset W$. It follows that W is open as required. □

⚠ The group structure here is important: in general for topological spaces the quotient map need not be open.

Example 4.6. The topological group $\mathbb{R}$ has a normal subgroup $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ is a topological group – it is the reals $\pmod 1$.

Although this group is not, there are more pathological examples.

Exercise 4.7. Show that if $\mathbb{R}$ is endowed with its usual topology then $\mathbb{Q}$ is a normal subgroup of $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ is (uncountable and) indiscrete.

Example 4.8. The map $\mathbb{R} \to S^1; x \mapsto \exp(2\pi i x)$ is a continuous homomorphism.

Lemma 4.9. *Suppose that G is a topological group and H is a normal subgroup of G. Then G/H is Hausdorff if and only if H is closed.*

Proof. By Proposition 4.5 G/H is a topological group and so by Proposition 3.2 it suffices to note that $\{1_{G/H}\} = \{H\}$ is closed in G/H if and only if H is closed in G by definition. □

Corollary 4.10. *Suppose that G is a topological group. Then $G/\overline{\{1_G\}}$ is a Hausdorff topological group.*

Proof. Since $\{1_G\}$ is a normal subgroup of G we have that $\overline{\{1_G\}}$ is a closed normal subgroup by Lemma 4.3. The result follows by Lemma 4.9. □

If $f : G \to \mathbb{C}$ is continuous then f is constant on cosets of $\overline{\{1_G\}}$ so if we are interested in continuous complex-valued functions on a group we lose nothing by supposing that the group is Hausdorff. This is a common convention.

5. Direct sums and products

Given a family of sets $(U_i)_{i\in I}$ we write $\prod_{i\in I} U_i$ for the cartesian product of the U_is which we think of as the set of choice functions $f : I \to \bigcup_{i\in I} I_i$ with $f(i) \in U_i$ for all $i \in I$; sometimes we write f_i for $f(i)$. If $I = \{i_1, \dots, i_n\}$ then we will frequently write $U_{i_1} \times \cdots \times U_{i_n}$.

Given a family $(G_i)_{i\in I}$ of groups indexed by a set I the **direct product**, denoted $\prod_{i\in I} G_i$, is the cartesian product of the G_is endowed with point-wise operations so

$$xy = (x_i y_i)_{i\in I} \text{ and } x^{-1} := (x_i^{-1})_{i\in I}.$$

This product is itself a group with these operations, and we write $p_j : \prod_{i\in I} G_i \to G_j; x \mapsto x_j$ for each $j \in I$ – these maps are called the **projection maps** – and p_j is a surjective homomorphism.

If the G_is are topological groups then $\prod_{i\in I} G_i$ is naturally endowed with the product topology, and when so endowed we call it the **topological direct product**. We have defined this for products of two topological spaces. More generally a base for the product is given by the sets

$$\prod_{i\in I} U_i \text{ where } \begin{cases} U_i = G_i \text{ for all } i \in I\setminus J \\ U_i \text{ is open in } G_i \text{ for all } i \in J \end{cases} \tag{5.1}$$

where J ranges all finite subsets of I.

Proposition 5.1. *Suppose that $(G_i)_{i\in I}$ is a family of topological groups. Then the topological direct product $\prod_{i\in I} G_i$ is a topological group and the projection maps are continuous open maps.*

Proof. These are routine checks similar to previous arguments. □

If the G_is are all the same we write G^I for the product $\prod_{i\in I} G_i$.

Exercise 5.2. Suppose that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and consider the map $\psi : \mathbb{R} \to S^1 \times S^1; t \mapsto (\exp(2\pi it), \exp(2\pi i\alpha t))$. Show that ψ is a continuous injective homomorphism but that $\operatorname{Im}\phi$ is not isomorphic (as a topological group) to $\mathbb{R}$.

Example 5.3. The topological group $\mathbb{Z}^{\mathbb{R}}$ (where $\mathbb{Z}$ is seen as discrete) is *not* normal.

Proof. We shall view the elements of $\mathbb{Z}^{\mathbb{R}}$ as functions and begin by noting that the sets $U(g) := \{f \in \mathbb{Z}^{\mathbb{R}} : f(s) = g(s) \text{ for all } s \in \operatorname{supp} f\}$ where $S \subset \mathbb{R}$ is finite and $g : S \to \mathbb{Z}$ form a base for the topology.

For $z \in \mathbb{Z}$ let A_z be the set of $f \in \mathbb{Z}^{\mathbb{R}}$ such that f is injective on $\{x \in \mathbb{R} : f(x) \neq z\}$. We shall show that A_0 and A_1 are disjoint closed sets but that they cannot be contained in disjoint open sets.

First, $A_0 \cap A_1 = \emptyset$ since $\mathbb{R}$ is uncountable but $\mathbb{Z}$ is countable. Secondly, A_z is closed for $z \in \mathbb{Z}$: For all $x, y \in \mathbb{R}$ and $w \in \mathbb{Z}$ we write $g_{x,y;w} : \{x, y\} \to \mathbb{Z}$ taking the constant value w. Then

$$A_z = \bigcap_{x,y\in\mathbb{R};x\neq y;w\in\mathbb{Z}\backslash\{z\}} U(g_{x,y;w})^c$$

so that A_z is closed.

Perhaps surprisingly $\mathbb{Z}^{\mathbb{R}}$ has a countable dense subset, meaning that there is a countable set D such that every non-empty open set in $\mathbb{Z}^{\mathbb{R}}$ intersects D. To see this simply note that the maps $\mathbb{R} \to \mathbb{Z}; x \mapsto \lfloor p(x) \rfloor$ where p is a polynomial with rational coefficients are dense in $\mathbb{Z}^{\mathbb{R}}$.

Now, suppose that $A_0 \subset U$ with U open. For each $x \in D \cap U$ let $S(x) \subset \mathbb{R}$ be finite and $g_x : S(x) \to \mathbb{Z}$ be such that $U(g_x) \subset U$. Write

$$V := \bigcup_{x\in D\cap U} U(g_x) \text{ and } S := \bigcup_{x\in D\cap U} \operatorname{supp} g_x.$$

It may happen that $V \neq U$, however we do have $V \subset U$ and since D is dense that $\overline{V} = \overline{U}$. S is countable since $\operatorname{supp} g_x$ is finite for all $x \in D \cap U$, and D is countable. Let $g : S \to \mathbb{Z}$ be an injection, and h an extension of g to $\mathbb{R}$ such that $h \in A_0$. Finally let $k : \mathbb{R} \to \mathbb{Z}$ be the function that is 0 on S and 1 elsewhere.

With this, note that $U(g_x) + k \subset U(g_x)$ for all $x \in D \cup U$, and hence $V + k \subset V$ and since addition is separately continuous we have $\overline{V} + k \subset \overline{V}$. But $h \in A_0 \subset U$ and so $h + k \in \overline{U} + k \subset \overline{U}$, while at the same time $h + k \in A_1$. Thus $A_1 \cap \overline{U} \neq \emptyset$. The result is proved. □

Exercise 5.4. Show that $\mathbb{Z}^{\mathbb{R}}$ has a countable dense subset – a topology with a countable dense subset is called **separable**.

Given a family $(G_i)_{i\in I}$ of groups indexed by a set I the **direct sum**, denoted $\oplus_{i\in I}G_i$, is the set of $x \in \prod_{i\in I} G_i$ such that x_i is the identity of G_i for all but finitely many $i \in I$. The direct sum is a subgroup of the direct product $\prod_{i\in I} G_i$ and so if the G_is are topological groups we *could* give the direct sum the subspace topology. However, this turns out not to be quite the right thing to do and to understand why it is instructive to return to the direct product.

We can think of the direct product of the topological groups $(G_i)_{i\in I}$ as the product in the category of groups – that is the usual direct product of groups – endowed with the weakest topology so that all the projection maps p_i are continuous. (This is sometimes called the **initial topology** induced by the maps p_i.) It is particularly easy to make sense of 'weakest' here because the intersection of two topologies on the same base set is, itself, a topology and if both contain all sets of the form $p_i^{-1}(S)$ for S open in G_i, then so does the intersection.

On the other hand, we think of the direct sum as a coproduct. In this case we need to take some care with the ambient category: the coproduct in the category of groups is what is usually called the free product, and that is not what we have here. We are interested in the coproduct in the category of *Abelian* groups. Now instead of projection maps we have embeddings $\iota_j : G_j \to \oplus_{i\in I}G_i$ defined so that $p_j\circ\iota_j(x) = x$ for all $x \in G_j$ and $p_i\circ\iota_j(x) = 1_{G_i}$ for all $x \in G_j$ and $i \neq j$.

Now suppose that the groups G_i are Abelian topological groups. The embeddings ι_j map the groups G_j *into* the direct sum $\oplus_{i\in I}G_i$, and since any sufficiently weak topology on the latter will make the embeddings continuous, we would like to endow $\oplus_{i\in I}G_i$ with the strongest topology so that all the maps ι_i are continuous. This topology is sometimes called the **final topology** induced by the maps ι_i, and the open sets are the sets

$$U \subset \oplus_{i\in I}G_i \text{ such that } \iota_i^{-1}(U) \text{ is open in } G_i \text{ for all } i \in I. \tag{5.2}$$

Since preimages preserve unions and (finite) intersections it is easy to see that this is a topology and any topology on $\oplus_{i\in I}G_i$ such that the maps ι_i are all continuous must be contained in this topology. We call $\oplus_{i\in I}G_i$ endowed with this topology the **topological direct sum**.

Proposition 5.5. *Suppose that $(G_i)_{i\in I}$ is a family of Abelian topological groups. Then the topological direct sum $\oplus_{i\in I}G_i$ is a topological group and the embeddings ι_i are continuous for all $i \in I$.*

Proof. These are also routine checks similar to previous arguments. □

It was relatively easy for us to write down a base for the topology of the topological direct product as we did in (5.1), but for the direct sum (5.2) is much more indirect and giving a direct characterisation is rather more complicated – see [Hig77] for more details.

Example 5.6. Suppose that $(G_i)_{i\in I}$ are discrete Abelian topological groups. Then the topological direct sum is also discrete.

Proof. This is immediate since certainly the embeddings are continuous if the topology on $\bigoplus_{i\in I} G_i$ is discrete, and there is no stronger topology than the discrete topology. □

Example 5.7. Suppose that $G_n = \mathbb{Z}/2\mathbb{Z}$ endowed with the discrete topology for all $n \in \mathbb{N}$. Then the topology on the (algebraic) direct sum $\bigoplus_{n\in\mathbb{N}} G_n$ endowed with the subspace topology, α, when it is considered as a subgroup of $\prod_{n\in\mathbb{N}} G_n$, is strictly weaker than the topology τ on the topological direct sum $\bigoplus_{n\in\mathbb{N}} G_n$.

Proof. From Example 5.6 we know that τ is discrete. On the other hand write $e_i := (0,\dots,0,1,0,\dots)$, that is the element of $\bigoplus_{n\in\mathbb{N}} G_n$ with 1 in the ith coordinate and 0 (*i.e.* the identity of $\mathbb{Z}/2\mathbb{Z}$) elsewhere. We have $e_i \to (0,\dots)$ in $\prod_{n\in\mathbb{N}} G_n$ and since $(0,\dots,), e_1, e_2, \dots \in \bigoplus_{n\in\mathbb{N}} G_n$ we also have $e_i \to (0,\dots)$ in α, but the only sequences in τ that converge are eventually constant. Hence $\alpha \neq \tau$, and since $\alpha \subset \tau$ we have the claim. □

6. Compactness

Compactness is a tremendously powerful tool worth sacrificing almost any other property of a mathematical structure. We shall see in this section that a lot of the pathologies we have encountered so far can be eliminated by a suitable use of compactness, particularly when combined with enough separation to make the group Hausdorff.

We say that a topological space X is **locally compact** if every point has a compact neighbourhood.

Exercise 6.1. Suppose that G is a locally compact topological group and H is a subgroup of G. Show that if H is closed then H is a locally compact topological group with the subspace topology, and that if H is normal (but not necessarily closed) that G/H is locally compact.

While locally compact groups cannot be too large locally(!) they can still be very large – for example any group with the discrete topology – and it is useful to be able to restrict them.

Lemma 6.2. [6]*Suppose that G is a locally compact topological group. Then there is a compact symmetric neighbourhood of the identity V and an open symmetric neighbourhood of the identity $S \subset V$ such that $\bigcup_{n\in\mathbb{N}} S^n = \bigcup_{n\in\mathbb{N}} V^n$ is an open subgroup of G.*

Proof. Let K be a compact neighbourhood of the identity in G, and let I be the interior of K *i.e.* the union of the open subsets of K. Put $S := I \cup I^{-1}$ and $V := \overline{S}$. Then S is open since inversion is continuous and the union of two open sets is open. $K \cup K^{-1}$ is compact since inversion is continuous and the union of two compact sets is compact. Hence V is compact as a closed subset of a compact set. Finally both S and V are symmetric neighbourhoods of the identity.

Put $H := \bigcup_{n=1}^{\infty} S^n$. Since $1_G \in S$ we have $1_G \in H$. Moreover, if $x, y \in H$ then there are $n, m \in \mathbb{N}$ such that $x \in S^n$ and $S \in V^m$ so that $y^{-1} \in (S^{-1})^m = S^m$ (since S is symmetric)

[6]This lemma has been slightly corrected to account for an error in an earlier version of Lemma 4.2.

and so $xy^{-1} \in S^{n+m}$. By the subgroup test H is a subgroup of G – it is the subgroup generated by S. S is a neighbourhood so H is open and closed by Lemma 4.1. Finally, since H is closed and $S \subset H$ we have $V \subset H$ and hence $H = \bigcup_{n\in\mathbb{N}} V^n$ as claimed. □

⚠It might be natural to say that a group generated by a compact set is 'compactly generated'. We shall not use this terminology, partly because it means something else in topologies.

Local compactness ensures that the stronger separation axiom of normality holds *c.f.* Example 5.3.

