A2: COMPLEX ANALYSIS

These are notes for the later two-thirds of the course A2: Metric Spaces and Complex Analysis, covering the material on complex analysis. They are closely based on a previous version of the notes by Kevin McGerty. Contact Ben Green (Sections 1–3) or Panos Papasoglu (the rest of the course) if you have any comments or corrections.

Synopsis

Basic geometry and topology of the complex plane, including the equations of lines and circles. Extended complex plane, Riemann sphere, stereographic projection. Möbius transformations acting on the extended complex plane. Möbius transformations take circlines to circlines. [3]

Complex differentiation. Holomorphic functions. Cauchy-Riemann equations (including $z, \overline{z}$ version). Real and imaginary parts of a holomorphic function are harmonic. [2]

Recap on power series and differentiation of power series. Exponential function and logarithm function. Fractional powers–examples of multifunctions. The use of cuts as method of defining a branch of a multifunction. [3]

Path integration. Cauchy's Theorem. (Sketch of proof only–students referred to various texts for proof.) Fundamental Theorem of Calculus in the path integral/holomorphic situation. [2]

Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem. [4]

Laurent's expansion. Classification of isolated singularities. Calculation of principal parts, particularly residues. [2]

Residue Theorem. Evaluation of integrals by the method of residues (straightforward examples only but to include the use of Jordan's Lemma and simple poles on contour of integration). [3]

Conformal mappings. Riemann mapping theorem (no proof): Möbius transformations, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Joukowski's transformation). [3]

Date: November 22, 2020.

1. Geometry and topology of the complex plane

The aim of this part of the course is to study functions $f : \mathbb{C} \to \mathbb{C}$, asking what it means for them to be differentiable, how to integrate them, and looking at the applications of all this. Before we begin, we record some basic properties of the complex numbers $\mathbb{C}$, much of which is revision from Prelims.

1.1. $\mathbb{C}$ as a metric space.

We can identify $\mathbb{C}$ with the plane $\mathbb{R}^2$ by taking real and imaginary parts. Thus we have mutually inverse bijections

$$z \mapsto (\Re z, \Im z)$$

from $\mathbb{C}$ to $\mathbb{R}^2$, and

$$(x, y) \mapsto x + iy$$

from $\mathbb{R}^2$ to $\mathbb{C}$. As we saw in the first part of the course, $\mathbb{R}^2$ is a metric space with the metric induced from the Euclidean norm

$$\|(x, y)\|_2 = \sqrt{x^2 + y^2}.$$

This gives a metric on $\mathbb{C}$ by the identification $\mathbb{C} \cong \mathbb{R}^2$ described above.

If $z = \Re z + i\Im z$ is a complex number we write $|z|$ (called the *modulus*) for this Euclidean norm, that is,

$$|z| = \sqrt{(\Re z)^2 + (\Im z)^2}.$$

The distance between two points $z, w \in \mathbb{C}$ is then $|z - w|$.

Let us write down some basic properties of the modulus $|z|$. Recall that $e^{i\theta} = \cos\theta + i\sin\theta$ when $\theta \in \mathbb{R}$. For now, we will take this as the *definition* of $e^{i\theta}$, which is more-or-less what was done in Prelims Complex Analysis. Later on we will define the complex exponential function e^z and link the two concepts.

Lemma 1.1. *Let $z, w \in \mathbb{C}$. Then*

1. *$|z|^2 = z\bar{z}$, where $\bar{z}$ is the complex conjugate of z;*
2. *If $z = re^{i\theta}$, where $r \in [0, \infty)$ and $\theta \in \mathbb{R}$, then $|z| = r$;*
3. *$|zw| = |z||w|$.*

Proof. (1) If $z = a + ib$ then $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$.

(2) We have $z = r\cos\theta + ir\sin\theta$ and so

$$|z| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r.$$

(3) One can calculate directly, writing $z = a + ib$ and $w = c + id$. Alternatively, write $z = re^{i\theta}$, $w = r'e^{i\alpha}$, and then observe that $zw = rr'e^{i(\theta+\alpha)}$ and use (2). □

1.2. **Topological properties of $\mathbb{C}$.** One can make sense of the notion of open set, closure, interior and so on by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

In the language of this part of the course, $U \subset \mathbb{C}$ being open means that if $z \in U$ then some ball $B(z, \varepsilon)$, $\varepsilon > 0$, also lies in U, where

$$B(z, \varepsilon) := \{w \in \mathbb{C} : |z - w| < \varepsilon\}.$$

In complex analysis it is often convenient to work with connected open sets, and these are called domains.

Definition 1.2. A connected open subset $D \subseteq \mathbb{C}$ of the complex plane will be called a *domain*.

We saw in the metric spaces part of the course (Chapter 7) that, for open subsets of normed spaces (such as $\mathbb{R}^2$ with the Euclidean metric), the notions of connectedness and path-connectedness are the same thing. Therefore domains are always path-connected.

1.3. **Geometry of $\mathbb{C}$.** Let us take a closer look at the geometry of the complex plane in terms of the distance $|z - w|$. When we talk about lines and circles in $\mathbb{C}$, we mean sets that are lines and circles in $\mathbb{R}^2$ (under the identification of $\mathbb{R}^2$ with $\mathbb{C}$).

Lemma 1.3 (Lines). *Let $a, b \in \mathbb{C}$ be distinct complex numbers. Then the set $\{z \in \mathbb{C} : |z - a| = |z - b|\}$ is a line. Conversely, every line can be written in this form.*

Proof. Given a and b, the set of z such that $|z - a| = |z - b|$ is the set of points equidistant from a and b, which is the perpendicular bisector of the line segment $\overline{ab}$. Conversely, every line is the perpendicular bisector of some line segment. □

Remarks. Sometimes, the set of all complex numbers satisfying some given equation is called a *locus*. Thus the locus of complex numbers satisfying $|z - a| = |z - b|$ is a line. The representation of lines in the above form is very much non-unique: for example, the x-axis (the set of z with zero imaginery part) can be described as $\{z : |z - a| = |z - \bar{a}|\}$ for any complex number a.

Now we turn to circles. Evidently, the set $\{z \in \mathbb{C} : |z - c| = r\}$, where $c \in \mathbb{C}$ and $r \in (0, \infty)$, is a circle centred on c and with radius r. Conversely, every circle can be written in this form. Less obvious is the following.

Lemma 1.4. *Let $a, b \in \mathbb{C}$ be distinct complex numbers, and let $\lambda \in (0, \infty)$, $\lambda \neq 1$. Then the locus of complex numbers satisfying $|z - a| = \lambda|z - b|$ is a circle. Conversely, every circle can be written in this form.*

Proof. Without loss of generality, $b = 0$ (a translate of a circle is a circle). Now observe the identity

$$|tz + a|^2 = t(t + 1)|z|^2 - t|z - a|^2 + (t + 1)|a|^2,$$

valid for all $a, z \in \mathbb{C}$ and all $t \in \mathbb{R}$. This can be checked by a slightly tedious calculation. Applying it with $t = \lambda^2 - 1$ gives

$$|(\lambda^2 - 1)z + a| = \lambda |a|,$$

which is clearly the equation of a circle. Taking $a = -c(\lambda^2 - 1)$ and $\lambda = r/|c|$, this gives $|z - c| = r$, and so every circle can be written in this form. $\square$

Remark. Lemma 1.4 is an interesting and non-obvious fact in classical Euclidean geometry. Phrased in that language, if A, B are points in the plane, and if $\lambda \in (0, \infty)$, $\lambda \neq 1$, then the set of all points P such that $|PA|/|PB| = \lambda$ is a circle. We have just proven that this is true using complex numbers.

2. THE EXTENDED COMPLEX PLANE $\mathbb{C}_\infty$

It is a remarkable fact that it is possible to "add the point at infinity" to $\mathbb{C}$ in such a way that the resulting space $\mathbb{C}_\infty$ has pleasant analytic properties. For instance, one can extend the function $f(z) = 1/z$ to a continuous bijection on this space, by setting $f(0) = \infty$ and $f(\infty) = 0$, and one can make rigorous sense of such statements as $\infty + 1 = \infty$. The aim of this section is to study $\mathbb{C}_\infty$, which is known as the extended complex plane.

There is more than one way to proceed, and we will mostly be discussing the Riemann sphere model of $\mathbb{C}_\infty$. Another important model is the complex projective line $\mathbb{P}^1(\mathbb{C})$; we will mention this only briefly. Finally, we remark that $\mathbb{C}_\infty$ is a very basic example of a *Riemann surface*, one of the main objects of study in the Geometry of Surfaces course in Part B. We will say nothing more about this here.

2.1. **Stereographic projection.** Let

$$\mathbb{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

be the unit sphere of radius 1 centred at the origin in $\mathbb{R}^3$. View the complex plane $\mathbb{C}$ as the copy of $\mathbb{R}^2$ inside $\mathbb{R}^3$ given by the plane $\{(x, y, 0) \in \mathbb{R} : x, y \in \mathbb{R}\}$. Thus $z = x + iy$ corresponds to the point $(x, y, 0)$. Let N be the "north pole" $N = (0, 0, 1)$ of $\mathbb{S}$.

We can define a bijective map $S : \mathbb{C} \to \mathbb{S} \setminus \{N\}$ as follows. To determine $S(z)$, join z to N by a straight line, and let $S(z)$ be the point where this line meets the sphere $\mathbb{S}$. This map (or more accurately its inverse) is called stereographic projection.

It is not too hard to give an explicit formula for $S(z)$.

