Quantum Theory

Sheet 1 — MT20

1. The potential energy for an electron in a hydrogen atom is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

where $-e$ is the charge of the electron, r is its distance from the nucleus, and ϵ_0 is a constant. In a circular orbit the electron has angular momentum $L = mvr$, where m is the electron mass and v is its speed. In 1913 Bohr proposed that L is *quantized*, satisfying $L = n\hbar$ where n is a positive integer.

   (a) Given that Newton's second law for circular orbits is

$$m\frac{v^2}{r} = V'(r) \ ,$$

   show that Bohr's quantization implies $r = n^2 a$, where $a = \dfrac{4\pi\epsilon_0\hbar^2}{me^2}$ is called the *Bohr radius.*

   (b) Show that the *total energy* $E = \frac{1}{2}mv^2 + V(r)$ is given by

$$E = -\frac{\hbar^2}{2ma^2} \cdot \frac{1}{n^2} \ .$$

   [*This successfully reproduces the hydrogen atom energy levels (1.3) in the lecture notes, but a full quantum mechanical treatment will only appear at the end of our course.*]

2. A particle of mass m moves in the interval $[-a, a]$ where the potential $V = V_0$ is constant. Using the stationary state Schrödinger equation show that the energy levels of the system are

$$E_n = V_0 + \frac{n^2\pi^2\hbar^2}{8ma^2} \ ,$$

where n is a positive integer, and find the corresponding normalized wave functions. Show that the wave functions are all either even or odd functions of x.

3. A particle of mass m moving on the x-axis has a (non-normalized) ground state wave function $\mathrm{sech}^2\, x$ with energy $-2\hbar^2/m$.

   (a) Show that the potential is $V(x) = -\frac{3\hbar^2}{m}\,\mathrm{sech}^2\, x$.

   (b) An excited state wave function for the particle is $\psi(x) = \tanh x \,\mathrm{sech}\, x$. What is the energy of this state?

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4. Consider a particle of mass m confined to a box in three dimensions, with potential

$$V(x,y,z) = \begin{cases} 0\ , & 0 < x < a\ ,\ 0 < y < b\ ,\ 0 < z < c\ , \\ +\infty & \text{otherwise}\ , \end{cases}$$

where (x,y,z) are Cartesian coordinates. By separating variables in the stationary state Schrödinger equation, show that the allowed energies of the particle are

$$E_{n_1,n_2,n_3} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right)\ ,$$

where n_1, n_2, n_3 are positive integers, and find the corresponding normalized wave functions. [*You may use the results for the one-dimensional box.*]

5. A particle of mass m moves on the x-axis in a potential $V(x)$, where V is an *even function* (that is $V(x) = V(-x)$). Let $\psi(x)$ be a normalized wave function satisfying the stationary state Schrödinger equation with energy E.

   (a) Show that $\tilde{\psi}(x) \equiv \psi(-x)$ is also a normalized wave function.

   (b) By considering the wave functions $\psi_\pm = \psi \pm \tilde{\psi}$, or otherwise, deduce that there is either an even or an odd wave function (or both) satisfying the same Schrödinger equation.

6. Suppose that $\Psi(x,t)$ satisfies the one-dimensional time-dependent Schrödinger equation with potential $V(x)$ (assumed real). We define $\rho(x,t) = |\Psi(x,t)|^2$ and

$$j(x,t) = \frac{\mathrm{i}\hbar}{2m}\left(\Psi\frac{\partial\overline{\Psi}}{\partial x} - \overline{\Psi}\frac{\partial\Psi}{\partial x}\right)\ .$$

   (a) Show that, as a consequence of the Schrödinger equation,

$$\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial x} = 0\ .$$

   (b) Show further that j vanishes identically if and only if there exists a nowhere zero function $\lambda(t)$ such that $\lambda(t)\Psi(x,t)$ takes only real values.

7. * (*Optional*) Verify that the *Gaussian wave packet*

$$\Psi(x,t) = \frac{1}{\pi^{1/4}\sqrt{1 + (\mathrm{i}\hbar t/m)}}\exp\left[-\frac{x^2}{2[1 + (\mathrm{i}\hbar t/m)]}\right]\ ,$$

satisfies the free Schrödinger equation and is normalized for all times t.

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