Theorem 6.3. *Suppose that G is a locally compact topological group. Then G is normal* i.e. *for any pair of disjoint closed sets $A, B \subset G$ there disjoint open sets $U, V \subset G$ with $A \subset U$ and $B \subset V$.*

Proof. We begin by establishing the result in the case where A is compact instead of closed.[7]

Claim. *Suppose that A is compact and B is closed with $A \cap B = \varnothing$. Then there is a symmetric open neighbourhood of the identity U such that $AU \cap BU = \varnothing$*

Proof. Since $A \cap B = \varnothing$, we have $A \subset B^c$. The latter is open and so for every $a \in A$ there is some open set $U_a \subset B^c$ containing a. Thus by Lemma 2.2 there is a symmetric open neighbourhood of the identity V_a such that $aV_a^2 \subset U_a \subset B^c$. Since A is compact there is a finite set $a_1, \dots, a_n$ such that $\bigcup_{i=1}^n a_iV_{a_i} \supset A$. The set $\bigcap_{i=1}^n V_{a_i}$ is a open neighbourhood of the identity and so by Lemma 2.2 again there is a symmetric open neighbourhood of the identity U such that $U^2 \subset \bigcap_{i=1}^n V_{a_i}$. Now, suppose that $z \in AU \cap BU$. Let $1 \leqslant i \leqslant n$ such that $z \in a_iV_{a_i}U$, and since $z \in BU$ there is some $b \in B$ such that $b^{-1}z \in U$ so $b \in a_iV_{a_i}UU^{-1} \subset a_iV_{a_i}^2 \subset B^c$ – a contradiction. □

If A is not compact then things are a little trickier: the example to have in mind is $G = \mathbb{R}$ and $A = \{n + 1/n : n \in \mathbb{N}\}$ and $B = \mathbb{N}\setminus\{2\}$. Here for any interval I we have $(B + I) \cap A \neq \varnothing$ – so any open set containing B that is disjoint from A must contain narrower intervals around n as n increases.

By[8] Lemma 6.2 there is a compact symmetric neighbourhood of the identity K and an open symmetric neighbourhood of the identity $S \subset K$ with $H := \bigcup_{n\in\mathbb{N}} S^n = \bigcup_{n\in\mathbb{N}} K^n$ an open subgroup. The partition G/H of G is into open and closed sets and left multiplication is continuous so the result will follow if we can prove it for the group H; from now on we assume that $G = H$.

By induction K^n is compact since it is the continuous image of $K \times K^{n-1}$ under the multiplication map, and the product $K \times K^{n-1}$ is compact being a topological product of two compact sets. Since K is a compact symmetric neighbourhood of the identity it contains a symmetric open neighbourhood of the identity S, and we have $\bigcup_{n\in\mathbb{N}} S^n = \bigcup_{n\in\mathbb{N}} K^n = G$.

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[7] ⚠While compact sets in Hausdorff spaces are closed, in general topological spaces they need not be.

[8] This paragraph has been slightly corrected from lectures to account for an error in the original Lemma 6.2.

Let $A_0 := K \cap A$ and $A_n := K^{n+1} \cap (A\backslash S^n)$ for $n > 0$. Since S^n is open and A is closed we have that $A\backslash S^n$ is closed. As noted above K^n is compact for $n \in \mathbb{N}$ and so A_n is compact for $n \in \mathbb{N}_0$. Since $A_n \cap B = \varnothing$ and B is closed we may apply the claim to get an open symmetric neighbourhood of the identity U_n such that $A_nU_n \cap BU_n = \varnothing$.

Let $W_n := \bigcap_{m\leqslant n} U_m$ and

$$U := \bigcup_{n\in\mathbb{N}_0} A_n(U_n \cap S) \text{ and } V := \bigcup_{n\in\mathbb{N}} BW_n \cap S^n.$$

First we note that U and V are open: The set U is open since it is a union of translates of the open sets $U_n \cap S$. Since the sets U_m are open neighbourhoods of the identity so are the sets W_n, and hence the set V is open.

Secondly, we check that $A \subset U$ and $B \subset V$: If $a \in A$ then since $\bigcup_{n\in\mathbb{N}} K^n = G$ there is some $n \in \mathbb{N}$ such that $a \in K^n$ and $a \notin K^{n-1}$. Since $S \subset K$ we have $a \in A_{n-1}$ and since $1_G \in U_n \cap S$ we conclude that $a \in U$ *i.e.* $A \subset U$. Since $\bigcup_{n\in\mathbb{N}} S^n = G$ we see that $B = \bigcup_{n\in\mathbb{N}} BS^n \subset V$ since $1_G \in W_n$.

Finally we show that $U \cap V = \varnothing$. To see this we show that $A_n(U_n \cap S) \cap V = \varnothing$ for each $n \in \mathbb{N}$ by showing in turn that $A_n(U_n \cap S) \cap (BW_m \cap S^m) = \varnothing$ for all $m \in \mathbb{N}$. We have two cases:

(i) For $m \geqslant n$ we have $W_m \subset U_n$ so $A_n(U_n \cap S) \cap (BW_m \cap S^m) \subset A_nU_n \cap BU_n = \varnothing$.

(ii) For $m < n$ we have $A_n(U_n\cap S)\cap(BU_m\cap S^m) \subset A_n(U_n\cap S)\cap S^{n-1}$, but $A_n\cap S^n = \varnothing$, hence $A_n(U_n \cap S) \cap (BW_m \cap S^m) = \varnothing$.

The result is proved. □

Exercise 6.4. Suppose that G is a topological group and $H \leqslant G$ is compact. Show that the quotient map $q : G \to G/H$ is closed.

Local compactness can also be used to give a partial response to Example 4.4.

Proposition 6.5. *Suppose that* $G = \bigcup_{n\in\mathbb{N}} K^n$ *where* K *is a compact symmetric neighbourhood of the identity,* H *is a locally compact Hausdorff group, and* $\pi : G \to H$ *is a continuous bijective homomorphism. Then* π *is an isomorphism of topological groups.*

Proof. We begin with a claim.

Claim. *There is some* $n \in \mathbb{N}$ *such that* $\pi(K^n)$ *is a neighbourhood.*

Proof. For those familiar with the Baire Category Theorem this is particularly straightforward. We shall proceed directly by what is essentially the proof of the BCT for locally compact Hausdorff spaces.

As in Theorem 6.3 the sets K^n are compact and so $\pi(K^n)$ is compact. Since H is Hausdorff the sets $\pi(K^n)$ are therefore closed. We construct a nested sequence of closed neighbourhoods inductively: Let U_0 be a compact (and so closed since H is Hausdorff) neighbourhood in H, and for $n \in \mathbb{N}$ let $U_n \subset \pi(K^n)^c \cap U_{n-1}$ be a closed neighbourhood.

This is possible since (by the inductive hypothesis) U_{n-1} is a neighbourhood and so contains an open neighbourhood V_{n-1}. But then $\pi(K^n)^c \cap V_{n-1}$ is open and non-empty since otherwise $\pi(K^n)$ contains a neighbourhood. It follows that $\pi(K^n)^c \cap U_{n-1}$ contains an open neighbourhood and so it contains a closed neighbourhood by[9] Lemma 2.2.

Now by the finite intersection property of the compact space U_0, the set $\bigcap_n U_n$ is non-empty. This contradicts surjectivity of π since $G = \bigcup_{n\in\mathbb{N}} K^n$ and the claim is proved. $\square$

With this claim we show that if $X \subset H$ is compact then $\pi^{-1}(X)$ is compact. Since X is compact and $\pi(K^n)$ contains a neighbourhood and the set $\{x\pi(K^n) : x \in H\}$ covers X, there are elements $x_1, \ldots, x_m$ such that $X \subset \bigcup_{i=1}^m x_i\pi(K^n)$ and hence $\pi^{-1}(X) \subset \bigcup_{i=1}^m \pi^{-1}(x_i)K^n$. However for each $1 \leqslant i \leqslant m$ there is some $n_i \in \mathbb{N}$ such that $\pi^{-1}(x_i) \in K^{n_i}$, whence $\pi^{-1}(X) \subset K^{n+\max\{n_1,\ldots,n_m\}}$. However since H is Hausdorff, X is closed and so $\pi^{-1}(X)$ is closed and a subset of a compact set and so compact.

It remains to show that if $C \subset G$ is closed then $\pi(C)$ is closed (from which the result follows). To see this suppose that y is a limit point of $\pi(C)$. H is locally compact so y has a compact neighbourhood X. Now $\pi^{-1}(X)$ is compact and so $\pi^{-1}(X) \cap C$ is compact. But then $X \cap \pi(C)$ is compact since π is continuous. However its closure contains y and hence it contains y. $\square$

Again, we need something like the given hypothesis on G since otherwise we can take $G = \mathbb{R}$ (as in Example 4.4) with the discrete topology on the domain and the usual topology on the codomain. Both are locally compact Hausdorff groups, and the identity map between them is a bijective topological homomorphism but this is not a topological isomorphism

In Example 1.5 we had a group endowed with a topology such that multiplication (called addition there) was jointly continuous but inversion was not. By way of contrast we have the following result.

Theorem 6.6 (Ellis, [Ell57b, Theorem])**.** *Suppose that G is a locally compact Hausdorff topological space and a group such that multiplication is jointly continuous. Then G is a topological group.*

In Example 1.6 we had a group endowed with a topology such that multiplication (called addition there) was separately (although we did not show this) but not jointly continuous and inversion was continuous. By way of contrast we have the following result.

Theorem 6.7 (Ellis, [Ell57a, Theorem 2])**.** *Suppose that G is a locally compact Hausdorff topological space and a group such that inversion is continuous and multiplication is separately continuous. Then G is a topological group.*

Finally we mention that the coproduct topology on countable direct sums has a nice base when the groups are locally compact.

Theorem 6.8 ([BHM75, Proposition 1])**.** *Suppose that $(G_i)_{i\in\mathbb{N}}$ is a sequence of locally compact Abelian topology groups. Then the sets $\prod_{i=1}^\infty U_i$ where U_i is open in G_i for each*

[9]This may deserve a word or two more: for any open neighbourhood U there is an open neighbourhood B of the identity and $x \in U$ such that $xBB^{-1} \subset U$. Then $xB \subset (U^cB)^c \subset U$ but U^cB is open and so $\overline{xB} \subset (U^cB)^c \subset U$.

$i \in \mathbb{N}$ is a base for a topology called the ***box topology*** *on the algebraic direct product $\prod_{i=1}^{\infty} G_i$. The algebraic direct sum $\bigoplus_{i=1}^{\infty} G_i$ endowed with the subspace topology inherited from the box topology is the topological direct sum $\bigoplus_{i=1}^{\infty} G_i$.*

7. Totally disconnected groups

A topological space is said to be **totally disconnected** if the only connected components are singletons. ⚠ Although any discrete space is totally disconnected, the converse is not true as we can see with a closer examination of Example 5.7.

Proposition 7.1. *Suppose that G is a closed subgroup of a product of finite Hausdorff topological groups. Then G is compact and totally disconnected.*

Proof. A finite Hausdorff topological group is necessarily endowed with the discrete topology and the properties of being compact and totally disconnected are preserved under passing to closed subsets so it suffices for us to show that a product of finite groups endowed with the discrete topology is totally disconnected. Let $(G_i)_{i\in I}$ be a sequence of such groups indexed by a set I.

Suppose that $X \subset \prod_{i\in I} G_i$ is connected and $x, y \in X$ have $x \neq y$. Then there is some $j \in I$ such that $x_j \neq y_j$ and the set $U := \{z \in \prod_{i\in I} G_i : z_j = x_j\}$ is open and closed since G_j is finite so X is a disjoint union of the the two open and closed sets $X \cap U$ and $X \cap U^c$. x is in the former of these and y is in the latter contradicting the connectivity of X. The result is proved. □

One of the results of this section is a converse to the above, but before going down this path it is helpful to have a more concrete example.

The p-adic integers are an important object in number theory for a variety of reasons, and they will provide us with an interesting class of examples of topological groups. For $p \in \mathbb{N}$ we define the p-adic integers to be the (closed) subgroup

$$\mathbb{Z}_p := \left\{ x \in \prod_{n\in\mathbb{N}_0} \mathbb{Z}/p^{n+1}\mathbb{Z} : x_{i+1} \equiv x_i \pmod{p^i} \text{ for all } i \in \mathbb{N} \right\},$$

where the group operation is inherited from the product. This is a compact totally disconnected group by Proposition 7.1. It can be helpful to think of the p-adic numbers more concretely as numbers written in base p extended infinitely to the left with addition defined in the same way as for the integers, and negation defined so as to ensure that $x + (-x)$ sum to the all-0s string *e.g.* for the sum and negation in $\mathbb{Z}_7$ we have

$$\begin{array}{r} \dots 1\,6\,2\,3 \\ +\dots 2\,4\,3\,4 \\ \hline \dots 4\,3\,6\,0 \end{array} \qquad \text{-}(\dots 325) = \dots 342$$

The integers embed into $\mathbb{Z}_p$ by writing a number in base p and prefixing it with a countable infinity of leading 0s on the left, and in particular while the integers are countable the p-adics are uncountable.

The p-adic integers can be extended to the p-adic numbers, denoted $\mathbb{Q}_p$. This is a locally compact and totally disconnected Hausdorff group in which the p-adic integers are a compact open subgroup; we shall not concern ourselves with its particular nature now, but the main result of this section is a characterisation of totally disconnected locally compact Hausdorff groups.

Theorem 7.2 (Van Dantzig's Theorem)**.** *Suppose that G is a Hausdorff topological group. Then G is locally compact and totally disconnected if and only if every neighbourhood of the identity contains a compact open subgroup.*

Proof of $\Leftarrow$. If x and y are distinct points then since G is Hausdorff there is an open set U containing x and not y; let H be a compact open subgroup (though set is enough here) contained in $x^{-1}U$. Since H is open it is closed and so if C is a set containing x and y then $C \cap xH$ and $C \cap (xH)^c$ is a partition of C into (relatively) open and (relatively) closed sets containing x and y respectively. Thus C is *not* connected and hence G is totally disconnected. $\square$

The proof in the other direction requires more work, with the next lemma being the key driver.

Lemma 7.3. *Suppose that G is a totally disconnected Hausdorff group, K is a compact neighbourhood of the identity, and $y \neq 1_G$. Then there is a relatively open and closed subset of K containing the identity and not containing y.*

Proof. It is most convenient for our topological language to refer to K as a topological space *i.e.* when we say open set we shall mean relatively open in K so a set of the form $K \cap U$ for U open in G. K is certainly totally disconnected in the relative topology. Moreover K is normal: this follows from Theorem 6.3, since any relatively closed set in K is also closed in G (since K is closed as a result of G being Hausdorff).