Lemma 2.1. *Suppose that $z = x + iy$. Then*

$$S(z) = \Big(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\Big).$$

Proof. The general point on the line joining z and N is $t(0, 0, 1) + (1 - t)(x, y, 0)$. There is a unique value of t for which this point lies on the sphere, namely $t = (x^2 + y^2 - 1)/(x^2 + y^2 + 1)$, as can be easily checked. $\square$

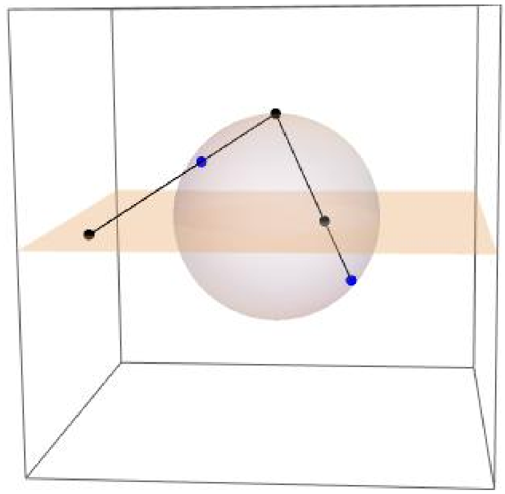

FIGURE 1. The stereographic projection map.

We remark that the same formula can be written in the alternative form

$$S(z) = \frac{1}{1+|z|^2}\big(2\Re(z), 2\Im(z), |z|^2 - 1\big).$$

As we have seen, $\mathbb{C}$ may be identified with $\mathbb{S} \setminus \{N\}$ by stereographic projection. The set $\mathbb{S} \setminus \{N\}$ has a natural metric, namely the one induced from the Euclidean metric on $\mathbb{R}^3$. This induces a metric d on $\mathbb{C}$, the unique metric on $\mathbb{C}$ such that S is an isometry. To spell it out,

$$d(z, w) := \|S(z) - S(w)\|.$$

Here is a formula for this metric.

Lemma 2.2. *For any $z, w \in \mathbb{C}$ we have*

$$d(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

Proof. Since $\|S(z)\| = \|S(w)\| = 1$ we have $\|S(z) - S(w)\|^2 = 2 - 2\langle S(z), S(w)\rangle$, where $\langle , \rangle$ is the usual Euclidean inner product on $\mathbb{R}^3$. Using the formulæ (and after a little computation),

$$\langle S(z), S(w)\rangle = 1 - \frac{2|z-w|^2}{(1+|z|^2)(1+|w|^2)}.$$

Therefore

$$\|S(z) - S(w)\|^2 = \frac{4|z-w|^2}{(1+|z|^2)(1+|w|^2)}$$

as required. □

This d is not the same as the usual metric. However, it is very similar to it. For instance, on any bounded set $\{z \in \mathbb{C} : |z| \leq K\}$ we have

$$c_1|z-w| \leq d(z,w) \leq c_2|z-w|$$

for some $c_1, c_2 > 0$ depending on K. In fact, we could take $c_2 = 2$ and $c_1 = \frac{1}{K^2}$ for $K \geq 1$ (exercise). That is, d is *strongly equivalent* to the usual metric on any such set, in the sense described in Chapter 3 of the metric spaces notes. Therefore d is equivalent (but not strongly equivalent) to the usual metric on all of $\mathbb{C}$. Recall what this means: any ball $B(z,\varepsilon)$ in the usual metric is contained in some ball $B_d(z,\varepsilon')$ in the metric d, and vice versa.

Therefore, as remarked in the metric spaces notes, notions such as limit and continuity are the same whether we work with the usual metric or with d.

2.2. **Adding in ∞.** Now it is time to add in the point at infinity, which we will call ∞ (note this is just a symbol).

Now (exercise) as $|z| \to \infty$, $S(z) \to N$. Therefore, once we have identified $\mathbb{C}$ with $\mathbb{S} \setminus \{N\}$, it is natural to identify ∞ with N, and hence $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ with the whole sphere $\mathbb{S}$. We extend the map S to a map $S : \mathbb{C}_\infty \to \mathbb{S}$ by defining $S(\infty) = N$.

Using, once again, the Euclidean metric on $\mathbb{S}$, we can extend d to a metric on $\mathbb{C}_\infty$, the unique metric for which the map S is an isometry.

Lemma 2.3. *For any $z \in \mathbb{C}$ we have*

$$d(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}.$$

Proof. By definition, $d(z,\infty) = \|S(z) - S(\infty)\| = \|S(z) - N\|$, where N is the north pole on the sphere. We may now proceed in much the same way as before, except the calculation is easier this time. The details are left as an exercise. □

We turn now to a few examples, which show that adding ∞ to $\mathbb{C}$ in this way leads to a space with nice analytic properties.

Example 2.4 (Translations)**.** Let $a \in \mathbb{C}$. Define $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $f(z) = z + a$ for $z \in \mathbb{C}$ and $f(\infty) = \infty$. Then f is a continuous bijection.

Proof. Clearly f is continuous with respect to the usual metric on $\mathbb{C}$. Therefore, restricted to $\mathbb{C}$, it is also continuous with respect to d, since d is equivalent to the usual metric.

It remains to check continuity at ∞. Let $\varepsilon > 0$. Now if $\delta > 0$ and if $d(z,\infty) < \delta$ then $|z| > \sqrt{\frac{4}{\delta^2} - 1}$ and so $|f(z)| > \sqrt{\frac{4}{\delta^2} - 1} - |a|$. This tends to ∞ as $\delta \to 0$, so by choosing δ small enough in terms of ε it will follows that $d(f(z),\infty) = \frac{2}{\sqrt{1+|f(z)|^2}} < \varepsilon$. □

Example 2.5 (Dilations)**.** Let $b \in \mathbb{C}$. Define $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $f(z) = bz$ for $z \in \mathbb{C}$ and $f(\infty) = \infty$. Then f is a continuous bijection.

Proof. This is very similar to the argument for translations and we leave the details as an exercise. □

The final example is the most interesting one.

Example 2.6 (Inversion)**.** Define $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $f(z) = 1/z$ for $z \in \mathbb{C}\backslash\{0\}$, $f(0) = \infty$ and $f(\infty) = 0$. Then f is a continuous bijection.

Proof. As before, the equivalence of d and the usual metric on $\mathbb{C}$ means that f is continuous except possibly at 0 and ∞.

We prove that f is continuous at 0, leaving the continuity at ∞ as an exercise (similar to Example 2.4).

Let $\varepsilon > 0$ be small. Then there is δ such that $\frac{2t}{\sqrt{1+t^2}} \le \varepsilon$ for all $t \in [0, \delta]$. If $|z| < \delta$, then

$$d(f(z), f(0)) = d(\frac{1}{z}, \infty) = \frac{2}{\sqrt{1 + \frac{1}{|z|^2}}} = \frac{2|z|}{\sqrt{1 + |z|^2}} \le \varepsilon.$$

This indeed shows that f is continuous at 0. □

There is a nice way to analyse Example 2.6, by considering what f looks like under the identification of $\mathbb{C}_\infty$ with the unit sphere $\mathbb{S}$. One can easily check using Lemma 2.1 that if $S(z) = (t, u, v) \in \mathbb{S}$ then $S(f(z)) = (t, -u, -v)$. That is, under the identification $S : \mathbb{C}_\infty \to \mathbb{S}$, f corresponds to the (obviously continuous) map $(t, u, v) \mapsto (t, -u, -v)$, that is to say rotation by π about the x-axis.

2.3. **Möbius maps.** In this subsection and subsequent ones we look at an important class of maps from $\mathbb{C}_\infty$ to itself, the Möbius maps.

Definition 2.7. The general linear group $\mathrm{GL}_2(\mathbb{C})$ consists of all nonsingular 2×2 matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$, with the group operation being matrix multiplication.

Each element $g \in \mathrm{GL}_2(\mathbb{C})$ gives a Möbius map $\Psi_g : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Roughly, this is given by the formula

$$\Psi_g(z) := \frac{az + b}{cz + d},$$

but one needs to be careful about ∞, as follows:

- If $c \neq 0$ then we define $\Psi_g(-d/c) = \infty$ and $\Psi_g(\infty) = a/c$;
- If $c = 0$ then we define $\Psi_g(\infty) = \infty$.

We remark that two elements $g, g' \in \mathrm{GL}_2(\mathbb{C})$ give the same Möbius map if $g = \lambda g'$ for some $\lambda \neq 0$.

An important fact about Möbius maps is that composing them corresponds to multiplying the relevant matrices. That is to say, we have the following.

Proposition 2.8 (Composition of Möbius maps)**.** *We have* $\Psi_{g_1 g_2} = \Psi_{g_1} \circ \Psi_{g_2}$. *That is,* $\mathrm{GL}_2(\mathbb{C})$ *acts on* $\mathbb{C}_\infty$ *via Möbius maps.*

One could prove this by direct calculation, but it becomes tedious to worry about ∞. The "correct" context in which to prove this result is by looking at a second model for $\mathbb{C}_\infty$, the complex projective line.

2.4. The complex projective line $\mathbb{P}^1(\mathbb{C})$.

In this subsection we briefly discuss the complex projective line, in order that we can explain a clean proof of Proposition 2.8. We begin with the definition.

Definition 2.9. Define an equivalence relation on $\mathbb{C}^2 \setminus \{0\}$ as follows. We write $(z_1, z_2) \sim (z_1', z_2')$ iff there is some $\lambda \neq 0$ such that $z_1 = \lambda z_1'$, $z_2 = \lambda z_2'$. The set of equivalence classes is denoted by $\mathbb{P}^1(\mathbb{C})$ and is called the *complex projective line*. The equivalence class of (z_1, z_2) is traditionally denoted $[z_1 : z_2]$.

