Quantum Theory

Sheet 3 — MT20

1. Define a linear operator R acting on wave functions ψ on the x-axis by

$$(R\psi)(x) = \psi(-x) \ .$$

This is called the *parity operator*.

(a) Show that R is self-adjoint, and $R^2 = \mathbb{1}$.

(b) What are the possible eigenvalues of R, and how can its eigenspaces be characterized?

(c) Suppose now that a particle of mass m moves under an even potential $V(x)$, so that $V(x) = V(-x)$.

(i) Show that R commutes with the Hamiltonian H, *i.e.* $(RH - HR)\psi = 0$ for all $\psi(x)$.

(ii) Show that $R\psi$ is an eigenstate of H with energy E if and only if ψ is. By considering $\psi \pm R\psi$, deduce that there is either an even or an odd eigenstate (or both) with energy E.

2. Show that for any infinitely differentiable function $\psi(x)$ whose Taylor series converges to $\psi(x)$, one has for all real s

$$\left(\mathrm{e}^{-\mathrm{i}sP/\hbar}\,\psi\right)(x) = \psi(x-s) \ ,$$

where P is the momentum operator. Deduce that on the subspace of such functions one has the equality of operators

$$\mathrm{e}^{-\mathrm{i}sP/\hbar}\,X\,\mathrm{e}^{\mathrm{i}sP/\hbar} = X - s\mathbb{1} \ ,$$

where X is the position operator and $\mathbb{1}$ is the identity operator.

3. (a) Show that the expectation value $\mathbb{E}_\psi(A) = \langle\psi|A\psi\rangle$ of an observable A in a state ψ is necessarily real.

(b) Show the converse result: if $\langle\psi|A\psi\rangle$ is real for all ψ then A satisfies

$$\langle\psi_1|A\psi_2\rangle = \langle A\psi_1|\psi_2\rangle \ ,$$

for all ψ_1, ψ_2, implying that A is self-adjoint.

[*Hint: look at* $\psi = \psi_1 \pm \psi_2$ *and* $\psi = \psi_1 \pm \mathrm{i}\psi_2$.]

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4. Consider the state space $\mathcal{H} = \mathbb{C}^3$, so that a wave function is a three-component column vector $\psi = (\psi_1(t), \psi_2(t), \psi_3(t))^T$. The Hamiltonian is

$$H = \hbar\omega \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix} ,$$

with Schrödinger equation $\mathrm{i}\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = H\psi$, and stationary state equation $H\psi = E\psi$.

(a) Find the stationary states of this quantum system.

(b) Consider the observable

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,$$

and suppose that at time $t = 0$ the eigenvalue 1 has just been measured.

(i) Find $\psi(t)$ at subsequent times t by solving the Schrödinger equation.

(ii) What is the probability that when A is measured at time t one again obtains the eigenvalue 1?

5. (a) Prove *Ehrenfest's Theorem*: for any observable A,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = -\frac{\mathrm{i}}{\hbar}\langle [A, H]\rangle + \langle \frac{\partial A}{\partial t}\rangle ,$$

where we have denoted expectation value $\langle A\rangle \equiv \mathbb{E}_\psi(A)$, and ψ is arbitrary. Note here that A might potentially depend explicitly on time t, hence the last term.

(b) Hence show that for the Hamiltonian $H = P^2/2m + V(X)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X\rangle = \frac{1}{m}\langle P\rangle , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle P\rangle = -\langle V'(X)\rangle ,$$

and deduce that $m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle X\rangle = -\langle V'(X)\rangle$. Do you recognize this equation?

6. The state $\psi = \psi_n$ is a normalized eigenvector for the energy level $E = E_n = (n + \frac{1}{2})\hbar\omega$ of the harmonic oscillator with Hamiltonian $H = P^2/2m + \frac{1}{2}m\omega^2X^2$.

(a) Show that

$$E = \frac{1}{2m}\mathbb{E}_\psi(P^2) + \frac{1}{2}m\omega^2\mathbb{E}_\psi(X^2) .$$

(b) By considering $\langle\psi|(P \pm \mathrm{i}m\omega X)^k\psi\rangle$ for $k = 1, 2$, and using orthogonality of eigenstates, or otherwise, show that

$$\mathbb{E}_\psi(P) = 0 = \mathbb{E}_\psi(X) , \qquad \mathbb{E}_\psi(P^2) = m^2\omega^2\mathbb{E}_\psi(X^2) = mE .$$

(c) Deduce that $\Delta_\psi(X)\Delta_\psi(P) = E/\omega$, and discuss how this relates to Heisenberg's uncertainty principle.

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