Let C be the intersection of all the closed and open subsets of K containing the identity, and suppose that $C = A \sqcup B$ for closed sets A and B with $1_G \in B$. By normality of K there is an open set U containing A whose closure is disjoint from B. Thus $\partial U := \overline{U}\setminus U$ is closed and disjoint from C, and by the definition of C, for each $x \in \partial U$ there is closed and open set U_x containing the identity such that $x \notin U_x$. Thus $\{K\setminus U_x : x \in \partial U\}$ is an (open) cover of ∂U, but the latter is a closed subset of the compact set K. Thus it is compact and there are elements $x_1, \dots, x_m$ such that $\partial U \subset \bigcup_{i=1}^m K\setminus U_{x_i}$, and hence $\bigcap_{i=1}^m U_{x_i} \subset K\setminus(\overline{U}\setminus U) = U \cup (K\setminus\overline{U})$. It follows that $V := \bigcap_{i=1}^m U_{x_i}\setminus\overline{U} = \bigcap_{i=1}^m U_{x_i}\setminus U$, and hence V is both open and closed. By design $A \cap V = \emptyset$ and also $1_G \in V$. But then $C \subset V$ and so $A = \emptyset$. We have shown that C is connected, but then since $1_G \in C$ and K is totally disconnected we have $C = \{1_G\}$. Since $y \neq 1_G$ there is some closed and open set U in K such that $1_G \in U$ and $y \notin U$ as required. $\square$

⚠ In a general topological space X the intersection of all closed and open sets containing an element x is called a **quasi-component** and it is not necessarily a connected component.

Exercise 7.4. Suppose that X is a compact Hausdorff space. Show that X is normal.

Theorem (Theorem 7.2 $\Rightarrow$). *Suppose that G is a totally disconnected locally compact Hausdorff topological group. Then every neighbourhood of the identity contains a compact open subgroup.*

Proof. Suppose S is the given neighbourhood; let T be an open neighbourhood of the identity in S; let K be a compact neighbourhood of the identity (which exists by local compactness); and finally let N be an open neighbourhood of the identity in K. Then $V := N \cap T$ is an open neighbourhood of the identity contained in S, and $\overline{V} \subset K$, so V has compact closure.

By Lemma 7.3 applied to $\overline{V}$, for each $x \in \partial V := \overline{V} \backslash V$ there is a relatively open and closed set V_x such that $x \notin V_x$ and $1_G \in V_x$. As before, by compactness there are elements $x_1, \ldots, x_m$ such that $\{V_{x_1}^c, \ldots, V_{x_m}^c\}$ is a finite open cover of ∂V and then $U := V_{x_1} \cap \cdots \cap V_{x_m}$ contains 1_G, is relatively open and closed in $\overline{V}$ and is contained in V. The set U so defined is relatively open in $\overline{V}$ and so $U = \overline{V} \cap L$ for some open set L, hence $U = U \cap V = \overline{V} \cap L \cap V = L \cap V$ is open in G. Similarly, U so defined is relatively closed so that $\overline{V} \backslash U = \overline{V} \cap L$ for some (other) open L and hence $V \backslash U = (\overline{V} \backslash U) \cap V = V \cap L$ is open and hence U is closed (seeing as $U \subset V$). We conclude that U is an open and closed (so compact) neighbourhood of the identity contained in S.

For each $x \in U$ there is an open neighbourhood of the identity V_x such that $xV_x \subset U$, and by Lemma 2.2 a further open neighbourhood W_x such that $W_x^2 \subset V_x$. The set $\{xW_x : x \in U\}$ is then an open cover of U and so there is a finite sub-cover such that $U \subset \bigcup_{i=1}^k x_i W_{x_i}$; put $W := \bigcap_{i=1}^k W_{x_i}$ so that $UW \subset U$. Now let H be the open subgroup generated by W so that $H \subset UH \subset U$ which is an open, and so closed, subgroup of V which itself has a compact closure. It follows that H is a compact open subgroup as required. □

Exercise 7.5. Suppose that G is a totally disconnected locally compact Hausdorff group. Show that every compact subgroup of G is contained in an open compact subgroup.

When we have compactness not just local compactness the examples given in Proposition 7.1 turn out to be the only ones – these are called the **profinite groups**. To show this we need an additional lemma.

Lemma 7.6. *Suppose that G is a compact Hausdorff topological group and H is an open subgroup of G. Then there is an open normal subgroup of G of finite index contained in H.*

Proof. Since G/H is an open over of G, and G is compact, it contains a finite subcover and so G/H is finite – write $G/H = \{x_1H, \ldots, x_mH\}$. Let $N := \bigcap_{i=1}^m x_i H x_i^{-1}$ which is a finite intersection of open subgroups and so an open subgroup. On the other hand, if $x \in G$ then for each j there is some i (depending on j and x) such that $xx_iH = x_jH$. But then $(Hx_i^{-1})x^{-1} = Hx_j^{-1}$ and so $x(x_iHx_i^{-1})x^{-1} \subset x_jHx_j^{-1}$, whence $xNx^{-1} \subset x_jHx_j^{-1}$. However, j was arbitrary and so $xNx^{-1} \subset N$. But this is true for all $x \in G$ and hence N is normal as required. Finally, N is an open subgroup of G so G/N is an open cover of the compact group G. Since G/N is a partition the only subcover is the whole set and so we conclude that the index of N in G is finite. The lemma is proved. □

Exercise 7.7. Suppose that $H \leqslant G$ are groups and $|G/H| = d$. Show that there is a subgroup $N \leqslant H$ which is normal in G such that $|G : N| \leqslant d!$.

Theorem 7.8. *Suppose that G is a topological group. Then G is a compact Hausdorff totally disconnected group if and only if it is (topologically isomorphic to) a closed subgroup of a product of finite groups.*

Proof. The if direction is Proposition 7.1. In the other direction write $\mathcal{C}$ for the set of open subgroups of G, and for each $H \in \mathcal{C}$ let $N(H)$ be a finite index (open) normal subgroup guaranteed by Lemma 7.6. Then consider the continuous homomorphism

$$\phi : G \to \prod_{H \in \mathcal{C}} G/N(H); x \mapsto (xN(H))_{H\in\mathcal{C}}.$$

The product on the right is a product of finite groups. Moreover, the map is an injection: if $x \neq 1_G$ then since G is Hausdorff there is an open neighbourhood U of 1_G with $x \notin \overline{U}$. By Theorem 7.2 there is a compact open subgroup H contained in U so $x \notin H$. Then $xN(H) \neq N(H)$ and so $\phi(x) \neq 1$.

Since G is compact and the map continuous, the image is compact and the product is Hausdorff so the image is closed. The result is proved. □

We saw in Lemma 4.2 that if G is a topological group then the connected component of the identity L is a closed normal subgroup. This gives rise to a short exact sequence

$$0 \to L \to G \to G/L \to 0,$$

and if G is locally compact and Hausdorff then by Exercise 6.1 L and G/L are also locally compact (and Hausdorff). The group L is a connected locally compact Hausdorff group and the group G/L is a totally disconnected locally compact Hausdorff group. Although this seems very promising the groups L and G/L can still fit together in many varied ways.

8. The Haar integral

We now turn to one of the most beautiful aspects of the theory of topological groups. This describes the way the topology and the algebra naturally give rise to a measure. First, given a group G and a function $f : G \to \mathbb{C}$ we write

$$\lambda_x(f)(y) := f(x^{-1}y) \text{ and } \rho_x(f)(y) := f(yx) \text{ for all } x, y \in G.$$

These are left actions in the sense that $\lambda_{xy}(f) = \lambda_x(\lambda_y(f))$ and $\rho_{xy}(f) = \rho_x(\rho_y(f))$ for all $x, y \in G$.

Given a topological space X we write $C_{\text{CPCT}}(X)$ for the set of continuous compactly supported functions on X, and $C^+_{\text{CPCT}}(X)$ for the set of non-negative elements of $C_{\text{CPCT}}(X)$.

If G is a topological group then λ and ρ both restrict to action on the set $C_{\text{CPCT}}(G)$, and we say that a non-zero linear map $I : C_{\text{CPCT}}(G) \to \mathbb{C}$ with $I(f) \geqslant 0$ whenever $f \in C^+_{\text{CPCT}}(G)$ and $I(\lambda_x(f)) = I(f)$ for all $x \in G$ and $f \in C_{\text{CPCT}}(G)$ is a **left invariant Haar integral**, and similarly on the right.

To provide a sufficient supply of continuous compactly supported functions we shall need a result called Urysohn's Lemma. The proof is sufficiently similar to the proof of Theorem 3.4 that we shall not give it.

Theorem 8.1 (Urysohn's Lemma)**.** *Suppose that X is a normal topological space. Then for every pair of disjoint closed sets A and B there is a continuous function $f : X \to [0,1]$ with $f(a) = 1$ for all $a \in A$ and $f(b) = 0$ for all $b \in B$.*

A key tool for us will be approximating continuous functions in two variables by sums of products of continuous functions in one variable.

Lemma 8.2. *Suppose that G is a locally compact Hausdorff topological group and $K \subset G$ is compact and $F : G \times G \to \mathbb{C}$ is continuous with support in $K \times K$. Then for all $\epsilon > 0$ there are elements $u_1, \dots, u_k, v_1, \dots, v_k \in C(G)$ with support in K such that*

$$\left\| F - \sum_{j=1}^{k} u_j v_j \right\|_\infty < \epsilon.$$

Proof. We define some auxiliary sets: for $i \in \{1, 2\}$ put

$$A_i := \{x_i : |F(x_1, x_2)| \geqslant 2\epsilon\}, B_i := \{x_i : |F(x_1, x_2)| > \epsilon\},$$

and

$$C_i := \{x_i : |F(x_1, x_2)| \geqslant \epsilon\}, D_i := \{x_i : |F(x_1, x_2)| > \epsilon/2\}.$$

Then the sets A_1, A_2, C_1, C_2 are closed while B_1, B_2, D_1, D_2 are open. Urysohn's Lemma gives continuous functions $H_1, H_2 : G \to [0,1]$ with $H_i(x) = 1$ for all $x \in A_i$ and $H_i(x) = 0$ for all $x \in G\backslash B_i$, where $i \in \{1, 2\}$. And similarly, continuous functions $H_1', H_2' : G \to [0,1]$ with $H_i'(x) = 1$ for all $x \in C_i$ and $H_i'(x) = 0$ for all $x \in G\backslash D_i$, where $i \in \{1, 2\}$.

Let

$$V := \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : n \in \mathbb{N}, \alpha_i, \beta_i \in C(G), \operatorname{supp} \alpha_i, \operatorname{supp} \beta_i \subset K \right\},$$

and $V' := \{f|_{C_1 \times C_2} : f \in V\}$. Then V' is a conjugation-closed subalgebra of $C(C_1 \times C_2)$. It contains the constant functions since $\lambda H_1' H_2' \in V$ for all $\lambda \in \mathbb{C}$, and it separates points by Urysohn's Lemma (or even just Theorem 3.4) since given $(x_1, x_2), (x_1', x_2') \in C_1 \times C_2$ distinct then either $x_1 \neq x_1'$ and so there is a continuous function u such that $u(x_1) \neq u(x_1')$ whence $uH_1'H_2' \in V$; or $x_2 \neq x_2'$ and we argue similarly. Given this we may apply the Stone-Weierstrass Theorem (see *e.g.* [Pri17, Theorem 5.10] for the real case) to see that V' is dense in $C(C_1 \times C_2)$ – let $u_1, \dots, u_k, v_1, \dots, v_k \in C(G)$ with $\operatorname{supp} u_i, \operatorname{supp} v_i \subset K$ be such that

$$\left| F(x_1, x_2) - \sum_{i=1}^{k} u_i(x_1) v_i(x_2) \right| \leqslant \epsilon \text{ for all } x_1 \in C_1, x_2 \in C_2. \tag{8.1}$$

Put

$$F'(x_1, x_2) := \sum_{i=1}^{k} H_1(x_1) u_i(x_1) H_2(x_2) v_i(x_2),$$

and note that

$$F'(x_1,x_2) = \begin{cases} \sum_{i=1}^k u_i(x_1)v_i(x_2) & \text{if } x \in A_1 \times A_2 \\ H_1(x_1)H_2(x_2)\sum_{i=1}^k u_i(x_1)v_i(x_2) & \text{if } x \in (B_1 \times B_2)\backslash(A_1 \times A_2) \\ 0 & \text{if } x \notin B_1 \times B_2 \end{cases}.$$

First, since $A_1 \times A_2 \subset C_1 \times C_2$ we have $|F(x) - F'(x)| \leqslant \epsilon$ for all $x \in A_1 \times A_2$. Secondly, by definition of B_1 and B_2 we have $|F(x)| \leqslant \epsilon$ if $x \notin B_1 \times B_2$ and hence $|F(x) - F'(x)| \leqslant \epsilon$ there. Finally, if $x \in (B_1 \times B_2)\backslash(A_1 \times A_2)$ then $x \in C_1 \times C_2$ and so (by (8.1))

$$\left|\sum_{i=1}^k u_i(x_1)v_i(x_2)\right| \leqslant |F(x_1,x_2)| + \epsilon,$$

whence $|F'(x)| \leqslant |F(x)| + \epsilon$. However $x \notin A_1 \times A_2$ and so $|F(x)| < 2\epsilon$, whereupon $|F'(x) - F(x)| < 5\epsilon$. It follows that F' is a suitable approximation (at least after rescaling ϵ) and the result is proved. □

The first result we are leading up to is the uniqueness of Haar integrals. To establish this we shall need to understand how two Haar integrals interact, and to do this we need some notation: given a linear functional $I : C_{\text{CPCT}}(G) \to \mathbb{C}$ and a function $F(x,y)$ with $y \mapsto F(x,y)$ in $C_{\text{CPCT}}(G)$ then we write $I_yF(x,y)$ for the functional I applied to the function $y \mapsto F(x,y)$, and similarly if $x \mapsto F(x,y)$ is in $C_{\text{CPCT}}(G)$ then we write $I_xF(x,y)$ for the functional I applied to the function $x \mapsto F(x,y)$.