The following lemma, which is easily checked, shows that one may think of $\mathbb{P}^1(\mathbb{C})$ as $\mathbb{C}$ together with an extra point.

Lemma 2.10. *Every element of* $\mathbb{P}^1(\mathbb{C})$ *is equivalent to precisely one of the elements* $[z : 1]$, $z \in \mathbb{C}$ *and* $[1 : 0]$.

It is natural to interpret $[1 : 0]$ as the point at infinity. Being a little more formal, we identify $\mathbb{C}_\infty$ with $\mathbb{P}^1(\mathbb{C})$ using the map $\iota : \mathbb{C}_\infty \to \mathbb{P}^1(\mathbb{C})$ defined by

$$\iota(z) = [z : 1], \quad \iota(\infty) = [1 : 0].$$

$\mathbb{P}^1(\mathbb{C})$ can be given a natural metric structure and it turns out that this identification is a homeomorphism, but we will not discuss this here.

Möbius maps extend naturally to maps on $\mathbb{P}^1(\mathbb{C})$.

Lemma 2.11. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. *Then is a well defined map* $\tilde{\Psi}_g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ *defined by*

$$\tilde{\Psi}_g([z_1 : z_2]) = [az_1 + bz_2 : cz_1 + dz_2]. \tag{2.1}$$

Moreover, $\tilde{\Psi}_g \circ \iota = \iota \circ \Psi_g$*; that is,* $\tilde{\Psi}_g$ *"is" the Möbius map* Ψ_g *under the identification of* $\mathbb{C}_\infty$ *with* $\mathbb{P}^1(\mathbb{C})$.

Proof. One must first check that $\tilde{\Psi}_g$ is well-defined, that is to say it does not matter which representative of the equivalence class we choose. This is straightforward.

To check that $\tilde{\Psi}_g \circ \iota = \iota \circ \Psi_g$, i.e. $\tilde{\Psi}_g(\iota(z)) = \iota(\Psi_g(z))$, there are a few different cases (corresponding to the different cases in the definition of Möbius map).

For instance, suppose $c \neq 0$ and $z = -d/c$. Then

$$\tilde{\Psi}_g(\iota(z)) = \tilde{\Psi}_g([-\frac{d}{c} : 1]) = \tilde{\Psi}_g([d : -c]) = [ad - bc : 0] = [1 : 0] = \iota(\infty) = \iota(\Psi_g(z)).$$

Here we used the fact that $ad - bc \neq 0$ (since g is nonsingular). We leave the other cases as an exercise. □

Now we prove Proposition 2.8. It may be checked by a routine direct calculation using (2.1) that $\tilde{\Psi}_{g_1 g_2} = \tilde{\Psi}_{g_1} \circ \tilde{\Psi}_{g_2}$. (The "reason" that this is true is that the action of $\tilde{\Psi}_g$ is very closely related to the usual linear action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$. Indeed, if $g\binom{z}{w} = \binom{z'}{w'}$, where $\binom{z}{w} \neq \binom{0}{0}$, where here g is acting on vectors in $\mathbb{C}^2$ via matrix multiplication in the familiar way, then $\tilde{\Psi}_g([z : w]) = [z' : w']$.)

We therefore have, for any $z \in \mathbb{C}_\infty$,

$$\iota(\Psi_{g_1 g_2}(z)) = \tilde{\Psi}_{g_1 g_2}(\iota(z)) = \tilde{\Psi}_{g_1}(\tilde{\Psi}_{g_2}(\iota(z))) = \tilde{\Psi}_{g_1}(\iota(\Psi_{g_2}(z)) = \iota(\Psi_{g_1}(\Psi_{g_2}(z)).$$

It follows that $\Psi_{g_1 g_2}(z) = \Psi_{g_1}(\Psi_{g_2}(z))$, as we were required to prove.

2.5. **Decomposing Möbius maps.** Earlier in this section, we looked at translations, dilations and inversion from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. It turns out that these are all Möbius maps, and moreover that an arbitrary Möbius map can be built from maps of these types.

That they are all Möbius maps is straightforward. Indeed,

- The translation $z \mapsto z + a$ is the Möbius map $\Psi_{T(a)}$, where $T(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$;
- The dilation $z \mapsto bz$ is the Möbius map $\Psi_{D(b)}$, where $D(b) = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$;
- The inversion $z \mapsto 1/z$ is the Möbius map Ψ_J, where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Lemma 2.12. *Every Möbius map can be written as a composition of translations, dilations and inversions.*

Proof. Let Ψ_g, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, be the Möbius map we are interested in. Suppose first that $c \neq 0$. Then, putting aside any worries about ∞, we have the following chain of compositions:

$$z \xrightarrow{\Psi_{D(c)}} cz \xrightarrow{\Psi_{T(d)}} cz + d \xrightarrow{\Psi_J} \frac{1}{cz + d} \xrightarrow{\Psi_{D(\frac{bc-ad}{c})}} \frac{b - \frac{ad}{c}}{cz + d} \xrightarrow{\Psi_{T(\frac{a}{c})}} \frac{az + b}{cz + d}.$$

This certainly suggests (very strongly!) that

$$\Psi_g = \Psi_{T(\frac{a}{c})} \circ \Psi_{D(\frac{bc-ad}{c})} \circ \Psi_J \circ \Psi_{T(d)} \circ \Psi_{D(c)}.$$

A rigorous proof follows from Proposition 2.8 and the following identity of matrices (which is of course an easy check):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T(\frac{a}{c}) \cdot D(\frac{bc-ad}{c}) \cdot J \cdot T(d) \cdot D(c).$$

The case $c = 0$ is much easier and we leave this as an exercise (in this case only a dilation and a translation are required). □

2.6. **Basic geometry of Möbius maps.** In this final subsection, we look at one key example of how Möbius maps transform $\mathbb{C}_\infty$.

Definition 2.13. A *circline* is either

- A circle in $\mathbb{C}$ (considered as a subset of $\mathbb{C}_\infty$) or
- A line in $\mathbb{C}$ (considered as a subset of $\mathbb{C}_\infty$) together with the point $\{\infty\}$.

We will see on Example Sheet 4 that circlines correspond, under stereographic projection, to circles on $\mathbb{S}$; this is one reason they are natural.

Proposition 2.14. *Möbius maps take circlines to circlines.*

Proof. By Lemma 2.12 it is enough to check this for translations, dilations and inversions. The first two are easy and left as exercises; it remains to show that inversion preserves circlines.

We consider the case of circles first. Suppose we have a circle $|z-a| = r$.

Case 1: $r \neq |a|$ and $a \neq 0$. Then 0 is not on the circle and under inversion it becomes the set of points $\{z \in \mathbb{C} : |\frac{1}{z} - a| = r\}$, or equivalently $|z - \frac{1}{a}| = \frac{r}{|a|}|z|$. By Lemma 1.4 this is also the equation of a circle (note $\frac{r}{|a|} \neq 1$).

Case 2: $r = |a|$ and $a \neq 0$. Then 0 is on the circle and under inversion it becomes the set of points $\{z \in \mathbb{C} : |z - \frac{1}{a}| = \frac{r}{|a|}|z|\}$ together with ∞. The first set is a line (Lemma 1.4) and so this is a circline.

Case 3: $a = 0$. Under inversion, the circle $|z| = r$ becomes $|z| = \frac{1}{r}$, still a circle.

Now we look at lines (plus ∞). Suppose we have a line $|z-a| = |z-b|$ together with the point ∞. (Recall from Lemma 1.3 that this is the general form for a line.) Note that the line is the perpendicular bisector of the segment $\overline{ab}$, so by extending this segment if necessary we can always choose $a, b \neq 0$.

Case 1: $|a| \neq |b|$. Then 0 does not lie on the line and under inversion it becomes the set of points $\{z \in \mathbb{C} \setminus \{0\} : |a||\frac{1}{a} - z| = |b||\frac{1}{b} - z|\}$. The point ∞ maps to 0. This set of points is a circle, by another application of Lemma 1.4.

Case 2: $|a| = |b|$. Then the line passes through 0. Under inversion it maps to $\{z \in \mathbb{C} \setminus \{0\} : |a||\frac{1}{a} - z| = |b||\frac{1}{b} - z|\} \cup \{\infty\}$. The point ∞ maps to 0. By Lemma 1.3 this is a line through 0 together with the point ∞, and hence a circline. □

3. Complex differentiability

Now we come to a crucial juncture in the course – the discussion of what it means for a function $f : \mathbb{C} \to \mathbb{C}$ to be differentiable. We begin with a quick refresher on limits, material which may be found (in the real case) in Prelims.

Suppose that $a \in \mathbb{C}$, and that U is a neighbourhood of a. That is, U contains some ball $B(a, \eta)$, $\eta > 0$, but U itself need not be open. Suppose that $F : U \setminus \{a\} \to \mathbb{C}$ is a function: that is, F is defined on U, except not at a. Then we say that $\lim_{z \to a} F(z) = L$ if the following is true: for all $\varepsilon > 0$, there is some $\delta > 0$ such that if $0 < |z - a| < \delta$ then $|F(z) - L| < \varepsilon$ (and we assume $\delta < \eta$ so that F is defined when $|z - a| < \delta$).

Exercise: $\lim_{z \to a} F(z) = L$ if and only if the function $\tilde{F}$ defined to equal L at a and F on $U \setminus \{a\}$ is continuous at a.

