The University of Oxford

MSc (Mathematics and Foundations of Computer Science)

Topological Groups

Trinity Term 2020

The steps of (each) mini project are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the mini project, but should make this assumption clear in your presentation.

The aim of this project is to explore a different argument for establishing the existence and uniqueness of Haar measure in a slightly different setting.

Suppose that (T, d) is a compact metric space with metric d. We write $\mathrm{Isom}(T)$ for the set of isometries of T, that is maps $g : T \to T$ such that

$$d(x, y) = d(g(x), g(y)) \text{ for all } x, y \in T.$$

A (left) isometric action of a group G on T is a left group action of G on T such that for each $g \in G$ the map $T \to T; x \mapsto g(x)$ given by the action is an isometry. Such an action induces an action on $C(T)$ via $g(f) := f \circ g^{-1}$.

1. Verify that $\mathrm{Isom}(T)$ is a group under composition and that the induced action on $C(T)$ really is an action.

We say that a linear map $I : C(T) \to \mathbb{C}$ is a G**-Haar expectation** if $|I(f)| \leqslant \|f\|_\infty$ for all $f \in C(T)$;

$$I(g(f)) = I(f) \text{ for all } f \in C(T) \text{ and } g \in G;$$

and $I(1_T) = 1$, where 1_T denotes the indicator function of T.

Our first aim is to show the following result.

Theorem (Existence of Haar expectations)**.** *Suppose that G is a group acting isometrically on a compact metric space (T, d). Then there is a G-Haar expectation on T.*

To prove this we shall use a lovely result from combinatorics called Hall's Marriage Theorem. Suppose that $\mathcal{G}$ is a finite bipartite graph with vertex sets V and W and write $v \sim w$ to mean that there is an edge between v and w, so that for $S \subset V$ the set $\Gamma(S) := \{w \in W : v \sim w \text{ for some } v \in S\}$ is the set of neighbours of S.

Theorem (Hall's Marriage Theorem)**.** *Suppose that $\mathcal{G}$ has $|\Gamma(S)| \geqslant |S|$ for all $S \subset V$. Then there is an injective choice function $\psi : V \to W$ such that $v \sim \psi(v)$.*

2. Prove Hall's Marriage Theorem by induction on the number of edges in $\mathcal{G}$ (or otherwise).
In the metric space T and $s \in T$ we write

$$B(s, \delta) := \{t \in T : d(s, t) \leqslant \delta\} \text{ and } U(s, \delta) := \{t \in T : d(s, t) < \delta\}.$$

We say that $\mathcal{S} \subset T$ is **δ-covering** if $T \subset \bigcup_{s \in \mathcal{S}} B(s, \delta)$.

3. Suppose that $\delta > 0$, and $\mathcal{S}$ and $\mathcal{T}$ are δ-coverings of T, and $\mathcal{S}$ has minimum size amongst all δ-coverings. Then there is a function $\psi : \mathcal{S} \to \mathcal{T}$ such that $d(\psi(s), s) < 2\delta$.

Since T is compact the open cover $\{U(t, 1/n) : t \in T\}$ has a finite sub-cover $\mathcal{C}_n$, and hence T has a finite $1/n$-covering subset. It follows that there is a $1/n$-covering subset of T of minimum size; let $\mathcal{T}_n$ be such a set and let ϕ_n be the linear functional

$$C(T) \to \mathbb{C}; f \mapsto \frac{1}{|\mathcal{T}_n|} \sum_{t \in \mathcal{T}_n} f(t). \tag{1.1}$$

We can apply the sequential Banach-Alaoglu theorem to the sequence $(\phi_n)_n$ to get a subsequence $(\phi_{n_j})_j$ and some $\phi : C(T) \to \mathbb{C}$ with $|\phi(f)| \leqslant \|f\|_\infty$ and $\phi_{n_j}(f) \to \phi(f)$ for all $f \in C(T)$.

4. Show that ϕ is a G-Haar expectation.
With this we have established the existence of Haar expectations.

A δ-covering set is said to be minimal if it is δ-covering and no proper subset is δ-covering. Obviously if a set is a δ-covering set of minimum size then it is minimal, but the converse does not hold.

5. Give an example to show that if the sets $\mathcal{T}_n$ in (1.1) had been taken to be minimal instead of minimum size then the resulting limit might not be a G-Haar expectation.

For the second part of this project we should like to show that the G-Haar expectations constructed in the first part are unique in the sense of the following theorem.

Theorem (Uniqueness of G-Haar expectations)**.** *Suppose that T is a compact metric space, G acts transitively and isometrically on T, and I and J are G-Haar expectations on T. Then $I = J$.*

Given a compact metric space (T, d) we can define a metric on $\text{Isom}(T)$ by

$$d_\infty(g, h) := \sup\{d(g(x), h(x)) : x \in T\}.$$

6. Show that $\text{Isom}(T)$ is a compact topological group under the topology induced by d_∞

7. Suppose that G acts isometrically on T. Show that if I is a G-Haar expectation then I is a $\overline{G}$-Haar expectation where $\overline{G}$ is the closure of G in $\text{Isom}(T)$.

8. Suppose that T is a compact metric space, G acts transitively and isometrically on T, and I and J are G-Haar expectations on T. Then $I = J$.

One might wonder about the extent to which reference to $\overline{G}$ is necessary in the above argument.

9. Show that there are transitive faithful actions of a group G on a compact metric space T where G is *not* a compact subspace of $\mathrm{Isom}(T)$.