Lemma 8.3. *Suppose that G is a locally compact Hausdorff group, I and J are left Haar integrals on G, and $F \in C_{\text{CPCT}}(G^2)$. Then the map $x \mapsto J_yF(x,y)$ is continuous and compactly supported, so that $I_xJ_yF(x,y)$ exists. Similarly $y \mapsto I_xF(x,y)$ is continuous and compactly supported, so that $J_yI_xF(x,y)$ exists and moreover*

$$I_xJ_yF(x,y) = J_yI_xF(x,y).$$

Proof. Since $F \in C_{\text{CPCT}}(G^2)$ has compact support, and the projection functions are continuous, there is a compact set K such that F has support in $K \times K$. By Lemma 8.2 for all $\epsilon > 0$ there are continuous functions $u_1,\dots,u_k,v_1,\dots,v_k$ supported in K such that

$$-\epsilon + \sum_{j=1}^{j} u_j(x)v_j(y) \leqslant F(x,y) \leqslant \epsilon + \sum_{j=1}^{j} u_j(x)v_j(y) \text{ for all } x,y \in G.$$

Let U be an open neighbourhood of the identity with compact closure. Apply Urysohn's Lemma to the disjoint closed sets K and $(KU)^c$ to get a continuous function $g : G \to [0,1]$ supported on $K\overline{U}$ and with $g(x) = 1$ for all $x \in K$. Then

$$-\epsilon g(x)g(y) + \sum_{j=1}^{j} u_j(x)v_j(y) \leqslant F(x,y) \leqslant \epsilon g(x)g(y) + \sum_{j=1}^{j} u_j(x)v_j(y) \text{ for all } x,y \in G.$$

Thus

$$-\epsilon g(x)Jg+\sum_{j=1}^{k}u_j(x)Jv_j \leqslant J_yF(x,y) \leqslant \epsilon g(x)Jg+\sum_{j=1}^{k}u_j(x)Jv_j. \tag{8.2}$$

Now $0 \leqslant g(x) \leqslant 1$ and so $x \mapsto J_yF(x,y)$ is a uniform limit of continuous functions and so continuous. Moreover, it is supported on $\overline{KU}$ which is compact and so $x \mapsto J_yF(x,y)$ has compact support. Similarly $y \mapsto I_xF(x,y)$ is continuous and has compact support. Finally, from (8.2) we have

$$-\epsilon IgJg+\sum_{j=1}^{k}Iu_jJv_j \leqslant I_xJ_yF(x,y) \leqslant \epsilon IgJg+\sum_{j=1}^{k}Iu_jJv_j$$

and similarly for $J_yI_xF(x,y)$ and so the last equality of the lemma holds since ϵ was arbitrary. □

The integral of a non-negative continuous function that is not identically 0 is positive, and this already follows from the axioms of a Haar integral:

Lemma 8.4. *Suppose that G is a locally compact Hausdorff group, I is a left Haar integral on G, and $f \in C^+_{\mathrm{CPCT}}(G)$ has $If = 0$. Then $f \equiv 0$.*

Proof. Suppose that $f \not\equiv 0$ so that there is some $x_0 \in G$ such that $f(x_0) > c > 0$ and hence an open neighbourhood of the identity U such that $f(x_0y) > c/2$ for all $y \in U$. Now for any $g \in C^+_{\mathrm{CPCT}}(G)$ there is a compact set K containing the support of g and $\{xU : x \in K\}$ is an open cover of K. It follows that it has a finite subcover $x_1U, \dots, x_mU$. But then

$$0 \leqslant g(x) \leqslant 2c^{-1}\|g\|_\infty \sum_{i=1}^{m} f(x_0x_i^{-1}x),$$

and hence

$$0 \leqslant Ig \leqslant 2c^{-1}\|g\|_\infty \sum_{i=1}^{m} I\lambda_{x_ix_0^{-1}}(f) = 2c^{-1}\|g\|_\infty mIf = 0.$$

Finally, any $h \in C_{\mathrm{CPCT}}(G)$ can be written in the form $h = h_1 - h_2 + ih_3 - ih_4$ where $h_1, h_2, h_3, h_4 \in C^+_{\mathrm{CPCT}}(G)$, and hence we have that $Ih = 0$ *i.e.* I is identically 0 contradicting the fact that it is a left Haar integral. The lemma follows. □

Compactly supported continuous functions on topological groups have a notion of uniform continuity captured in the next lemma.

Lemma 8.5. *Suppose that G is a locally compact Hausdorff group and $f \in C_{\mathrm{CPCT}}(G)$. Then for all $\epsilon > 0$ there is a symmetric open neighbourhood of the identity V such that*

$$|f(xy) - f(y)| < \epsilon \text{ and } |f(yx) - f(y)| < \epsilon \text{ for all } x \in V, y \in G$$

Proof. Let H be an open symmetric neighbourhood of the identity with compact closure, and K a compact set supporting f. Since f is continuous for all $y \in G$ there is an open neighbourhood U_y of y such that $|f(x) - f(y)| < \epsilon/2$ for all $x \in U_y$. For each $y \in G$ let V_y

be a symmetric neighbourhood of the identity such that $yV_y^2 \subset U_y$ and $V_y^2 y \subset U_y$ – such a set exists by Lemma 2.2 since $(y^{-1}U_y) \cap (U_y y^{-1})$ is a neighbourhood of the identity.

Since $\overline{H}K\overline{H}$ is compact, the open cover $\{yV_y \cap V_y y : y \in \overline{H}K\overline{H}\}$ has a finite subcover $\{y_1V_{y_1} \cap V_{y_1}y_1, \dots, y_mV_{y_m} \cap V_{y_m}y_m\}$. We let V be a symmetric open neighbourhood of the identity such that $V \subset H \cap \bigcap_{i=1}^m V_{y_i}$. Now, suppose that $x \in V$ and $y \in G$. If any of xy, yx or y are in the support of f then $y \in HK \cup KH \cup K \subset \overline{H}K\overline{H}$ and so there is some y_i such that $y \in y_iV_{y_i} \cap V_{y_i}y_i$. Then $xy \in V_{y_i}^2 y_i \subset U_{y_i}$, $yx \in y_iV_{y_i}^2 \subset U_{y_i}$ and $y \in y_iV_{y_i}^2 \subset U_{y_i}$. The result follows by the triangle inequality. □

With these preparatory tools in hand we are ready to prove the important result that, if it exists, the Haar integral is unique up to dilation by a positive scalar.

Theorem 8.6 (Uniqueness of the Haar Integral)**.** *Suppose that G is a locally compact Hausdorff group and I and J are left Haar integrals on G. Then there is some $\lambda > 0$ such that $I = \lambda J$.*

Proof. Suppose that $f_1, f_2 \in C^+_{\text{CPCT}}(G)$ are not identically 0 and write K for a compact set containing the support of f_1 and f_2 (which exists since finite unions of compact sets are compact). Let H be a compact symmetric neighbourhood of the identity in G, and let F be a continuous function with compact support such that $F|_{HKH}$ is identically 1. Such a function exists since the product of compact sets is compact so HKH is compact and hence closed (since G is Hausdorff), and if U is a neighbourhood of the identity in H then $HKHU$ is open and so its complement is closed and disjoint from HKH so we can apply Urysohn's Lemma to get a continuous function that is 1 on HKH and 0 on $KHKU^c$, and so supported on $HKHU \subset HKH^2$; the latter set is compact.

Since the support of f_i is compact and f_i is continuous we may apply Lemma 8.5 to get a symmetric neighbourhood of the identity V_i such that $|f_i(xy) - f_i(y)| < \epsilon$ and $|f_i(yx)-f_i(y)| < \epsilon$ for all $y \in G$, $x \in V_i$. By Lemma 2.2 let V be a symmetric neighbourhood of the identity with $V^2 \subset H \cap V_1 \cap V_2$.

By Urysohn's Lemma (and in fact our proof of complete regularity is enough) there is a continuous function $k : G \to [0, 1]$ with $k(x) = 0$ for all $x \in V^c$ and $k(1_G) = 1$. Put $h(x) := k(x)k(x^{-1})$ so that $h(x) = h(x^{-1})$, $h \not\equiv 0$ is non-negative and $h(x) = 0$ for all $x \notin V^c$.

Now, by translation invariance of J we have $J_xh(y^{-1}x) = J_xh(x)$ and since $(x, y) \mapsto f_i(y)h(x)$ and $(x, y) \mapsto f_i(y)h(y^{-1}x)$ are both continuous and compactly supported we have

$$I_yf_i(y)J_xh(x) = I_yJ_xf_i(y)h(y^{-1}x).$$

Since $h(z) = h(z^{-1})$, Lemma 8.3 and the translation invariance of I then give

$$I_yJ_xf_i(y)h(y^{-1}x) = I_yJ_xf_i(y)h(x^{-1}y) = J_xI_yf_i(y)h(x^{-1}y) = J_xI_yf_i(xy)h(y).$$

Now,

$$|f_i(xy) - f_i(x)|h(y) \leqslant \epsilon F(x)h(y) \text{ for all } x, y \in G$$

and so

$$f_i(x)I_yh(y) - \epsilon I_yF(x)h(y) \leqslant I_yf_i(xy)h(y) \leqslant f_i(x)I_yh(y) + \epsilon I_yF(x)h(y).$$

We conclude that
$$|If_iJh - Jf_iIh| \leqslant \epsilon JFIh.$$
Since h and f_i are not identically 0 we have $Ih, If_i > 0$ by Lemma 9.1 and so these combine to give
$$\left|\frac{Jf_i}{If_i} - \frac{Jh}{Ih}\right| \leqslant \epsilon\frac{JF}{If_i},$$
and hence
$$\left|\frac{Jf_1}{If_1} - \frac{Jf_2}{If_2}\right| \leqslant 2\epsilon JF\left(\frac{1}{If_1} + \frac{1}{If_2}\right).$$
Since ϵ is arbitrary and JF, If_1 and If_2 are independent of ϵ we conclude that there is some $\lambda > 0$ such that $Jf = \lambda If$ for all $f \in C^+_{\text{CPCT}}(G)$. This extends to all $f \in C_{\text{CPCT}}(G)$ by writing f as a linear combination of four elements of $C^+_{\text{CPCT}}(G)$. □

It is now useful to consider some examples.

Exercise 8.7. Let
$$G := \left\{\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix} : x > 0, y \in \mathbb{R}\right\}.$$
Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{R})$, and that
$$If := \int_{-\infty}^{\infty}\int_0^{\infty} f\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix}\frac{1}{x^2}dxdy$$
us a left Haar integral while
$$If := \int_{-\infty}^{\infty}\int_0^{\infty} f\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix}\frac{1}{x}dxdy$$
is a right Haar integral.

Given a topological group G and $f \in C_{\text{CPCT}}(G)$ we write $\widetilde{f}(x) = \overline{f(x^{-1})}$ so that $\widetilde{\cdot}$ is a conjugate-linear involution on $C_{\text{CPCT}}(G)$. The reason for making it conjugate-linear will become clearer later.

Lemma 8.8. *Suppose that G is a locally compact Hausdorff group and I is a left (resp. right) Haar integral on G then $f \mapsto \overline{I\widetilde{f}}$ is a right (resp. left) Haar integral on G.*

Proof. Note that
$$\rho_x(f)(y) = f(yx) = \overline{\widetilde{f}(x^{-1}y^{-1})} = \overline{\lambda_x(\widetilde{f})(y^{-1})} = (\lambda_x(\widetilde{f}))^{\sim}(y).$$
Hence
$$\overline{I\widetilde{\rho_x(f)}} = \overline{I\lambda_x(\widetilde{f})} = \overline{I\widetilde{f}}$$
and so the given map is invariant under right translation. It is also linear, non-trivial, and non-negative on non-negative functions and hence it is a right Haar integral as required. A similar argument works if I is a right Haar integral. □

Proposition 8.9 (Modular function)**.** *Suppose that G is a locally compact Hausdorff group supporting a left Haar integral I. Then there is a (unique) continuous homomorphism $\Delta : G \to \mathbb{R}_{>0}$ such that for any left Haar integral J on G we have*

$$J\rho_t(f) = \Delta(t)Jf \text{ for all } t \in G, \text{ and } f \in C_{\text{CPCT}}(G).$$

Proof. The maps λ_x and ρ_t commute[10] so that $\rho_t(\lambda_x(f)) = \lambda_x(\rho_t(f))$ hence the map $f \mapsto I\rho_t(f)$ is also a left Haar integral. It follows that there is some $\Delta(t)$ such that $I\rho_t(f) = \Delta(t)If$ for all $f \in C_{\text{CPCT}}(G)$. Furthermore, if J is another left Haar integral then there is some $\mu > 0$ such that $Jf = \mu If$ for all $f \in C_{\text{CPCT}}(G)$, and hence $J\rho_t(f) = \mu I\rho_t(f) = \mu\Delta(t)If = \Delta(t)Jf$ and so Δ does not depend on I.

Δ is a homomorphism since $\Delta(st)If = I\rho_{st}(f) = I\rho_s(\rho_t(f)) = \Delta(s)I\rho_t(f) = \Delta(s)\Delta(t)If$ for all $s, t \in G$, and I is not identically 0 so there is some f such that $If \neq 0$.

Finally, Δ is continuous: To establish this, first let U be an open symmetric neighbourhood of the identity with compact closure and suppose that $f \in C_{\text{CPCT}}(G)$ is supported on the compact set K. Then $K\overline{U}$ so by Urysohn's Lemma there is $g \in C_{\text{CPCT}}(G)$ mapping into $[0, 1]$ with $g(x) = 1$ for all $x \in K\overline{U}$. Now, let f be such that $If > 0$ (we know there is an f such that $If \neq 0$, and if $If < 0$ then replace f by $-f$) and for $y \in G$ and $\delta > 0$ put

$$S := \{x \in G : |\Delta(x) - \Delta(y)| < \delta\} = \{x \in G : |I\rho_x(f) - I\rho_y(f)| < \delta If\};$$

suppose that $x \in S$. Let $\epsilon > 0$ be such that $|I\rho_x(f) - I\rho_y(f)| < (\delta - \epsilon)If$. By Lemma 8.5 there is an open neighbourhood V of the identity such that $\|\rho_{xz}(f) - \rho_x(f)\|_\infty < \epsilon If/I\rho_x(g)$ for all $z \in V$. But then if $z \in V \cap U$ we have

$$\begin{aligned} |I\rho_{xz}(f) - I\rho_y(f)| &\leqslant |I\rho_{xz}(f) - I\rho_x(f)| + |I\rho_x(f) - I\rho_y(f)| \\ &\leqslant \|\rho_{xz}(f) - \rho_x f\|_\infty I\rho_x(g) + |I\rho_x(f) - I\rho_y(f)| < \delta If. \end{aligned}$$

It follows that $xz \in S$ so that S contains an open neighbourhood of x as required. The result is proved. □

We call the function Δ of this proposition the **modular function** and a group where Δ is identically 1 is called **unimodular**. Once we have shown existence of Haar integrals we shall be able to conclude that every locally compact Hausdorff group has a modular function.