3.1. Complex differentiability.

With the relevant notions of limit having been recalled, we can give the definition of (complex) derivative. In fact, it is the same as the definition of real derivative, but with complex numbers in place of reals.

Definition 3.1 (Complex differentiability)**.** Let $a \in \mathbb{C}$, and suppose that $f : U \to \mathbb{C}$ is a function, where U is a neighbourhood of a. In particular, f is defined on some ball $B(a, \eta)$. Then we say that f is (complex) differentiable at a if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. If the limit exists, we write $f'(a)$ for it and call this the derivative of f at a.

Since we will be talking exclusively about functions on $\mathbb{C}$, we just use the terms differentiable/derivative and omit the word 'complex'. The following lemma collects the basic facts about derivatives. We omit the proof, which is essentially identical to the real case.

Lemma 3.2. *Let $a \in \mathbb{C}$, let U be a neighbourhood of a and let $f, g : U \to \mathbb{C}$.*

1. (Sums, products) *If f, g are differentiable at a then $f + g$ and fg are differentiable at a and $(f+g)'(a) = f'(a)+g'(a)$, $(fg)'(a) = f'(a)g(a)+f(a)g'(a)$.*
2. (Quotients) *If f, g are differentiable at a and $g(a) \neq 0$ then f/g is differentiable at a and*
$$(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$
3. (Chain rule) *If U and V are open subsets of $\mathbb{C}$ and $f : V \to U$ and $g : U \to \mathbb{C}$ are functions, where f is differentiable at $a \in V$ and g is differentiable at $f(a) \in U$, then $g \circ f$ is differentiable at a with*
$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Just as in the real case, the basic rules of differentiation stated above allow one to check that polynomial functions are differentiable: using the product rule and induction one sees that z^n has derivative nz^{n-1} for all $n \geq 0$ (as a constant obviously has derivative 0, and $f(z) = z$ has derivative 1). Then by linearity it follows every polynomial is differentiable.

Just as in the real-variable case (Prelims Analysis II) one can formulate complex differentiability in the following form, which is in fact the better form to use in most instances including the proof of Lemma 3.2.

Lemma 3.3. *Let $a \in \mathbb{C}$, let U be a neighbourhood of a and let $f : U \to \mathbb{C}$. Then f is differentiable at a, with derivative $f'(a)$, if and only if we have*

$$f(z) = f(a) + f'(a)(z - a) + \varepsilon(z)(z - a),$$

where $\varepsilon(z) \to 0$ as $z \to a$.

It is an easy exercise to check that this definition is indeed equivalent to (really just a reformulation of) the previous one.

Finally, we give an important definition.

Definition 3.4 (Holomorphic function)**.** Let $U \subseteq \mathbb{C}$ be an open set (for example, a domain). Let $f : U \to \mathbb{C}$ be a function. If f is complex differentiable at every $a \in U$, we say that f is *holomorphic* on U.

Sometimes one says that f is holomorphic at a point a; this means that there is some open set U containing a on which f is holomorphic. Some authors (particularly those from Cambridge) use the term *analytic* instead of holomorphic at this point in the development of the subject. We will use the term analytic later on to mean that f has a Taylor expansion about every point of U, but (even later) it turns out that this is the same notion as holomorphic. The situation here is rather similar to that with the terms sequentially compact and compact in the theory of metric spaces.

3.2. **Partial derivatives. Cauchy-Riemann equations.** A function from $\mathbb{C}$ to $\mathbb{C}$ may also be thought of as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, and it is useful to study what differentiability means in this language.

Let $a \in \mathbb{C}$, and let U be a neighbourhood of a. Let $f : U \to \mathbb{C}$ be a function. We abuse notation and identify $\mathbb{C} \cong \mathbb{R}^2$ in the usual way, and identify a with (a_1, a_2) (thus $a = a_1 + ia_2$). Then (again with some abuse of notation) we may think of U as an open subset of $\mathbb{R}^2$ and write $f = (u, v)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$ (the letters u, v are quite traditional in this context, and sometimes we call these the *components* of f). Another way to think of this is that $f(x + iy) = u(x, y) + iv(x, y)$.

Example 3.5. Consider the function $f(z) = z^2$ (which is holomorphic on all of $\mathbb{C}$). Since $(x + iy)^2 = (x^2 - y^2) + 2ixy$, we see that the components of f are given by $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$.

We have the partial derivatives

$$\partial_x u(a) := \lim_{h\to 0} \frac{u(a_1+h, a_2) - u(a_1, a_2)}{h}$$

(if the limit exists) and

$$\partial_y u(a) := \lim_{k\to 0} \frac{u(a_1, a_2+k) - u(a_1, a_2)}{k},$$

and similarly for v. It is important to note that h, k in these limits are *real*.

An important fact is that if f is differentiable then these partial derivatives *do* exist, and moreover they are subject to a constraint.

Theorem 3.6 (Cauchy-Riemann equations)**.** *Let $a \in \mathbb{C}$, let U be a neighbourhood of a, and let $f : U \to \mathbb{C}$ be a function which is complex differentiable at a. Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be the components of f. Then the four partial derivatives $\partial_x u$, $\partial_y u$, $\partial_x v$, $\partial_y v$ exist at a. Moreover, we have the* Cauchy-Riemann equations

$$\partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u,$$

and $f'(a) = \partial_x u(a) + i\partial_x v(a)$.

Remark. By the Cauchy-Riemann equations, there are in fact four different expressions for $f'(a)$ using the partial derivatives.

The important point to take away from Theorem 3.6 is that a complex differentiable function is much more than simply a pair of real differentiable functions. For instance, the function $f = (u, v)$ with $u(x, y) = xy$ and $v(x, y) = 0$ is as differentiable as one could wish for from the real point of view, but it is *not* a complex differentiable function since the Cauchy-Riemann equations fail to hold.

Let us turn to the proof of Theorem 3.6.

Proof. We have

$$f(z) = f(a) + f'(a)(z-a) + \varepsilon(z)(z-a),$$

where $\varepsilon(z) \to 0$ as $z \to a$. Identifying $\mathbb{C} \cong \mathbb{R}^2$ and writing $a = (a_1, a_2)$, $z = (a_1+h, a_2+k)$, $f = (u, v)$ and $f'(a) = (b_1, b_2)$, this gives

$$\begin{aligned}&(u(a_1+h, a_2+k), v(a_1+h, a_2+k))\\ &\quad = (u(a_1, a_2), v(a_1, a_2)) + (b_1, b_2)\cdot(h, k) + (\varepsilon_1(h,k), \varepsilon_2(h,k))\cdot(h,k).\end{aligned}$$

The $\cdot$ here means complex multiplication, under the identification of $\mathbb{C}$ and $\mathbb{R}^2$: thus

$$(b_1, b_2)\cdot(h, k) = (b_1 h - b_2 k, b_1 k + b_2 h)$$

(because $(b_1 + ib_2)(h + ik) = (b_1 h - b_2 k) + i(b_2 h + b_1 k)$). The functions $\varepsilon_1(h, k), \varepsilon_2(h, k)$ both tend to 0 as $\|(h, k)\| \to 0$.

Looking at the first component, we have

$$u(a_1+h, a_2+k) = u(a_1, a_2) + b_1 h - b_2 k + \varepsilon_1(h,k)h - \varepsilon_2(h,k)k.$$

In particular,

$$u(a_1+h, a_2) = u(a_1, a_2) + b_1 h + \varepsilon_1(h, 0)h.$$

Since $\varepsilon_1(h, 0) \to 0$ as $|h| \to 0$, it follows that $\partial_x u(a)$ exists and equals b_1.

Very similar arguments may be used for the other partial derivatives and we see that they all exist and that

$$\partial_y u(a) = -b_2, \ \partial_x v(a) = b_2, \ \partial_y v(a) = b_1.$$

Everything stated in the theorem now follows. □

Let us pause to give a simple example using the Cauchy-Riemann equations, which shows that complex differentiation is a much more rigid property than one might think at first sight.

Example 3.7. The function $f(z) = \overline{z}$ is not (complex) differentiable anywhere.

Proof. Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be the components of f. Then clearly $u(x, y) = x$, $v(x, y) = -y$ and so $\partial_x u = 1$, $\partial_y u = 0$, $\partial_x v = 0$, $\partial_y v = -1$. Thus $\partial_x u$ is never equal to $\partial_y v$, so the Cauchy-Riemann equations are never satisfied. □

By contrast, as a real-variable function f is as smooth as one could possibly wish.

The following basic fact will be established again later in the course in a different way and in greater generality, but we will need it in this section when establishing the basic properties of the exponential function.

Lemma 3.8. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and that f' is identically zero. Then f is constant.*

Proof. Let the components of f be (u, v). By Theorem 3.6, the partial derivative $\partial_x u$ exists and is zero. This means that, for fixed y, the function $x \mapsto u(x, y)$ is differentiable with derivative zero. By the real-variable version of the lemma we are trying to prove (which is a simple consequence of the mean value theorem) we see that $u(x, y)$ is constant as a function of x, for fixed y. Similarly, since $\partial_y u$ exists and is zero, $u(x, y)$ is constant as a function of y, for fixed x. Therefore, for arbitrary (x, y) and (x', y') we have $u(x, y) = u(x', y) = u(x', y')$, which means that u is constant. By an identical argument, v is constant. □

The Cauchy-Riemann equations are essentially the only requirement for complex differentiability. For instance one has the following converse result, which is not examinable in this course.