Note that if G is compact and supports a left Haar integral then $\Delta(G)$ is compact since Δ is continuous, and hence G is unimodular since the only compact subgroup of $\mathbb{R}_{>0}$ is $\{1\}$. If G is discrete then it supports a left Haar integral:

$$I : C^+_{\text{CPCT}}(G) \to \mathbb{C}; f \mapsto \sum_{x \in G} f(x),$$

which is also a right Haar integral so any discrete group is unimodular. Even more easily if G is Abelian then G is unimodular since then $\rho_t(f) = \lambda_{t^{-1}}(f)$ for all $t \in G$ and $f \in C_{\text{CPCT}}(G)$, and hence $I\rho_t(f) = I\lambda_{t^{-1}}(f) = If$ for all $t \in G$ and $f \in C_{\text{CPCT}}(G)$.

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[10] ⚠ This is just *associativity* of the group operation since $\lambda_x(\rho_t(f))(y) = \rho_t(f)(x^{-1}y) = f((x^{-1}y)t) = f(x^{-1}(yt)) = \lambda_x(f)(yt) = \rho_t(\lambda_x(f))(y)$ for all $x, y, t \in G$.

On the other hand, there are non-Abelian groups that are neither compact nor discrete that are unimodular, for example $\mathrm{GL}_n(\mathbb{R})$ where a left Haar integral is given by

$$If = \int f(A)|\det A|^{-n} \prod_{1\leqslant i,j\leqslant n} dA_{ij}.$$

and dA_{ij} is Lebesgue measure on $\mathbb{R}$.

9. Existence of a Haar Integral

In this section our aim is to show that locally compact Hausdorff groups all support a Haar integral. We begin by defining a sort of approximation: for $f, \phi \in C^+_{\text{CPCT}}(G)$ with ϕ not identically 0 put

$$(9.1)\quad (f;\phi) := \inf\left\{\sum_{j=1}^n c_j : n \in \mathbb{N}; c_1,\dots,c_n \geqslant 0; y_1,\dots,y_n \in G; \text{ and } \sum_{j=1}^n c_j\lambda_{y_j^{-1}}(\phi) \geqslant f\right\}.$$

We think of this as a sort of covering number and have a lemma to record some of the basic properties.

Lemma 9.1. *Suppose that $f, g, \phi, \psi \in C^+_{\text{CPCT}}(G)$ with ϕ and ψ non-zero. Then*

(i) $(f;\phi)$ *is well-defined;*
(ii) $(f;\phi) \leqslant (g;\phi)$ *whenever* $f \leqslant g$*;*
(iii) $(f+g;\phi) \leqslant (f;\phi) + (g;\phi)$*;*
(iv) $(\mu f;\phi) = \mu(f;\phi)$ *for* $\mu \geqslant 0$*;*
(v) $(\lambda_x(f);\phi) = (f;\phi)$ *for all* $x \in G$*;*
(vi) $(f;\psi) \leqslant (f;\phi)(\phi;\psi)$*;*
(vii) $(f;\phi) \geqslant \|f\|_\infty/\|\phi\|_\infty$.

Proof. To show that $(f;\phi)$ is well-defined requires that ϕ is not identically 0 so that there is some $x_0 \in G$, $c > 0$ and some open neighbourhood U of the identity such that $\phi(x) > c$ for all $x \in x_0U$. Then since the support of f is compact it is covered by a set $\{x_1U,\dots,x_nU\}$ and so

$$f(x) \leqslant \sum_{i=1}^n \|f\|_\infty c^{-1}\phi(x_0x_i^{-1}x) = \sum_{i=1}^n \|f\|_\infty c^{-1}\lambda_{x_ix_0^{-1}}(\phi)(x),$$

whence the set on the right of (9.1) is non-empty and it is bounded below by 0 and so has an infimum.

(ii), (iii), (iv), and (v) are all immediate. Finally, for (vi) and (vii) suppose $c_1,\dots,c_n \geqslant 0$ are such that $f \leqslant \sum_{j=1}^n c_j\lambda_{y_j^{-1}}(\phi)$, so that by (ii), (iii), (iv), and (v) we have $(f;\psi) \leqslant \sum_{j=1}^n c_j(\phi;\psi)$; while by non-negativity of c_js and the fact that $\|\lambda_x(\phi)\|_\infty = \|\phi\|_\infty$ for all $x \in G$ we have $|f(x)| \leqslant \sum_{j=1}^n c_j\|\phi\|_\infty$, and (vi) follows by taking infima, while (vii) follows by taking suprema and infima. □

To make use of $(\cdot;\cdot)$ we need to fix a non-zero reference function $f_0 \in C^+_{\text{CPCT}}(G)$ and we put

$$I_\phi(f) := \frac{(f;\phi)}{(f_0;\phi)}$$

which is well-defined in view of Lemma 9.1 (vii).

Many of the properties of Lemma 9.1 translate into properties of I_ϕ. In particular, we have $I_\phi(f_1+f_2) \leqslant I_\phi(f_1)+I_\phi(f_2)$; for suitable ϕ we also have the following converse.

Lemma 9.2. *Suppose that $f_1, f_2 \in C^+_{\text{CPCT}}(G)$ and $\epsilon > 0$. Then there is a symmetric open neighbourhood of the identity V such that if $\phi \in C^+_{\text{CPCT}}(G)$ is not identically 0 and has support in V then $I_\phi(f_1)+I_\phi(f_2) \leqslant I_\phi(f_1+f_2)+\epsilon$.*

Proof. Let L be a compact symmetric neighbourhood of the identity; K be compact such that $L \operatorname{supp} f_j \subset K$ for $j \in \{1,2\}$; and by Urysohn's Lemma let $F : G \to [0,1]$ be continuous, compactly supported, and have $F(x) = 1$ for all $x \in K$. For $j \in \{1,2\}$ put

$$g_j(x) := \begin{cases} \frac{f_j(x)}{f_1(x)+f_2(x)+\epsilon F(x)} & \text{if } x \in \operatorname{supp} f_j \\ 0 & \text{otherwise} \end{cases}.$$

The functions g_j are continuous and so by Lemma 8.5 (applied twice and taking the intersection of the open sets) for $\epsilon > 0$ there is a symmetric open neighbourhood of the identity V such that

$$|g_j(yx) - g_j(x)| < \epsilon \text{ for all } y \in V, x \in G, j \in \{1,2\}. \tag{9.2}$$

Now suppose that $\phi \in C^+_{\text{CPCT}}(G)$ is not identically 0 and has support in V, and that $c_1,\dots,c_n \geqslant 0$ and $y_1,\dots,y_n \in G$ are such that

$$f_1(x)+f_2(x)+\epsilon F(x) \leqslant \sum_{i=1}^n c_i\phi(y_ix) \text{ for all } x \in G.$$

Then by (9.2) we have

$$f_j(x) \leqslant \sum_{i=1}^n c_i\phi(y_ix)g_j(x) \leqslant \sum_{i=1}^n c_i(g_j(y_i^{-1})+\epsilon)\phi(y_ix) \text{ for all } x \in G, j \in \{1,2\}.$$

However, $g_1(y_i^{-1})+g_2(y_i^{-1}) \leqslant 1$ for all $1 \leqslant i \leqslant n$, whereupon

$$(f_1;\phi)+(f_2;\phi) \leqslant \sum_{i=1}^n c_i(1+2\epsilon),$$

and so by Lemma 9.1 (iii) and (iv) and then (vi) (which gives $I_\phi(h) \leqslant (h;f_0)$ for all $h \in C^+_{\text{CPCT}}(G)$)

$$\begin{aligned} I_\phi(f_1)+I_\phi(f_2) &\leqslant (1+2\epsilon)I_\phi(f_1+f_2+\epsilon F) \\ &\leqslant (1+2\epsilon)(I_\phi(f_1+f_2)+\epsilon I_\phi(F)) \\ &\leqslant I_\phi(f_1+f_2)+(2(f_1+f_2;f_0)+(F;f_0)+2(f_1+f_2;f_0)(F;F_0))\epsilon. \end{aligned}$$

The result then follows since $\epsilon > 0$ was arbitrary. □

With this we can establish the existence of a Haar integral.

Theorem 9.3. *Suppose that G is a locally compact Hausdorff group. Then there is a left Haar integral on $C_{\text{CPCT}}(G)$.*

Proof. By complete regularity and the fact G has a compact neighbourhood there is some $f_0 \in C^+_{\text{CPCT}}(G)$ with $f_0 \not\equiv 0$. The set of maps $I : C^+_{\text{CPCT}}(G) \to \mathbb{R}_{\geqslant 0}$ such that $|I(f)| \leqslant (f; f_0)$ is compact by Tychonoff's Theorem, and the set of such maps I with $I(f_0) = 1$,

$$I(f) \leqslant I(g) \text{ for all } , f, g \in C^+_{\text{CPCT}}(G) \text{ with } f \leqslant g, \tag{9.3}$$

$$I(\mu f) = \mu I(f) \text{ for all } \mu \geqslant 0, f \in C^+_{\text{CPCT}}(G), \tag{9.4}$$

and

$$I(\lambda_x(f)) = I(f) \text{ for all } x \in G, f \in C^+_{\text{CPCT}}(G), \tag{9.5}$$

is closed and so also compact – denote it X.

For $\epsilon > 0$ and $f, f' \in C^+_{\text{CPCT}}(G)$ consider the sets

$$B(f, f'; \epsilon) := \{I \in X : |I(f + f') - I(f) - I(f')| \leqslant \epsilon\}.$$

For any $f_1, f_1', f_2, f_2', \dots, f_n, f_n' \in C^+_{\text{CPCT}}(G)$ and $\epsilon_1, \dots, \epsilon_n$, by Lemma 9.2 there are symmetric open neighbourhoods of the identity $V_1, \dots, V_n$ such that if $\phi \in C^+_{\text{CPCT}}(G)$ is not identically 0 and is supported in V_i then

$$|I_\phi(f_i + f_i') - I_\phi(f_i) - I_\phi(f_i')| < \epsilon_i. \tag{9.6}$$

The set $V := \bigcap_{i=1}^n V_i$ is also a symmetric open neighbourhood of the identity and by complete regularity there is $\phi \in C^+_{\text{CPCT}}(G)$ that is not identically 0. Then ϕ is supported in V and is not identically 0 and so I_ϕ enjoys (9.6) for all $1 \leqslant i \leqslant n$. Moreover, I_ϕ then enjoys (9.3) by Lemma 9.1 (ii); (9.4) by Lemma 9.1 (iv); (9.5) by Lemma 9.1 (v); $I_\phi(f_0) = 1$ by design; and $I_\phi(f) \leqslant (f; f_0)$ by Lemma 9.1 (vi).

We conclude that $\bigcap_{i=1}^n B(f_i, f_i', \epsilon_i)$ is non-empty which is to say the set $\{B(f, f'; \epsilon) : f, f' \in C^+_{\text{CPCT}}(G), \epsilon > 0\}$ has the finite intersection property. The sets $B(f, f'; \epsilon)$ are closed and so by compactness of X there is some I in the intersection of all the $B(f, f'; \epsilon)$s. We extend this I to $C_{\text{CPCT}}(G)$ by putting $I(f) := I(f_1) - I(f_2) + iI(f_3) - iI(f_4)$ where $f = f_1 - f_2 + if_3 - if_4$ and $f_1, f_2, f_3, f_4 \in C^+_{\text{CPCT}}(G)$. □

Given a left Haar integral we can go ahead and define a regular Borel measure on G by setting

$$\mu(A) := \inf\{I(f) : 1_A \leqslant f \text{ where } f \in C^+_{\text{CPCT}}(G)\}$$

but we shall not discuss that at much length. This is what is called a left **Haar measure**.

10. The dual group

Suppose that G is a topological group and recall that S^1 is the group of complex numbers of modulus 1 under multiplication. We write $\widehat{G}$ for the set of continuous homomorphisms $G \to S^1$, and call the elements of $\widehat{G}$ **characters**. This is naturally endowed with the structure of an Abelian group with multiplication and inversion defined by

$$(\gamma, \gamma') \mapsto (x \mapsto \gamma(x)\gamma'(x)) \text{ and } \gamma \mapsto (x \mapsto \overline{\gamma(x)}).$$

We write $1_{\widehat{G}}$ for the character taking the constant value 1 and call it the **trivial character**. On the face of it it is not clear whether or not there are *any* non-trivial characters in $\widehat{G}$, but it will turn out that (in general) there are. Indeed, it will turn out that much more than this is true.

The set $\widehat{G}$ – as a set of continuous, but not necessarily compactly supported functions – is a subset of $C(G)$ and so can be endowed with the subspace topology, when that space is considered as endowed with the **compact-open topology**, that is the topology generated by translates of the sets

$$U(K, \epsilon) := \{\gamma \in \widehat{G} : |\gamma(x) - 1| < \epsilon \text{ for all } x \in K\}$$

as K ranges compact subsets of G.

Proposition 10.1. *Suppose that G is a topological group. Then $\widehat{G}$ is an Abelian Hausdorff topological group.*

Proof. Since $|\gamma(x)-1| = |\overline{\gamma(x)}-1|$ the inversion is certainly continuous. On the other hand if $|(\gamma\lambda)(x) - 1| < \epsilon$ for all $x \in K$ then since $\gamma\lambda$ is continuous and K is compact $|\gamma\lambda - 1|$ achieves its bounds on K and hence there is some $\delta > 0$ such that $|(\gamma\lambda)(x) - 1| < \epsilon - \delta$ for all $x \in K$. But then if $\lambda' \in \lambda U(K, \delta/2)$ and $\gamma' \in \gamma U(K, \delta/2)$ we have

$$\begin{aligned} |(\gamma'\lambda')(x) - 1| &\leqslant |(\gamma'\lambda')(x) - (\gamma\lambda')(x)| + |(\gamma\lambda')(x) - (\gamma\lambda)(x)| + |(\gamma\lambda)(x) - 1| \\ &< \delta/2 + \delta/2 + \epsilon - \delta = \epsilon. \end{aligned}$$

The joint continuity of multiplication follows. The group is clearly commutative, and it is Hausdorff since if $\gamma \neq \lambda$ then there is some $x \in G$ such that $\gamma(x) \neq \lambda(x)$; put $\epsilon := |\gamma(x) - \lambda(x)|/2$ and note that $\gamma U(\{x\}, \epsilon)$ and $\lambda U(\{x\}, \epsilon)$ are disjoint open sets containing γ and λ respectively. □

We call $\widehat{G}$ endowed with the above topology the **dual group** of G.