Theorem 3.9. *Suppose that $U \subseteq \mathbb{C}$ is open and that $f : U \to \mathbb{C}$ is a function. Let the components of f be (u, v), where $u, v : \mathbb{R}^2 \to \mathbb{R}$. Suppose that all four partial derivatives $\partial_x u$, $\partial_y u$, $\partial_x v$, $\partial_y v$ exist, are continuous in U, and satisfy the Cauchy-Riemann equations. Then f is holomorphic on U, with derivative $\partial_x u + i\partial_x v$.*

Wirtinger derivatives. Although we will not use them again in this course, we briefly mention another way to state the Cauchy-Riemann equations.

Definition 3.10. Let $f : U \to \mathbb{C}$ be a function with components (u, v), and suppose that the partial derivatives of these exist. Then we define the *Wirtinger* (partial) derivatives by

$$\partial_z f := \frac{1}{2}(\partial_x - i\partial_y)u + i\frac{1}{2}(\partial_x - i\partial_y)v$$

and

$$\partial_{\bar{z}} f := \frac{1}{2}(\partial_x + i\partial_y)u + i\frac{1}{2}(\partial_x + i\partial_y)v.$$

Lemma 3.11. *Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$. Then f satisfies the Cauchy-Riemann equations if and only if $\partial_{\bar{z}} f = 0$.*

Proof. Straightforward calculation. □

3.3. ***Real-differentiability of complex-differentiable functions.** The notion of what it means for a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ to be differentiable (occasionally called "totally differentiable") is not covered until the ASO course Introduction to Manifolds. Therefore the following remarks are nonexaminable, but you may care to read them once you have studied that ASO course, or indeed anyway.

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. As we have seen it can also be regarded as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ which, for the purposes of this section, we shall denote by $F : \mathbb{R}^2 \to \mathbb{R}^2$ to avoid confusion. Thus

$$F(x, y) = (\Re f(x + iy), \Im f(x + iy)).$$

Now F being differentiable at a point (a_1, a_2) means the following (see ASO Introduction to Manifolds): there is a linear map $F'(a_1, a_2) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$F(a_1 + h, a_2 + k) = F(a_1, a_2) + F'(a_1, a_2)(h, k) + \varepsilon(h, k)\|(h, k)\|, \tag{3.1}$$

where $\varepsilon(h, k) \to 0$ as $\|(h, k)\| \to 0$.

On the other hand if f is (complex) differentiable at $a = a_1 + ia_2$ then, by Lemma 3.3, we have

$$\begin{aligned} &F(a_1 + h, a_2 + k) = \\ &= F(a_1, a_2) + (\Re(f'(a)(h + ik)), \Im(f'(a)(h + ik))) + \varepsilon'(h, k)|h + ik|, \end{aligned} \tag{3.2}$$

where $\varepsilon'(h, k) \to 0$ as $|h + ik| \to 0$. Comparing (3.1) and (3.2), we see that (3.2) implies that indeed F is differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, with derivative given by

$$F'(a_1, a_2)(h, k) = (\Re(f'(a)(h + ik)), \Im(f'(a)(h + ik))).$$

More explicitly, if $f'(a) = b_1 + ib_2$ then we have

$$F'(a_1, a_2)(h, k) = (b_1 h - b_2 k, b_1 k + b_2 h)$$

(as in the proof of the Cauchy-Riemann equations). In other words, the matrix of the linear map $F'(a_1, a_2)$ is $\begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix}$.

The geometric idea of the derivative F' is that, near (a_1, a_2), $F(a_1+h, a_2+k)$ looks like the (affine) linear map $F(a_1, a_2) + F'(a_1, a_2)(h, k)$. The key point to observe here is that, in the case that F is a complex differentiable map (considered as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$), this linear map is not arbitrary: in fact, $\begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix}$ is a scalar multiple of a rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. That is, locally F looks like a rotation composed with a scaling. This means that F is *conformal* (preserves angles): we will be examining this concept in more detail later on.

3.4. **Harmonic functions.** In this brief section, we introduce the notion of a harmonic function and the relation of this concept to complex differentiability. We will return to this in much more detail later in the course.

We begin with the basic definitions.

Definition 3.12. Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ is a function on some open set $U \subseteq \mathbb{R}^2$ which is twice differentiable (that is, the partial derivatives themselves have partial derivatives). Then we define the *Laplacian* $\Delta u = \partial_{xx} u + \partial_{yy} u$, where $\partial_{xx} u = \partial_x(\partial_x u) = \partial_x^2 u$ and similarly for $\partial_{yy} u$.

Definition 3.13. Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ is a function on some open set $U \subseteq \mathbb{R}^2$ which is twice differentiable. Then we say that u is *harmonic* if $\Delta u = 0$.

The reason for introducing this notion here is the following important result.

Theorem 3.14. *Let $U \subseteq \mathbb{C}$ be open, and suppose that $f : U \to \mathbb{C}$ is holomorphic. Let the components of f be (u, v), and suppose that they are both twice continuously differentiable. Then u and v are harmonic.*

Proof. From the Cauchy-Riemann equations,

$$\partial_{xx} u = \partial_{xy} v \ (= \partial_x \partial_y v), \quad \partial_{yy} u = -\partial_{yx} v.$$

However, one knows (Prelims) that under the stated conditions we have the symmetry property of partial derivatives

$$\partial_{xy} v = \partial_{yx} v,$$

and the result follows. □

Let us make some further comments on this result:

- We will show later in the course that a holomorphic function such as f is in fact infinitely (complex) differentiable. (This is a rather remarkable and important fact, not true at all in real-variable analysis.) Therefore the assumption that u, v be twice differentiable is automatically satisfied and can be omitted in the statement of the theorem, once one has established that result later in the course.

- The symmetry of mixed partial derivatives means that we can factorise $\Delta = (\partial_x - i\partial_y)(\partial_x + i\partial_y)$.
- If $u, v : \mathbb{R}^2 \to \mathbb{R}$ are harmonic functions such that $f(z) = u(z) + iv(z)$ is holomorphic, we say that u and v are *harmonic conjugates*.

3.5. **Power series.** In this subsection we look at power series of a complex variable. Much of the theory parallels the real-variable theory as seen in Prelims Analysis II and the proofs go over verbatim. For the most part we will omit them.

A (formal) power series is really just a sequence $(a_n)_{n=0}^{\infty}$ of complex numbers, but we call it a power series because we are interested in understanding $\sum_{n=0}^{\infty} a_n z^n$. *A priori*, however, this sum may not converge for even a single nonzero z; nonetheless, it is conventional to write $\sum_{n=0}^{\infty} a_n z^n$, rather than be technically formal and correct and refer to the sequence $(a_n)_{n=0}^{\infty}$.

We say that a power series $\sum_{n=0}^{\infty} a_n z^n$ *converges* at a point z if the sequence of partial sums $\sum_{n=0}^{k} a_n z^n$ tends to a limit as $k \to \infty$. For such z, $\sum_{n=0}^{\infty} a_n z^n$ makes sense as an actual complex number.

Definition 3.15 (Radius of convergence)**.** Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series, and let S be the set of $z \in \mathbb{C}$ at which it converges. The *radius of convergence* of the power series is $\sup\{|z| : z \in S\}$, or ∞ if the set S is unbounded. Note that S is always nonempty since $0 \in S$.

The following result is mostly, but not entirely, in Prelims Analysis I. We will prove it again, albeit at a moderately high speed.

Proposition 3.16. *Let $\sum_{n=0} a_n z^n$ be a power series, let S be the subset of $\mathbb{C}$ on which it converges and let R be its radius of convergence. Then we have*

$$B(0, R) \subseteq S \subseteq \bar{B}(0, R). \tag{3.3}$$

The series converges absolutely on $B(0, R)$ and if $0 \le r < R$ then it converges uniformly on $\bar{B}(0, r)$. Moreover, we have

$$\frac{1}{R} = \limsup_n |a_n|^{1/n}. \tag{3.4}$$

Remark. The statement is uncontroversial when $0 < R < \infty$. Suitably interpreted, the proposition makes sense when $R = 0$ and $R = \infty$ as well, and we consider the statement to include these cases:

- When $R = 0$, one should take $B(0, R) = \emptyset$ and $\bar{B}(0, R) = \{0\}$, so (3.3) is the statement that $S \subseteq \{0\}$ in this case (which is trivial). Statement (3.4) should be taken to mean that $\limsup_n |a_n|^{1/n} = \infty$ (which is not so trivial).
- When $R = \infty$, one should take $B(0, R) = \bar{B}(0, R) = \mathbb{C}$, so (3.3) is the statement that $S = \mathbb{C}$. Statement (3.4) should be taken to mean that $\lim_{n\to\infty} |a_n|^{1/n} = 0$.

Proof. We begin with (3.3), which was essentially proven in Prelims. The containment $S \subseteq \bar{B}(0, R)$ is immediate from the definition of radius of convergence (even when $R = \infty$). The other containment $B(0, R) \subseteq S$, as well as the statement that the series converges absolutely on $B(0, R)$, are both consequences of the statement that the series converges uniformly on $\bar{B}(0, r)$ when $0 \leq r < R$. This is because $B(0, R) = \bigcup_{r<R} \bar{B}(0, r)$ (this is also true when $R = \infty$).

Let us, then, prove this statement. By definition of R, there is some w, $|w| > r$, such that $\sum_{n=0}^{\infty} a_n w^n$ converges. In particular, the terms of the sum are bounded: $|a_n w^n| \leq M$ for some M. But then if $|z| \leq r$ we have

$$|a_n z^n| = |a_n w^n||\frac{z}{w}|^n \leq M|\frac{r}{w}|^n.$$

The geometric series $\sum_n |\frac{r}{w}|^n$ converges, since $|w| > r$. Therefore, by the Weierstass test (for series) $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly for $|z| \leq r$.

Now we turn to the formula (3.4), which is not always covered in Prelims. Suppose the radius of convergence is R. Let $0 \leq r < R$. By the above, there is some w, $|w| > r$, such that $|a_n w^n| \leq M$ for all n. We may clearly assume that $M \neq 0$. Taking nth roots gives $|a_n|^{1/n}|w| \leq M^{1/n}$. Since $M^{1/n} \to 1$ as $n \to \infty$, this implies that $\limsup_n |a_n|^{1/n} \leq \frac{1}{|w|} < \frac{1}{r}$. Since $r < R$ was arbitrary, it follows that $\limsup_n |a_n|^{1/n} \leq \frac{1}{R}$. (This is perfectly legitimate when $R = \infty$ as well, with the interpretation that $\frac{1}{R} = 0$ in this case.)

In the other direction, suppose that $\limsup_n |a_n|^{1/n} = L$ and that $L \in (0, \infty)$. If $L' > L$, this means that $|a_n|^{1/n} \leq L'$ for all sufficiently large n. Therefore $|a_n z^n| \leq |L' z|^n$ (for sufficiently large n), and by the geometric series formula the series $\sum_n a_n z^n$ converges provided $|z| < \frac{1}{L'}$. Therefore $R \geq \frac{1}{L'}$. Since $L' > L$ was arbitrary, $R \geq \frac{1}{L}$, that is to say $\limsup_n |a_n|^{1/n} \geq \frac{1}{R}$.

The argument is valid with minimal changes when $L = 0$; we have shown that $R > \frac{1}{L'}$ for all $L' > 0$, and so $R = \infty$, and so the inequality $\limsup_n |a_n|^{1/n} \geq \frac{1}{R}$ remains true (with the interpretation discussed above).

When $L = \infty$, the inequality is vacuously true. Putting all this together concludes the proof. □

The next lemma is about sums and products of power series.

Lemma 3.17. *Let $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ be power series with radii of convergence R_1 and R_2 respectively. For $|z| < \min(R_1, R_2)$, write $s(z), t(z)$ for the functions to which these series converge.*

1. *The power series $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ converges in $|z| < \min(R_1, R_2)$, to $s(z) + t(z)$.*
2. *The power series $\sum_{n=0}^{\infty}(\sum_{k+l=n} a_k b_l)z^n$ converges in $|z| < \min(R_1, R_2)$, to $s(z)t(z)$.*

Proof. See Prelims Analysis II (for the real variable case; the complex case is the same). Note that $\min(R_1, R_2)$ is only a lower bound for the radius of convergence in each case – it is easy to find examples where the actual radius of convergence of the sum or product is strictly larger than this. □

Next we differentiate power series term by term.

Proposition 3.18 (Differentiation of power series)**.** *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series, with radius of convergence R. Let $s(z)$ be the function to which this series converges on $B(0, R)$. Then power series $t(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ also has radius of convergence R and on $B(0, R)$ the power series s is complex differentiable with $s'(z) = t(z)$. In particular, a power series is infinitely complex differentiable within its radius of convergence.*

Proof. This is proved in Prelims Analysis II (in the real variable case); the proof adapts to the complex case with trivial changes. It was also proved in Analysis III using integration (at least when BG lectured that course), but adapting that proof is not quite so immediate so we refer the reader to Analysis II. □

3.6. **Power series about other points.** We conclude with some remarks about power series about points z_0 other than 0, which come up frequently in complex analysis.

Such power series are functions given by an expression of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

All the results we have shown above immediately extend to these more general power series, since if

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then the function f is obtained from g simply by composing with the translation $z \mapsto z - z_0$. In particular, the chain rule shows that

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

3.7. **The exponential and trigonometric functions.** With the basic facts about complex power series under our belt, we can define some of the most important functions in mathematics as functions of a complex variable.

Example 3.19. The functions

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

$$\cos(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!},$$

and

$$\sin(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

are all holomorphic on all of $\mathbb{C}$ and their derivatives are given by term-by-term differentiation of the series. In particular,

$$\exp' = \exp, \quad \cos' = -\sin, \quad \sin' = \cos.$$

Also

$$e^{iz} = \exp(iz) = \cos z + i \sin z.$$

Note in particular that we have now properly understood Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$. Note also that

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z$$

and so

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The exponential function also satisfies the following extremely important property.

Proposition 3.20. *We have* $\exp(z + w) = \exp(z)\exp(w)$.

Proof. Fix $a \in \mathbb{C}$, and consider the function $f(z) = \exp(z)\exp(a - z)$. Differentiating and using the product rule and chain rule, we see that

$$f'(z) = \exp(z)\exp(a - z) - \exp(z)\exp(a - z) = 0.$$

Therefore, by Lemma 3.8, f is constant. It follows that

$$f(z) = f(0) = \exp(a),$$

that is to say

$$\exp(z)\exp(a - z) = \exp(a).$$

Substituting $a = z + w$ gives the stated result. □

Corollary 3.21. *For* $x, y \in \mathbb{R}$ *we have* $e^{x+iy} = e^x(\cos y + i \sin y)$.

3.8. **Logarithms.** Roughly speaking, the logarithm function is the inverse of the exponential function. In the complex plane, its definition is problematic because if $e^w = z$ then $e^{w+2\pi in} = z$ for any $n \in \mathbb{Z}$. In other words, if z has a logarithm then it has infinitely many. It turns out that there is no canonical choice of logarithm and, worse still, there is no way to define the logarithm as a holomorphic function on all of $\mathbb{C}$. We will pay closer attention to these points in the coming lectures, but for now we record the following positive result.

Proposition 3.22. *Let $D = \mathbb{C} \setminus B$, where $B = \{x \in \mathbb{R} : x \le 0\}$. That is, D is the complex plane minus the negative real axis (and 0). Define the function* $\mathrm{Log} : D \to \mathbb{C}$ *as follows: if $z = |z|e^{i\theta}$ with $\theta \in (-\pi, \pi]$ then set*

$$\mathrm{Log}(z) := \log|z| + i\theta.$$

Then Log *is holomorphic on D.*

Remarks. Note that there is a unique choice of θ, sometimes known as the principle value of the argument of z. It is also obvious from the definition that

$$\exp(\mathrm{Log}(z)) = z. \tag{3.5}$$

Proof. Let $a \in D$. We will (to nobody's surprise) prove that $\mathrm{Log}'(a) = \frac{1}{a}$.

By looking at the series expansion of $\exp$, we have the following two inequalities when $|w|$ is sufficiently small:

$$|w| \le 2|e^w - 1| \tag{3.6}$$

and

$$|(1 + w)e^{-w} - 1| \le 2|w|^2. \tag{3.7}$$

To verify (3.6), note that

$$\begin{aligned} |e^w - 1| = |w + \frac{w^2}{2!} + \cdots| &\ge |w| - |\frac{w^2}{2!} + \frac{w^3}{3!} + \cdots| \\ &= |w|(1 - |\frac{w}{2!} + \frac{w^2}{3!} + \cdots|) \\ &\ge \frac{|w|}{2}, \end{aligned}$$

where the last step is true when $|w|$ is sufficiently small since the series here converges to something $< \frac{1}{2}$. Inequality (3.7) may be proven similarly and is left as an exercise.

Let $|h|$ be small. Then, applying (3.7) with $w = h/a$, we see that

$$\big|\exp(\mathrm{Log}(a + h) - \mathrm{Log}(a) - \frac{h}{a}) - 1\big| = \big|(1 + \frac{h}{a})e^{-h/a} - 1\big| \le 2|\frac{h}{a}|^2. \tag{3.8}$$

Now Log is easily seen to be continuous at a, and so if $|h|$ is small enough we may apply (3.6) with $w = \mathrm{Log}(a+h) - \mathrm{Log}(a) - \frac{h}{a}$. Combined with (3.8), this gives

$$|\mathrm{Log}(a + h) - \mathrm{Log}(a) - \frac{h}{a}| \le 4|\frac{h}{a}|^2.$$

It therefore follows, from the definition of complex differentiability, that Log is differentiable at a with derivative $\frac{1}{a}$. □

You may care, as an exercise, to prove that Log is differentiable using Theorem 3.9, that is to say by verifying the Cauchy-Riemann equations and checking that the derivatives are suitably continuous (note, however, that we have not proven that result in this course).

4. Branch cuts

It is often the case that we study a holomorphic function on a domain $D \subseteq \mathbb{C}$ which does not extend to a function on the whole complex plane.

Example 4.1. Consider the square root "function" $f(z) = z^{1/2}$. Unlike the case of real numbers, every complex number has a square root, but just as for the real numbers, there are two possiblities unless $z = 0$. Indeed if $z = x + iy$ and $w = u + iv$ has $w^2 = z$ we see that

$$u^2 - v^2 = x; \quad 2uv = y,$$

and so

$$u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}, v^2 = \frac{-x + \sqrt{x^2 + y^2}}{2}.$$

where the requirement that u^2, v^2 are nonnegative determines the signs. Hence taking square roots we obtain the two possible solutions for w satisfying $w^2 = z$. (Note it looks like there are four possible sign combinations in the above, however the requirement that $2uv = y$ means the sign of u determines that of v.) In polars it looks simpler: if $z = re^{i\theta}$ then $w = \pm r^{1/2}e^{i\theta/2}$. Indeed this expression gives us a continuous choice of square root except at the positive real axis: for any $z \in \mathbb{C}$ we may write z uniquely as $re^{i\theta}$ where $\theta \in [0, 2\pi)$, and then set $f(z) = r^{1/2}e^{i\theta/2}$. But now for θ small and positive, $f(z) = r^{1/2}e^{i\theta/2}$ has small positive argument, but if $z = re^{(2\pi-\epsilon)i}$ we find $f(z) = r^{1/2}e^{(\pi-\epsilon/2)i}$, thus $f(z)$ in the first case is just above the positive real axis, while in the second case $f(z)$ is just above the negative real axis. Thus the function f is only continuous on $\mathbb{C}\backslash\{z \in \mathbb{C} : \Im(z) = 0, \Re(z) > 0\}$. In fact f is also holomorphic on this domain since

$$\frac{f(a+h) - f(a)}{h} = \frac{f(a+h) - f(a)}{f(a+h)^2 - f(a)^2)} = \frac{1}{f(a+h) + f(a)} \to \frac{1}{2f(a)}$$

as $h \to 0$, where we calculate the last limit using the continuity of f.