Proposition 10.2. *Suppose that G is a compact topological group. Then $\widehat{G}$ is discrete.*

Proof. Suppose that $\gamma \in \widehat{G}$ and suppose that there is $x \in G$ is such that $\gamma(x) \neq 1$. Let $y \in G$ be such that $|\gamma(y) - 1|$ is maximal (which exists since G is compact and $x \mapsto |\gamma(x) - 1|$ is continuous) and note that by assumption this is positive. If $|\gamma(y) - 1| < 1$ then we have

$$\begin{aligned} |\gamma(y^2) - 1| = |\gamma(y)^2 - 1| &= |(2 + (\gamma(y) - 1))||\gamma(y) - 1| \\ &\geqslant (2 - |\gamma(y) - 1|)|\gamma(y) - 1| > |\gamma(y) - 1|, \end{aligned}$$

since $|\gamma(y) - 1| > 0$ by assumption. This is a contradiction. Hence $\gamma \notin U(G,1)$ *i.e.* $U(G,1) = \{1_{\widehat{G}}\}$ where $1_{\widehat{G}}$ denotes the trivial character. It follows that the group is discrete. □

Proposition 10.3. *Suppose that G is a discrete topological group. Then $\widehat{G}$ is compact.*

Proof. The set $\widehat{G}$ is a subset of the set of functions $G \to S^1$, and the latter space is compact when endowed with the product topology by Tychonoff's theorem. On the other hand since G is discrete the only compact sets in G are finite and hence the topology on $\widehat{G}$ is the topology induced by considering it as a subspace of the set of functions $G \to S^1$ with the product topology. It remains to check that $\widehat{G}$ is closed at which point it follows that it is compact. This last fact follows since the sets $\{f : G \to S^1 : f(xy) = f(x)f(y)\}$ are closed for each $x, y \in G$, and hence

$$\bigcap \{\{f : G \to S^1 : f(xy) = f(x)f(y)\} : x, y \in G\}$$

is closed. This is the set of all homomorphisms $G \to S^1$, but *all* homomorphisms are continuous since G is discrete and hence this set equals $\widehat{G}$. □

Exercise 10.4. Show that $\widehat{\mathbb{Z}} \cong \mathbb{T}$, $\widehat{\mathbb{T}} \cong \mathbb{Z}$ and $\widehat{\mathbb{R}} \cong \mathbb{R}$.

A key application of our Haar integral is then the following result establishing the fact that if G is locally compact and Hausdorff then so is $\widehat{G}$.

Theorem 10.5. *Suppose that G is a locally compact Hausdorff group. Then $\widehat{G}$ is locally compact.*

Proof. Let I be a left Haar integral on G and $f_0 \in C^+_{\mathrm{CPCT}}(G)$ have $I(f_0) = 1$. Write K for a compact set supporting f_0 and U for a compact neighbourhood of the identity (which exists since G is locally compact). Apply Urysohn's Lemma to get a continuous compactly supported $F : G \to [0,1]$ such that $F(x) = 1$ for all $x \in UK$. Consider

$$V := \{\gamma \in \widehat{G} : |\gamma(x) - 1| \leqslant \epsilon \text{ for all } x \in K\} \text{ where } \epsilon := 1/4IF\|f_0\|_\infty$$

so that V certainly contains an open neighbourhood of the identity: $U(K,\epsilon)$.

Note that if $\gamma \in V$ and $\lambda \in \overline{\gamma}V$ then by the triangle inequality we have

$$|I(f_0\lambda) - 1| = |I(f_0(\lambda - 1))| \leqslant 2\epsilon\|f_0\|_\infty IF \leqslant 1/2,$$

and hence $|I(f_0\lambda)| \geqslant 1/2$ by the triangle inequality again.

Claim. *Suppose that $\kappa, \delta > 0$. Then there is a symmetric open neighbourhood of the identity $L_{\delta,\kappa}$ such that if $|I(f_0\gamma)| \geqslant \kappa$ then $|1 - \gamma(y)| < \delta$ for all $y \in L_{\delta,\kappa}$.*

Proof. By Lemma 8.5 there is an open neighbourhood of the identity $L_{\delta,\kappa}$ (which we may assume is contained in U) such that $\|\lambda_y(f_0) - f_0\|_\infty < \delta\kappa/IF$. Importantly $L_{\delta,\kappa}$ does *not* depend on γ. But then if $y \in L_{\delta,\kappa}$ we have

$$\begin{aligned}|1 - \gamma(y)|\kappa \leqslant |(\gamma(y) - 1)I(f_0\gamma)| &= |I(f_0(\lambda_{y^{-1}}(\gamma) - \gamma)| \\ &= |I((\lambda_y(f_0) - f_0)\gamma)| \leqslant I(|\lambda_y(f_0) - f_0|) < \delta,\end{aligned}$$

since $\mathrm{supp}(\lambda_y(f_0) - f_0) \subset UK$. The claim is proved. □

We write C for the set of functions $G \to S^1$ endowed with the product topology so that C is compact. Now V is a subset of C and is closed in C since it is an intersection of closed sets:

$$\begin{aligned} V = & \bigcap \{\{f \in C : f(xy) = f(x)f(y)\} : x, y \in G\} \\ & \cap \bigcap \{\{f \in C : |f(x) - 1| \leqslant \delta\} : \delta > 0, x \in L_{\delta,1/2}\} \\ & \cap \bigcap \{f \in C : |f(x) - 1| \leqslant \epsilon\} : x \in K\}. \end{aligned}$$

Certainly all the sets on the right are closed. To see the equality note that V is a set of homomorphisms and so contained in the first big intersection; by the claim (and the fact that the trivial character is in V) V is contained in the second big intersection; and by definition of V it is contained in the third big intersection. Moreover, if $f \in C$ and f is in the first big intersection then f is a homomorphism. If f is a homomorphism *and* in the second big intersection then f is continuous, and hence in $\widehat{G}$. Finally, the last big intersection then restricts to elements of V.

Compactness of V (in the compact-open topology on $\widehat{G}$) follows if every cover of the form $\{\gamma U(K_\gamma, \delta_\gamma) : \gamma \in V\}$ (where K_γ is compact and $\delta_\gamma > 0$) has a finite subcover. Write L_γ for $L_{\delta_\gamma/2,1/2}$ and note that by compactness of K_γ there is a finite set T_γ such that $K_\gamma \subset T_\gamma L_\gamma$.

Suppose that $\lambda \in \gamma U(T_\gamma, \delta_\gamma/2) \cap V$ then $\overline{\gamma}\lambda \in U(T_\delta, \delta_\gamma/2) \cap \overline{\gamma}V$. Thus $|I(f_0\overline{\gamma}\lambda)| \geqslant 1/2$ and so the claim gives $|1 - \overline{\gamma(y)}\lambda(y)| < \delta_\gamma/2$ for all $y \in L_\gamma$. But we also have $|1 - \overline{\gamma(y)}\lambda(y)| < \delta_\gamma/2$ for all $y \in T_\gamma$ and hence by the triangle inequality $|1 - \overline{\gamma(y)}\lambda(y)| < \delta_\gamma$ for all $y \in K_\gamma$. We conclude that

$$\gamma U(T_\gamma, \delta_\gamma/2) \cap V \subset \gamma U(K_\gamma, \delta_\gamma) \cap V.$$

But then $\{\gamma U(T_\gamma, \delta_\gamma/2) : \gamma \in V\}$ is a cover of V by sets that are open in C, and hence it has a finite subcover which leads to a finite subcover of our original cover. □

Exercise 10.6. Suppose that G is a topological group. Write $H(G)$ for the set of continuous homomorphisms $G \to \mathbb{C}^*$ and show that $H(G)$ is a topological group in the compact-open topology inherited from $C(G)$. On the other hand show that we may have G locally compact and Hausdorff while $H(G)$ is *not* locally compact.

As a last result of this section we have the following.

Proposition 10.7. *Suppose that G is a locally compact Hausdorff group. Then the map*

$$G \times \widehat{G} \to S^1; (x, \gamma) \mapsto \gamma(x)$$

is continuous.

Proof. Suppose that $x \in G$ and $\gamma \in \widehat{G}$ and let $L \subset \{x' \in G : |\gamma(x') - 1| < \delta\}$ be an open set with compact closure – such exists since G is locally compact. Now suppose that $(x', \gamma') \in xL \times \gamma U(\overline{xL}, \delta)$ – an open neighbourhood of (x, γ) – then

$$\begin{aligned} |\gamma'(x') - \gamma(x)| & \leqslant |\gamma'(x') - \gamma(x')| + |\gamma(x') - \gamma(x)| \\ & = |(\gamma'\overline{\gamma})(x') - 1| + |\gamma(x^{-1}x') - 1| < 2\delta. \end{aligned}$$

It follows that the map is continuous. □

⚠ Note that this map is *not* a homomorphism since $\gamma(x)\lambda(y)$ is not in general equal to $(\gamma\lambda)(xy)$ – it is a bihomomorphism.

11. Annihilators, Bohr sets, and Pontryagin's map

Suppose that G is a topological group and $S \subset G$. The **annihilator** of S is the set

$$S^\perp := \{\gamma \in \widehat{G} : \gamma(x) = 1 \text{ for all } x \in S\}$$

which is visibly a closed (since $(\gamma, x) \mapsto \gamma(x)$ is continuous in γ by Proposition 10.7) subgroup of $\widehat{G}$.

Proposition 11.1. *Suppose that G is a topological group and H is a closed normal subgroup of G. Then there is a continuous algebraic isomorphism $\phi : \widehat{G/H} \to H^\perp$. If G is locally compact and Hausdorff then ϕ is a topological isomorphism.*

Proof. We write $q : G \to G/H$ for the usual quotient map and consider the map

$$\phi : \widehat{G/H} \to H^\perp; \gamma \mapsto \gamma \circ q.$$

First, this map is well-defined: to see this simply note that for all $x \in H$ we have $\gamma(q(x)) = \gamma(H) = \gamma(1_{G/H}) = 1$. It is certainly a homomorphism and injective since q is surjective. On the other hand if $\gamma \in H^\perp$ then γ is constant on cosets of H and the map $\tilde{\gamma} : G/H \to S^1; xH \mapsto \gamma(x)$ is a well-defined continuous homomorphism and $\phi(\tilde{\gamma}) = \gamma$. We conclude that ϕ is surjective.

The map $q : G \to G/H$ is continuous and so ϕ is continuous since $q(K)$ is compact whenever K is compact. Finally, suppose that G is locally compact and Hausdorff. Then there is an open neighbourhood U of the identity with compact closure. For a compact $K \subset G/H$, the set $\{q(xU) : x \in G\}$ is an open cover of K (since q is open) and so has a finite subcover $\{q(x_1U), \dots, q(x_mU)\}$. Let $K' := (\overline{x_1U} \cup \dots \cup \overline{x_mU}) \cap q^{-1}(K)$ which is then the union of a finite number of compact sets intersected with a closed set, and hence compact. Moreover, $q(K') = K$ and the last part of the result is proved. □

Given a set $\Lambda \subset \widehat{G}$ the set $\Lambda^\perp$ is a subset of $\widehat{\widehat{G}}$, but there is also a set

$$\Lambda^\circ := \{x \in G : \gamma(x) = 1 \text{ for all } \gamma \in \Lambda\}.$$

It will turn out that Λ° and $\Lambda^\perp$ are essentially the same, and to show this we shall need a tighter handle on the topology on G.

Suppose that G is a locally compact group, Λ is a compact subset of $\widehat{G}$, and $\delta > 0$. Then we write

$$\mathrm{Bohr}(\Lambda, \delta) := \{x \in G : |\gamma(x) - 1| < \delta \text{ for all } \gamma \in \Lambda\}$$

and call this set a **Bohr set** with frequency set Λ. A **Bohr neighbourhood** is a translate of a Bohr set.

⚠ In the literature Bohr neighbourhood is sometimes used to mean what we are calling a Bohr set, and sometimes Bohr sets are defined to be sets of the form

$$\{x \in G : |\operatorname{Arg} \gamma(x)| < \delta \text{ for all } \gamma \in \Lambda\}$$

which leads to slight differences in some estimates.

Lemma 11.2. *Suppose that G is a locally compact Hausdorff group, Λ is a compact subset of $\widehat{G}$, and $\delta > 0$. Then* $\mathrm{Bohr}(\Lambda, \delta)$ *is open.*

Proof. Fix $x_0 \in \mathrm{Bohr}(\Lambda, \delta)$. For each $\lambda \in \Lambda$, Proposition 10.7 gives us open neighbourhoods of the respective identities $U_\lambda \subset G$ and $\Gamma_\lambda \subset \widehat{G}$ such that $x_0 U_\lambda \times \lambda\Gamma_\lambda$ is a subset of $\{(x, \gamma) : |\gamma(x) - 1| < \delta\}$. The sets $\{\lambda\Gamma_\lambda : \lambda \in \Lambda\}$ form an open cover of Λ and so there is a finite subcover $\lambda_1\Gamma_{\lambda_1}, \dots, \lambda_m\Gamma_{\lambda_m}$ of Λ; let $U' := U_{\lambda_1} \cap \cdots \cap U_{\lambda_m}$. Then $x_0 U' \subset \mathrm{Bohr}(\Lambda, \delta)$, and the set is open as required. □

⚠ When G is Abelian it turns out that Bohr neighbourhoods form a base for the topology on G, however in the present level of generality they do not:

Example 11.3. Suppose that G is a non-Abelian finite simple group with the Hausdorff topology. Then $\widehat{G}$ is trivial and the Bohr neighbourhoods form a base for the indiscrete topology on G and in particular *not* for the Hausdorff topology.

Proof. Any character is a homomorphism into S^1 so its kernel is either trivial or the whole of G. Since G is non-Abelian while S^1 is Abelian we conclude that the kernel is the whole of G. Since $\widehat{G}$ is trivial the Bohr neighbourhoods are all just the whole of G and so they generate the indiscrete topology which is not the Hausdorff topology on G since G has more than one element (the one-element group is Abelian). □

Given a topological group G we write

$$\alpha : G \to \widehat{\widehat{G}}; x \mapsto (\gamma \mapsto \gamma(x)),$$

which we shall call this the **Pontryagin duality map**. ⚠ This is *not* standard.

Theorem 11.4. *Suppose that G is a locally compact Hausdorff group. Then α is a continuous homomorphism.*

Proof. The map is visibly a homomorphism and it is continuous by Lemma 11.2 given the definition of the topology on $\widehat{\widehat{G}}$. □

It turns out that if G is Abelian then α is a topological isomorphism, but to prove this we shall need a better idea of the structure of locally compact Abelian Hausdorff groups.

12. The structure of locally compact Abelian groups

In this section we shall look to develop a more detailed picture of the structure of locally compact Abelian groups.

A topological group G is said to be **monothetic** if there is a (continuous) homomorphism $\mathbb{Z} \to G$ whose image is dense in G. (We regard $\mathbb{Z}$ as discrete here.)

Proposition 12.1. *Suppose that G is a locally compact Hausdorff monothetic group. Then G is compact or else G is topologically isomorphic to $\mathbb{Z}$.*

Proof. Write Z for the image of $\mathbb{Z}$ in G and let U be a symmetric open neighbourhood of the identity such that U^2 has compact closure. Suppose $U \cap Z$ is finite and $x \in U \backslash Z$. Then for all $z \in U \cap Z$ there is an open set W_z containing x and not z and so $x \in U \cap \bigcap_{z \in U \cap Z} W_z$ is an open set containing x and not containing any of Z. This contradicts the density of Z.

We have two possibilities: if $U \cap Z$ is finite and equal to U then, again since G is Hausdorff, we have that G is discrete and so G is a quotient of $\mathbb{Z}$ and these are either finite (and so compact) or the whole of $\mathbb{Z}$.

Alternatively, $U \cap Z$ is infinite. In this case let $x \in G$ be such that $Z = \{x^n : n \in \mathbb{Z}\}$ and $N := \{x^n : n \in \mathbb{N}\}$. Since $U \cap Z$ is infinite and U is symmetric we see that U contains x^n for some arbitrarily large values of n, and hence $Z \subset NU^{-1} = NU$ and $Z \subset N^{-1}U$. Since $ZU = G$ we then have that $G \subset NU^2$ and $G \subset N^{-1}U^2$, and so for each $z \in G$ we may let $n(z) \in \mathbb{N}$ be minimal such that $z \in x^{n(z)}\overline{U^2}$.

Since $\overline{U^2}$ is compact and $G \subset N^{-1}U^2$ there is some n_0 such that

$$\overline{U^2} \subset \{x^{-1}, x^{-2}, \ldots, x^{-n_0}\}U^2.$$

In view of the above there is some $1 \leqslant i \leqslant n_0$ such that $x^{-n(z)}z \in x^{-i}U^2$, whence $z \in x^{n(z)-i}U^2 \subset x^{n(z)-i}\overline{U^2}$. By minimality of $n(z)$ it follows that $n(z) - i \leqslant 0$ and so $n(z) \leqslant n_0$. We conclude that

$$G \subset \{x, x^2, \ldots, x^{n_0}\}\overline{U^2}$$

and as a finite union of compact spaces is compact. The result is proved. □

In the next lemma we make essential use of the fact that G is Abelian, and recall that we shall write 0_G for the identity of an Abelian group G, and nK for the n-fold sum of K with itself.

Lemma 12.2. *Suppose that G is an Abelian Haudorff topological group and K is a compact symmetric neighbourhood of the identity with $G = \bigcup_{n \in \mathbb{N}} nK$. Then there is some $m \in \mathbb{N}$ such that G contains a discrete subgroup L isomorphic to $\mathbb{Z}^m$, and G/L is compact with $K \cap L = \{0_G\}$.*

Proof. Since K is compact so is $K + K$, and since K is a neighbourhood it follows that there is a finite set $X \subset K$ such that $K + K \subset X + K$. Let H be the group generated by X and note by induction that $G = K + H$. Since H is finitely generated there is a maximal $n \in \mathbb{N}_0$ such that $\mathbb{Z}^n$ is a subgroup of H; let $L \leqslant H$ be free Abelian and discrete in G, and have maximal – say $m \in \mathbb{N}_0$ – generators of any subgroup with these properties. Such exists since any free Abelian subgroup of H has at most $|X|$ generators. Since L is discrete and K is compact $K \cap L$ is finite and so by passing to a finite index subgroup of L we may assume $K \cap L = \{0_G\}$.

Write $q : G \to G/L$ for the quotient map. Since H is finitely generated, so is $q(H)$ and we can write $q(L) = T + F$ where T is a finite torsion group and F is a free Abelian group generated by, say, $y_1 + L, \ldots, y_l + L$. (This is the structure theorem for finitely generated Abelian groups.) Let H_i be the group generated by $y_i + L$ and suppose that there is some $1 \leqslant i \leqslant l$ such that $\overline{H_i}$ (the closure of H_i in G/L) is *not* compact. Then by Proposition 12.1 H_i is topologically isomorphic to $\mathbb{Z}$ *i.e.* discrete in G/L. Consider L', the group generated by y_i and the elements of L. Since F is free, L' is free and has $m+1$ generators. Since H_i is discrete, $\{L\}$ is open in H_i and there is an open set $U \subset G/L$ such that $U \cap H_i = \{L\}$. But then $q^{-1}(U) \cap L' = L$ – $\supset$ follows since $L \subset L'$ and $L \in U$ so $L \subset q^{-1}(L)$; $\subset$ follows since if $u \in L'$ then $u = zy_i + l$ for some $z \in \mathbb{Z}$ and $l \in L$, and if $q(u) \in U$ the $zy_i + L = q(zy_i + l) \in U$, but also $qy_i + L \in L'/L = H_i$, so $zy_i + L = L$ and hence $z = 0$ and $u \in L$ as required. Since L is discrete there is an open set $V \subset G$ such that $V \cap L = \{0_G\}$, and hence $q^{-1}(U) \cap V \cap L' = \{0_G\}$ and so L' is discrete contradicting maximality of m.

Thus $\overline{H_i}$ is compact for every $1 \leqslant i \leqslant l$. We saw above that $G = K + H$ and so $G/L = q(K+H) = q(K) + T + \overline{H_1} + \cdots + \overline{H_l}$. The set $q(K)$ is compact since K is compact and q is continuous; T is compact since it is finite; and the $\overline{H_i}$s were shown to be compact above. It follows that G/L is a sum of compact sets and so compact as required. □

Note that since $K \cap L = \{0_G\}$ we must have that L is closed in G: otherwise there would be some $x \in G$ such that every neighbourhood of x containing infinitely many elements of L. Let U be a symmetric neighbourhood of the identity with $U + U \subset K$. Suppose that $z, w \in (x+U) \cap L$. Then $z - w = U - U = U + U \subset K$, but also L is a subgroup and so $z - w \in L$ and hence $z - w \in K \cap L$ and we have $z = w$. Thus $x + U$ is a neighbourhood of x intersecting L in at most 1 point.

We can also identity copies of the reals in certain groups.

Lemma 12.3. *Suppose that G is connected with no infinite compact subgroup and it is locally isomorphic to $\mathbb{R}^k$, meaning that there is some neighbourhood U of 0_G, an open ball B around the origin in $\mathbb{R}^k$, and a homeomorphism $\phi : B \to U$ such that $\phi(x+y) = \phi(x)+\phi(y)$ whenever $x, y, x+y \in B$. Then G is topologically isomorphic to $\mathbb{R}^k$.*

Proof. For each $x \in \mathbb{R}^k$ there is some $n(x) \in \mathbb{N}$ such that for all $n \geqslant n(x)$ we have $x/n(x) \in B$. Suppose that $n, m \geqslant n(x)$. Then

$$\begin{aligned} n\phi(x/n) &= n\phi(\overbrace{(x/nm) + \cdots + (x/nm)}^{m \text{ times}}) \\ &= n(\overbrace{\phi(x/nm) + \cdots + \phi(x/nm)}^{m \text{ times}}) = nm\phi(x/nm) \end{aligned}$$

and similarly we have $m\phi(x/m) = nm\phi(x/nm)$ and so $\{n\phi(x/n) : n \geqslant n(x)\}$ has one element – call this element $\psi(x)$. Thus ψ is a map $\mathbb{R}^k \to G$ and our aim is to show it is a topological isomorphism. First, it is a homomorphism: if $x, y \in \mathbb{R}^k$ then for $n \geqslant \max\{n(x), n(y), n(x+y)\}$ we have

$$\psi(x) + \psi(y) = n\phi(x/n) + n\phi(y/n) = n\phi((x+y)/n) = \psi(x+y).$$

ψ is also continuous: by translation invariance it suffices to check that for sufficiently small open neighbourhoods V of 0 in G we have $\psi^{-1}(V)$ open. This follows since for $V \subset U$ we have $\psi^{-1}(V) = \phi^{-1}(V)$ which is open since ϕ is continuous. Similarly ψ is open. But then $\psi(\mathbb{R}^k)$ is an open subgroup of G which is assumed connected and so ψ is surjective. Finally, it remains to check that ψ is injective.

If $\psi(x) = 0$ then $n\phi(x/n) = 0$ for all $n \geqslant n(x)$. If $x \neq 0$ then for $m \in \mathbb{N}_0$ write D_m for the group generated by $\phi(x/n(x)2^m)$. We have some important properties of the D_ms:

(i) For $0 \leqslant a \leqslant 2^m$ the elements $a\phi(x/n(x)2^m)$ are all distinct so D_m has size at least $2^m + 1$. To see this suppose that $0 \leqslant a, a' \leqslant 2^m$ and $a\phi(x/n(x)2^m) = a'\phi(x/n(x)2^m)$, so that $\phi((a - a')x/n(x)2^m) = 0$ (since $|a - a'| \leqslant 2^m$) and so by the homeomorphism property of ϕ we have $a = a'$.

(ii) $D_m \subset 2n(x)\phi(B)$. To see this, it suffices to show that $a\phi(x/n(x)2^m) \in n(x)\phi(B)$ whenever $0 \leqslant a < n(x)2^m$ since $n(x)2^m\phi(x/n(x)2^m) = 0$. Write $a = un(x) + v$ for $0 \leqslant v < n(x)$ and $0 \leqslant u < 2^m$. Then $a\phi(x/n(x)2^m) = n(x)\phi(ux/n(x)2^m) + v\phi(x/n(x)2^m) \in 2n(x)\phi(B)$.

Now $D_m \leqslant D_{m+1}$ for all m and so $D := \bigcup_m D_m$ is also a subgroup of G and $D \subset 2n(x)\phi(B)$. Since B has compact closure, D has compact closure and it is of course infinite since the groups D_m are of unbounded size. This contradicts the assumption that G has no compact infinite subgroup. □

We also need an analogue of Proposition 11.1.

Proposition 12.4. *Suppose that G is a locally compact Hausdorff Abelian group and $H \leqslant G$ is a closed subgroup. Then $\widehat{G}/H^\perp$ is topologically isomorphic to $\widehat{H}$.*

This would follow from Proposition 11.1 if we had Pontryagin duality, but otherwise it is hard to establish surjectivity of the natural map $\widehat{G} \to \widehat{H}; \gamma \mapsto \gamma|_H$. We shall revisit this after establishing our next key result.

Theorem 12.5 (The Principal Structure Theorem)**.** *Suppose that G is a locally compact Hausdorff Abelian group. Then G has an open subgroup G_1 which is (topologically isomorphic to) the direct sum of a compact group H and $\mathbb{R}^n$ for some $n \in \mathbb{N}_0$.*

Proof. Let L be the connected component of 0_G in G. The quotient group G/L is a locally compact Hausdorff totally disconnected group and so by van Dantzig's Theorem there is a compact open subgroup K in G/L. Write $q : G \to G/L$ for the usual quotient map and let $G_1 := q^{-1}(K)$ which is an open subgroup of G. Since K is compact it contains no open subgroup of infinite index (otherwise K would be a union of infinitely many disjoint open sets – the cosets of this subgroup). Every open subgroup of G_1 contains L since L is the connected component of 0_G and so G_1 contains no open subgroup of infinite index (since q is open).

As in the proof of Proposition 11.1 there is a compact neighbourhood V in G_1 such that $q(V) = K$. The group (algebraically) generated by V meets every coset of L in G_1 and is open and so is the whole of G_1 since L is connected. It follows that we may apply Lemma 12.2 to get a discrete subgroup $H \leqslant G_1$ with H free Abelian on n generators such that G_1/H is compact.

By Proposition 12.4 we have that $\widehat{G_1}/H^\perp$ is topologically isomorphic to $\widehat{H}$, which itself is topologically isomorphic to $\mathbb{T}^d$. By Proposition 11.1 we have that $H^\perp$ is topologically isomorphic to $\widehat{G_1/H}$ and is hence discrete. Hence $\widehat{G_1}$ is locally isomorphic to $\mathbb{R}^d$ and L', the component of the identity in $\widehat{G_1}$, is open in $\widehat{G_1}$. On the other hand if Γ were a compact subgroup of $\widehat{G_1}$ then Γ° would be[11] an open subgroup of G_1, and by Proposition 11.1 we see that G_1/Γ° is infinite if and only if $(\Gamma^\circ)^\perp$ is infinite, but $\Gamma \subset (\Gamma^\circ)^\perp$ and so if Γ is infinite then Γ° has infinite index in G_1 which we know not to be true. It follows by Lemma 12.3 that L' is topologically isomorphic to $\mathbb{R}^d$.

Let $\Lambda \leqslant \widehat{G_1}$ be maximal subject to the condition $L' \cap \Lambda = \{0_{\widehat{G_1}}\}$. Since Λ has at most one element in each coset of L' we see that Λ is discrete, and of course $\Lambda + L$ is a direct sum. Moreover, $\Lambda + L' = \widehat{G_1}$. To see this, suppose there were some $\gamma \in \widehat{G_1}\backslash(\Lambda + L')$. Then by maximality there would be $\gamma_0 \in \Lambda$ and $k \in \mathbb{Z}^*$ such that $\gamma_0 + k\gamma = x \in L'$ where $x \neq 0$. Then $k(x/k - \gamma) \in \Lambda$ and $\gamma_1 := x/k - \gamma \notin L' + \Lambda$, and so again there is $\gamma_2 \in \Lambda$ and $m \in \mathbb{Z}^*$ such that $\gamma_2 + m\gamma_1 = z \in L'$ and $z \neq 0$. But then

$$k\gamma_2 + km\gamma_1 = kz \neq 0$$

and since $k\gamma_1, \gamma_2 \in \Lambda$ we see that the left is in Λ but the right in L' contradicting the fact that $\Lambda + L'$ is a direct sum. We shall be done once we have completed the proof of the duality theorem. □

13. Completing the duality theorem

We now turn to the last parts we need for the proof of the duality theorem.