The positive real axis is called a *branch cut* for the *multi-valued function* $z^{1/2}$. If we set $g(z) = -r^{1/2}e^{i\theta/2}$ we obtain a different *branch* of this function.

By choosing different intervals for the argument (such as $(-\pi, \pi]$ say) we can take different cuts in the plane and obtain more *branches* of the function $z^{1/2}$ defined on their complements.

We formalize these concepts as follows:

Definition 4.2. A *multi-valued function* or *multifunction* on a subset $U \subseteq \mathbb{C}$ is a map $f : U \to \mathcal{P}(\mathbb{C})$ assigning to each point in U a subset[1] of the complex numbers. A *branch* of f on a subset $V \subseteq U$ is a function $g : V \to \mathbb{C}$ such that $g(z) \in f(z)$, for all $z \in V$. If g is continuous (or holomorphic) on V we refer to it as a continuous, (respectively holomorphic) branch of f.

[1] We use the notation $\mathcal{P}(X)$ to denote the *power set* of X, that is, the set of all subsets of X.

We will primarily be interested in branches of multifunctions which are holomorphic.

Remark 4.3. In order to distinguish between multifunctions and functions, it is sometimes useful to introduce some notation: if we wish to consider $z \mapsto z^{1/2}$ as a multifunction, then to emphasize that we mean a multifunction we will write $[z^{1/2}]$. Thus $[z^{1/2}] = \{w \in \mathbb{C} : w^2 = z\}$. Similarly we write $[\mathrm{Log}(z)] = \{w \in \mathbb{C} : e^w = z\}$. This is not a uniform convention in the subject, but is used, for example, in the text of Priestley.

Thus the square root $z \mapsto [z^{1/2}]$ is a multifunction, and we saw above that we can obtain holomorphic branches of it on a cut plane $\mathbb{C}\backslash R$ where $R = \{te^{i\theta} : t \in \mathbb{R}_{\geq 0}\}$. The point here is that both the origin and infinity as "branch points" for the multifunction $[z^{1/2}]$.

Definition 4.4. Suppose that $f\colon U \to \mathcal{P}(\mathbb{C})$ is a multi-valued function defined on an open subset U of $\mathbb{C}$. We say that $z_0 \in U$ is not a branch point of f if there is an open disk[2] $D \subseteq U$ containing z_0 such that there is a holomorphic branch of f defined on $D\backslash\{z_0\}$. We say z_0 is a *branch point* otherwise. When $\mathbb{C}\backslash U$ is bounded, we say that f does not have a branch point at ∞ if there is a holomorphic branch of f defined on $\mathbb{C}\backslash B(0,R) \subseteq U$ for some $R > 0$. Otherwise we say that ∞ is a branch point of f.

A *branch cut* for a multifunction f is a curve in the plane on whose complement we can pick a holomorphic branch of f. Thus a branch cut must contain all the branch points.

Example 4.5. An important example of a multi-valued function which we have already discussed is the complex logarithm: as a multifunction we have $[\mathrm{Log}(z)] = \{\log(|z|) + i(\theta + 2n\pi) : n \in \mathbb{Z}\}$ where $z = |z|e^{i\theta}$. To obtain a branch of the multifunction we must make a choice of argument function $\arg\colon \mathbb{C} \to \mathbb{R}$,we may define

$$\mathrm{Log}(z) = \log(|z|) + i\arg(z),$$

which is a continuous function away from the branch cut we chose. By convention, the *principal branch* of Log is defined by taking $\arg(z) \in (-\pi, \pi]$.

We note that $L(z)$ is also holomorphic. Indeed for small $h \neq 0$, $L(a+h) \neq L(a)$ and

$$\frac{L(a+h) - L(a)}{h} = \frac{L(a+h) - L(a)}{\exp(L(a+h) - \exp(L(a))},$$

since

$$\lim h \to 0 \frac{\exp(L(a+h)) - \exp(L(a)}{L(a+h) - L(a)} = \exp'(L(a)) = a$$

since when $h \to 0$, $L(a+h) - L(a) \to 0$ by the continuity of L. So we have $L'(a) = 1/a$.

[2]In fact any simply-connected domain – see our discussion of the homotopy form of Cauchy's theorem.

Another important class of examples of multifunctions are the *fractional power* multifunctions $z \mapsto [z^\alpha]$ where $\alpha \in \mathbb{C}$: These are given by

$$z \mapsto \exp(\alpha \cdot [\text{Log}(z)]) = \{\exp(\alpha \cdot w) : w \in \mathbb{C}, e^w = z\}$$

Note this is includes the square root multifunction we discussed above, which can be defined without the use of exponential function. Indeed if $\alpha = m/n$ is rational, $m \in \mathbb{Z}, n \in \mathbb{Z}_{>0}$, then $[z^\alpha] = \{w \in \mathbb{C} : w^m = z^n\}$. For $\alpha \in \mathbb{C}\backslash\mathbb{Q}$ however we can only define $[z^\alpha]$ using the exponential function. Clearly from its definition, anytime we choose a branch $L(z)$ of $[\text{Log}(z)]$ we obtain a corresponding branch $\exp(\alpha.L(z))$ of $[z^\alpha]$. If $L(z)$ is the principal branch of $[\text{Log}(z)]$ then the corresponding branch of $[z^\alpha]$ is called the *principal branch* of $[z^\alpha]$.

Example 4.6. Let $F(z)$ be the multi-function

$$[(1+z)^\alpha] = \{\exp(\alpha.w) : w \in \mathbb{C}, \exp(w) = 1+z\}.$$

Using $L(z)$ the principal branch of $[\text{Log}(z)]$ we obtain a branch $f(z)$ of $[(1+z)^\alpha]$ given by $f(z) = \exp(\alpha \cdot L(1+z))$. Let $\binom{\alpha}{k} = \frac{1}{k!}\alpha\cdot(\alpha-1)\dots(\alpha-k+1)$. We want to show that a version of the binomial theorem holds for this branch of the multifunction $[(1+z)^\alpha]$. Let

$$s(z) = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k,$$

By the ratio test, $s(z)$ has radius of convergence equal to 1, so that $s(z)$ defines a holomorphic function in $B(0,1)$. Moreover, you can check using the properties of power series established in a previous section, that within $B(0,1)$, $s(z)$ satisfies $(1+z)s'(z) = \alpha \cdot s(z)$.

Now $f(z)$ is defined on $\mathbb{C}\backslash(-\infty,-1)$, and hence on all of $B(0,1)$. Moreover $f'(z) = \alpha/1+z$. We claim that within the open ball $B(0,1)$ the power series $s(z) = \sum_{n=0}^{\infty}\binom{\alpha}{k}z^k$ coincides with $f(z)$. Indeed if we set

$$g(z) = s(z)\exp(-f(z))$$

then $g(z)$ is holomorphic for every $z \in B(0,1)$ and by the chain rule

$$g'(z) = (s'(z) - s(z)f'(z))\exp(-f(z)) = 0$$

since $s'(z) = \frac{\alpha \cdot s(z)}{1+z}$. Also $g(0) = 1$ so, since $B(0,1)$ is connected g is constant and $s(z) = f(z)$.

Here we use the fact that if a holomorphic function g has $g'(z) = 0$ on an open set then it is constant. We have already proven this when the open set is $\mathbb{C}$ and we will prove it soon in general.

Example 4.7. Another example of a natural multifunction to consider in this context is $[\arg(z)] := \{\theta \in \mathbb{R} : z = |z|e^{i\theta}\}$ defined on $\mathbb{C} \setminus \{0\}$. Clearly if $z = |z|e^{i\theta}$ then $\arg(z)$ is equal to the set $\{\theta + 2n\pi : n \in \mathbb{Z}\}$.

We claim that there is no continuous branch of $[\arg(z)]$ on $\mathbb{C}\setminus\{0\}$. Indeed consider the circle $S = \{z : |z| = 1\}$. Suppose that $f(z)$ is a continuous branch of [arg(z)] defined on S. Let's say that $f(1) = 2n\pi, n \in \mathbb{Z}$. Consider $g : [0, 2\pi) \to \mathbb{R}$ given by $g(t) = 2n\pi + t$. Then $f(e^{i0}) = g(0) = 2n\pi$. Since f is continuous there is some $\delta > 0$ such that $f(e^{it}) = g(t)$ for all $t \in [0, \delta)$. Indeed it suffices to pick δ so that when $|t-s| < \delta$ we have $|f(e^{it})-f(e^{is})| < 1$. Consider now the set $A = \{t : f(e^{it}) = g(t)\} \subseteq [0, 2\pi)$. This is an open and closed subset of $[0, 2\pi)$, so, since $[0, 2\pi)$ is connected, $A = [0, 2\pi)$. But then $\lim_{t\to 2\pi} f(e^{it}) = 2(n + 1)\pi \neq f(1)$, while $\lim_{t\to 2\pi} e^{it} = 1$, so f is not continuous.