Lemma 13.1. *Suppose that G is an Abelian topological group and H is an open subgroup of G. Then for every $\gamma \in \widehat{H}$ there is $\lambda \in \widehat{G}$ such that $\lambda|_H = \gamma$.*

Proof. The argument here is a typical Zorn's Lemma argument. We begin with the engine:

Claim. *Suppose that $\gamma \in \widehat{H}$, $x \in G\backslash H$, and K is the group generated by x and H. Then there is some $\lambda \in \widehat{K}$ such that $\lambda|_H = \gamma$.*

Proof. Let $k \in \mathbb{N}_0$ be minimal (when $\mathbb{N}_0$ is partially ordered by divisibility) such that $kx \in H$ (*i.e.* k is the order of $x + H$ as an element of G/H with the convention that infinite order is denoted 0), and let w be a kth root of $\gamma(kx)$ (with the convention that it is 1 if $k = 0$); define $\lambda(zx + h) := w^z\gamma(h)$ for all $z \in \mathbb{Z}$ and $h \in H$. We need to check this is well-defined so that if $zx + h = z'x + h'$ then $(z - z')x = h' - h \in H$ and so $k \mid z - z'$ (meaning $z = z'$ if $k = 0$, and hence $h = h'$) whence $w^z\gamma(h) = w^{z'}\gamma((z - z')x)\gamma(h) = w^{z'}\gamma(h')$ as required. λ is also visibly a homomorphism and the claim is proved since H is open in K, and so λ is continuous since γ is continuous and K/H is discrete. □

Let $\mathcal{L}$ be the set of pairs (K, λ) such that $H \leqslant K \leqslant G$ and $\lambda \in \widehat{K}$ has $\lambda|_H = \gamma$. This set is partially ordered by $(K, \lambda) \leqslant (K', \lambda')$ if $K \leqslant K'$ and $\lambda = \lambda'|_K$. If $\mathcal{C}$ is a chain in $\mathcal{L}$ then $K^* := \bigcup\{K : (K, \lambda) \in \mathcal{C}\}$ is a group containing H and all K with $(K, \lambda) \in \mathcal{C}$, and we can define $\lambda^*(x)$ for all $x \in K^*$ by setting $\lambda^*(x) = \lambda(x)$ whenever $(K, \lambda) \in \mathcal{C}$ and $x \in K$.

[11]This requires duality as well: in particular that $(\Gamma^\circ)^\perp = \Gamma$ when Γ is a closed subgroup. We shall revisit this when we discuss separation of characters.

Thus by Zorn's Lemma $\mathcal{L}$ has a maximal element (K, λ) and by the claim if $x \in G\backslash K$ then $\mathcal{L}$ would contain a larger element. This contradiction proves the result. □

The challenge with the above result is extending to the case where H is a closed subgroup of G, in which case we need to find a way to preserve continuity on the extension which is not a problem when H is open since then G/H is discrete.

Corollary 13.2. *Suppose that G is a discrete Abelian group. Then $\widehat{G}$ separates the points of G, and hence the Pontryagin duality map is an injection.*

Proof. Suppose that $x \neq 0_G$. Then the group generated by x, call it H, is cyclic and so there is $\gamma \in \widehat{H}$ such that $\gamma(x) \neq 1$. It follows by Lemma 13.1 that this character can be extended to G and we have the result. □

We can also use Lemma 13.1 to extend some of our other results.

Proposition 13.3. *Suppose that G and H are locally compact Hausdorff Abelian groups and $\phi : G \to H$ is a continuous homomorphism. Then the map $\phi^* : \widehat{H} \to \widehat{G}; \gamma \mapsto \gamma \circ \phi$ is a well-defined continuous homomorphism. If ϕ is surjective then ϕ^* is injective; if ϕ is open and injective then ϕ^* is surjective.*

Proof. Certainly ϕ^* is well-defined. To see that it is continuous note $(\phi^*)^{-1}(U(K, \delta)) = U(\phi(K), \delta)$, and if $K \subset G$ is compact then $\phi(K)$ is compact since ϕ is continuous. If ϕ is surjective and $\phi^*(\gamma) = 1_{\widehat{G}}$ then $\gamma(\phi(x)) = 1$ for all $x \in G$ and hence $\gamma(z) = 1$ for all $z \in H$ *i.e.* $\gamma = 1_{\widehat{H}}$.

Now if ϕ is injective and open and $\gamma \in \widehat{G}$, then $\phi(G)$ is open in H and so by Lemma 13.1 there is a continuous homomorphism $\lambda : H \to S^1$ such that $\gamma = \phi^*(\lambda)$ as required. □

Exercise 13.4. Suppose that G a finitely generated Abelian Hausdorff topological group. Use the structure theorem for finitely generated Abelian groups to show that the Pontryagin duality map is a topological isomorphism.

Proposition 13.5. *Suppose that G is a compact Abelian Hausdorff group. Then the Pontryagin duality map is a surjection.*

Proof. The map is a continuous homomorphism, and $\alpha(G)$ separates the points of $\widehat{G}$. It is also compact since G is compact and so to see it is onto we shall show that the image is dense. Since $\widehat{G}$ is discrete, the compact sets in $\widehat{G}$ are all finite so the open sets in $\widehat{\widehat{G}}$ are generated by the sets $U(K, \delta)$ where $K \subset \widehat{G}$ is finite.

Write L for the group generated by K and note that the embedding $L \to \widehat{G}$ is an open injective homomorphism and so by Proposition 13.3 there is a continuous surjective homomorphism $\phi^* : \widehat{\widehat{G}} \to \widehat{L}$. Now $\alpha(G)$ separates the points of $\widehat{G}$ and so $\alpha(G)$ separates the points of $\widehat{L}$. Since L is finitely generated the Pontryagin duality map is an isomorphism and so $\phi^*(\alpha(G))$ is dense in $\widehat{L}$ which tells us that $\alpha(G)$ is dense in $\widehat{\widehat{G}}$.

Of course the continuous image of a compact set is closed and so $\alpha(G) = \widehat{\widehat{G}}$ as required. □

It will be convenient to formalise what we have been doing and say that Pontryagin duality holds for a group G if Pontryagin's duality map is a topological isomorphism. At this point we have shown in various exercises that Pontryagin duality holds for finitely generated discrete Abelian groups; for $\mathbb{T}$; for $\mathbb{Z}$; and for $\mathbb{R}$.

Exercise 13.6. Show that if duality holds for $G_1, \dots, G_n$ then it holds for $G_1 \oplus \dots \oplus G_n$.

Rather harder that Corollary 13.2 we have the following which is usually established for compact groups as a consequence of the Peter-Weyl Theorem.

Theorem 13.7. *Suppose that G is a locally compact Hausdorff Abelian group. Then the characters on G separate points.*

Sketch proof: Suppose that $x_0 \neq 0_G$. Our aim is to construct a character $\gamma_0 \in \widehat{G}$ such that $\gamma_0(x_0) \neq 1$. We begin by choosing a continuous function with this property.

By Urysohn's Lemma and Lemma 2.2 there is a continuous function f_0 taking values in $[0, 1]$ with $f_0(0_G) = 1$ and support in a symmetric compact neighbourhood of the identity K such that $x_0 \notin K + K$.

Let I be a Haar integral for G and define an inner product on $C_{\text{CPCT}}(G)$:

$$\langle f, g\rangle := I(f\overline{g}) \text{ for all } f, g \in C_{\text{CPCT}}(G),$$

writing $\|\cdot\|$ for the induced norm; and a map

$$M : C_{\text{CPCT}}(G) \to C_{\text{CPCT}}(G); g \mapsto (y \mapsto I_x(f_0(x)g(-x + y))),$$

which is a well-defined linear map with $\|Mg\| \leqslant I|f_0|\|g\|$. In particular, if L contains the support of g then the support of Mg is contained in $K + L$.

This sort of operator is sometimes called a **convolution operator** and Mg is sometimes written $f * g$ in the literature. By design $M \not\equiv 0$ since *e.g.* $Mf \neq 0$ for any $f \in C^+_{\text{CPCT}}(G)$ that is not identically 0.

The space $C_{\text{CPCT}}(G)$ endowed with the inner product $\langle \cdot, \cdot\rangle$ has a completion which we denote H, and the operator M extends to a map $H \to H$ (which we also denote M) with the same norm so, in particular $\|Mh\| \leqslant I|f_0|\|h\|$ for all $h \in H$. We shall regard $C_{\text{CPCT}}(G)$ as a dense subset of H in the obvious way. We write $T_x : C_{\text{CPCT}}(G) \to C_{\text{CPCT}}(G); f \mapsto \lambda_x(f)$ which has $\|T_xf\| = \|f\|$ for all $f \in C_{\text{CPCT}}(G)$ by translation-invariance of the Haar integral. These maps too extend to maps $H \to H$ and also denoted T_x with $\|T_xh\| = \|h\|$ for all $h \in H$. Moreover, since G is commutative $T_xM = MT_x$ for all $x \in G$.

Since $M \not\equiv 0$ the operator norm M is not 0. Let $\epsilon > 0$ be optimised shortly and let $h \in H$ have unit norm and be such that $|\|Mh\|^2 - \|M\|^2| < \epsilon$ which is possible by definition. Then $|\langle h, M^*Mh\rangle - \|M\|^2| < \epsilon$, and so

$$\|M^*Mh - \|M\|^2h\|^2 \leqslant 2\|M\|^4 - 2\|M\|^2(\|M\|^2 - \epsilon) = 2\epsilon\|M\|^4.$$

If G is compact then the operator M is compact and we can go further than this and identify a non-trivial eigenspace of M^*M. The dimension of this space is finite and is closed under the action of the operators T_x. This gives a continuous homomorphism from G to $U_n(\mathbb{C})$, the $n \times n$ unitary matrices (where n is finite). Since G is Abelian these maps commute and can be simultaneously diagonalised which leads to the homomorphism we want. □

As a final element we have the next lemma which provides a way of bootstrapping almost homomorphism to give actual homomorphisms.

Lemma 13.8. *Suppose that G is a Hausdorff Abelian topological group containing a symmetric compact neighbourhood of the identity K with $G = \bigcup_{n\in\mathbb{N}} nK$, and $\gamma : G \to \mathbb{C}^*$ (not necessarily a homomorphism) is continuous and has*

$$\left|\frac{\gamma(x+y)}{\gamma(x)\gamma(y)} - 1\right| < \epsilon < \frac{1}{3} \text{ for all } x, y \in G.$$

Then there is $\lambda \in \widehat{G}$ such that $|\lambda(x) - \gamma(x)| = O(\epsilon)$ for all $x \in G$.

Sketch proof: The idea here is that we can take a map which is 'almost' a homomorphism and find a nearby map which is actually *is* a homomorphism.

By Urysohn's Lemma there is $F_n : G \to [0,1]$ be continuous with $F_n(x) = 1$ for all $x \in nK$ and $\operatorname{supp} F_n \subset (n+1)K$. In particular for all $x \in G$ and $\epsilon > 0$ there is n such that $I(|\lambda_x(F_n) - F_x|) < \epsilon$.

Since G is a locally compact Hausdorff group it supports a Haar integral I. Put

$$\lambda(x) := \gamma(x) \lim_{n\to\infty} \exp\left(\frac{I_w F_n(w) \log\left(\frac{\gamma(x+w)}{\gamma(w)\gamma(x)}\right)}{IF_n}\right) \text{ for all } x \in G.$$

This makes sense provided $\epsilon < 1$ since $\log z$ is well-defined and continuous on $\{z \in \mathbb{Z} : |z-1| < 1\}$, and it is a locally uniform limit of continuous functions and so continuous. Note that

$$|\lambda(x) - \gamma(x)| = O(\epsilon) \text{ for all } x \in G$$

by monotonicity of the Haar integral.

For course, for all $x, y, z \in G$ we have

$$\begin{aligned}\frac{\gamma((x+y)+w)}{\gamma(x+y)\gamma(w)} \cdot \frac{\gamma(x)\gamma(w)}{\gamma(x+w)} \cdot \frac{\gamma(y)\gamma(w)}{\gamma(y+w)} \\ = \frac{\gamma(x+(y+w))}{\gamma(x)\gamma(y+w)} \cdot \frac{\gamma(x)\gamma(w)}{\gamma(x+w)} \cdot \frac{\gamma(x)\gamma(y)}{\gamma(x+y)},\end{aligned}$$

and hence for $\epsilon < 1/3$ (which is used to ensure that $\log abc = \log a + \log b + \log c$ whenever $|a-1|, |b-1|, |c-1| < 1/3$) we have

$$\begin{aligned}\frac{\lambda(x+y)}{\lambda(x)\lambda(y)} &= \frac{\gamma(x+y)}{\gamma(x)\gamma(y)} \cdot \lim_{n\to\infty} \exp\left(\frac{I_wF_n(w)\log\left(\frac{\gamma((x+y)+w)}{\gamma(x+y)\gamma(w)}\cdot\frac{\gamma(x)\gamma(w)}{\gamma(x+w)}\cdot\frac{\gamma(y)\gamma(w)}{\gamma(y+w)}\right)}{IF_n}\right)\\ &= \frac{\gamma(x+y)}{\gamma(x)\gamma(y)} \cdot \lim_{n\to\infty} \exp\left(\frac{I_wF_n(w)\log\left(\frac{\gamma(x+(y+w))}{\gamma(x)\gamma(y+w)}\cdot\frac{\gamma(x)\gamma(w)}{\gamma(x+w)}\cdot\frac{\gamma(x)\gamma(y)}{\gamma(x+y)}\right)}{IF_n}\right)\\ &= \lim_{n\to\infty} \exp\left(\frac{I_wF_n(w)\left(\frac{\gamma(x+(y+w))}{\gamma(x)\gamma(y+w)}\right)}{IF_n} - \frac{I_wF_n(w)\log\left(\frac{\gamma(x+w)}{\gamma(x)\gamma(w)}\right)}{IF_n}\right) = 1.\end{aligned}$$

The result is proved. □

Corollary 13.9 (Pontryagin Duality)**.** *Suppose that G is a locally compact Abelian Hausdorff group. Then α is a topological isomorphism.*

Sketch proof: Theorem 13.7 gives that α is injective and then an application of Stone-Weierstrass will give that $\alpha(G)$ is dense from which one can establish the result. □

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