On the other hand one sees easily that it is possible to define a continuous branch $f(z)$ of [arg(z)] on $\mathbb{C} \setminus [0, -\infty)$, for example by choosing $f(z)$ to be the unique element of [arg(z)] $\cap (-\pi, \pi)$.

The argument multifunction is closely related to the logarithm. There is a continuous branch of $[\mathrm{Log}(z)]$ on a set U if and only if there is continuous branch of [arg(z)] on U. Indeed if $f(z)$ is a continuous branch of [arg(z)] on U we may define a continuous branch of $[\mathrm{Log}(z)]$ by $g(z) = log|z| + if(z)$, and conversely given $g(z)$ we may define $f(z) = \Im(g(z))$.

Example 4.8. A more interesting example is the function $f(z) = [(z^2 - 1)^{1/2}]$. Using the principal branch of the square root function, we obtain a branch f_1 of f on the complement of $E = \{z \in \mathbb{C} : z^2 - 1 \in (-\infty, 0]\}$, which one calculates is equal to $[-1, 1] \cup i\mathbb{R}$.

To find another branch, note that we may write $f(z) = \sqrt{z-1}\sqrt{z+1}$, thus we can take the principal branch of the square root for each of these factors. More explicitly, if we write $z = 1+re^{i\theta_1} = -1+se^{i\theta_2}$ where $\theta_1, \theta_2 \in (-\pi, \pi]$ then we get a branch of f given by $f_2(z) = \sqrt{rs}.e^{i(\theta_1+\theta_2)/2}$. Now the factors are discontinuous on $(-\infty, 1]$ and $(\infty, -1]$ respectively, however let us examine the behavior of their product: If z crosses the negative real axis at $\Im(z) < -1$ then θ_1 and θ_2 both jump by 2π, so that $(\theta_1 + \theta_2)/2$ jumps by 2π, and hence $\exp((\theta_1 + \theta_2)/2)$ is in fact continuous. On the other hand, if we cross the segment $(-1, 1)$ then only the factor $\sqrt{z-1}$ switches sign, so our branch is discontinuous there. Thus our second branch of f is defined away from the cut $[-1, 1]$.

Example 4.9. The branch points of the complex logarithm are 0 and infinity: indeed if $z_0 \neq 0$ then we can find a half-plane $H = \{z \in \mathbb{C} : \Im(az) > 0\}$, for some $a \in \mathbb{C}$, $|a| = 1$, such that $z_0 \in H$. We can chose a continuous choice of argument function on H, and this gives a holomorphic branch of Log defined on H and hence on the disk $B(z_0, r)$ for r sufficiently small. The logarithm also has a branch point at infinity, since we cannot chose a continuous argument function on $\mathbb{C}\backslash B(0, R)$ for any $R > 0$. (We will return to this point when discussing the winding number later in the course.)

Note that if $f(z) = [\sqrt{z^2-1}]$ then the second of our branches f_2 discussed above shows that f does not have a branch point at infinity, whereas both 1 and -1 are branch points – as we move in a sufficiently small circle around we cannot make a continuous choice of branch. One can given a rigorous proof of this using the branch f_2: given any branch g of $[\sqrt{z^2-1}]$ defined on $B(1,r)$ for $r<1$ one proves that $g=\pm f_2$ so that g is not continuous on $B(0,r)\cap(-1,1)$. See Problem Sheet 5, question 1, for more details.

Example 4.10. A more sophisticated point of view on branch points and cuts uses the theory of Riemann surfaces. As a first look at this theory, consider the multifunction $f(z) = [\sqrt{z^2-1}]$ again. Let $\Sigma = \{(z,w) \in \mathbb{C}^2 : w^2 = z^2-1\}$ (this is an example of a Riemann surface). Then we have two maps from Σ to $\mathbb{C}$, projecting along the first and second factor: $p_1(z,w) = z$ and $p_2(z,w) = w$. Now if $g(z)$ is a branch of f, it gives us a map $G\colon \mathbb{C} \to \Sigma$ where $G(z) = (z, g(z))$. If we take $f_2(z) = \sqrt{z-1}\sqrt{z+1}$ (using the principal branch of the square root function in each case, then let $\Sigma_+\{(z,f_2(z)) : z \notin [-1,1]\}$ and $\Sigma_- = \{(z,-f_2(z)) : z \notin [-1,1]\}$, then $\Sigma_+ \cup \Sigma_-$ covers all of Σ apart from the pairs (z,w) where $z\in[-1,1]$. For such z we have $w = \pm i\sqrt{1-z^2}$, and Σ is obtained by gluing together the two copies Σ_+ and Σ_- of the cut plane $\mathbb{C}\backslash[-1,1]$ along the cut locus $[-1,1]$. However, we must examine the discontinuity of g in order to see how this gluing works: the upper side of the cut in Σ_+ is glued to the lower side of the cut in Σ_- and similarly the lower side of the cut in Σ_+ is glued to the upper side of Σ_-.

Notice that on Σ we have the (single-valued) function $p_2(z,w) = w$, and any map $q\colon U \to \Sigma$ from an open subset U of $\mathbb{C}$ to Σ such that $p_1 \circ q(z) = z$ gives a branch of $f(z) = \sqrt{z^2-1}$ given by $p_2 \circ q$. Such a function is called a *section* of p_1. Thus the multi-valued function on $\mathbb{C}$ becomes a single-valued function on Σ, and a branch of the multifunction corresponds to a section of the map $p_1\colon \Sigma \to \mathbb{C}$. In general, given a multi-valued function f one can construct a Riemann surface Σ by gluing together copies of the cut complex plane to obtain a surface on which our multifunction becomes a single-valued function.

5. Paths and Integration

Paths will play a crucial role in our development of the theory of complex differentiable functions. In this section we review the notion of a path and define the integral of a continuous function along a path.

5.1. **Paths.** Recall that a *path* in the complex plane is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$. A path is said to be *closed* if $\gamma(a) = \gamma(b)$. If γ is a path, we will write γ^* for its image, that is

$$\gamma^* = \{z \in \mathbb{C} : z = \gamma(t),\ \text{some } t \in [a,b]\}.$$

Although for some purposes it suffices to assume that γ is continuous, in order to make sense of the integral along a path we will require our paths to be (at least piecewise) differentiable. We thus need to define what we mean for a path to be differentiable:

Definition 5.1. We will say that a path $\gamma\colon [a,b] \to \mathbb{C}$ is *differentiable* if its real and imaginary parts are differentiable as real-valued functions. Equivalently, γ is differentiable at $t_0 \in [a,b]$ if

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists, and then we denote this limit as $\gamma'(t_0)$. (If $t = a$ or b then we interpret the above as a one-sided limit.) We say that a path is C^1 if it is differentiable and its derivative $\gamma'(t)$ is continuous.

We will say a path is *piecewise* C^1 if it is continuous on $[a,b]$ and the interval $[a,b]$ can be divided into subintervals on each of which γ is C^1. That is, there is a finite sequence $a = a_0 < a_1 < \ldots < a_m = b$ such that $\gamma_{|[a_i, a_{i+1}]}$ is C^1. Thus in particular, the left-hand and right-hand derivatives of γ at a_i $(1 \leq i \leq m-1)$ may not be equal.

Remark 5.2. Note that a C^1 path may not have a well-defined tangent at every point: if $\gamma\colon [a,b] \to \mathbb{C}$ is a path and $\gamma'(t) \neq 0$, then the line $\{\gamma(t) + s\gamma'(t) : s \in \mathbb{R}\}$ is tangent to γ^*, however if $\gamma'(t) = 0$, the image of γ may have no tangent line there. Indeed consider the example of $\gamma\colon [-1,1] \to \mathbb{C}$ given by

$$\gamma(t) = \begin{cases} t^2 & -1 \leq t \leq 0 \\ it^2 & 0 \leq t \leq 1. \end{cases}$$

Since $\gamma'(0) = 0$ the path is C^1, even though it is clear there is no tangent line to the image of γ at 0.

If $s\colon [a,b] \to [c,d]$ is a differentiable map, then we have the following version of the chain rule, which is proved in exactly the same way as the real-valued case. It will be crucial in our definition of the integral of functions $f\colon \mathbb{C} \to \mathbb{C}$ along paths.

Lemma 5.3. *Let $\gamma\colon [c,d] \to \mathbb{C}$ and $s\colon [a,b] \to [c,d]$ and suppose that s is differentiable at t_0 and γ is differentiable at $s_0 = s(t_0)$. Then $\gamma \circ s$ is differentiable at t_0 with derivative*

$$(\gamma \circ s)'(t_0) = s'(t_0) \cdot \gamma'(s(t_0)).$$

Proof. Let $\epsilon\colon [c,d] \to \mathbb{C}$ be given by $\epsilon(s_0) = 0$ and

$$\gamma(x) = \gamma(s_0) + \gamma'(s_0)(x - s_0) + (x - s_0)\epsilon(x),$$

(so that this equation holds for all $x \in [c,d]$), then $\epsilon(x) \to 0$ as $x \to s_0$ by the definition of $\gamma'(s_0)$, *i.e.* ϵ is continuous at t_0. Substituting $x = s(t)$ into this we see that for all $t \neq t_0$ we have

$$\frac{\gamma(s(t)) - \gamma(s_0)}{t - t_0} = \frac{s(t) - s(t_0)}{t - t_0}\big(\gamma'(s(t)) + \epsilon(s(t))\big).$$

Now $s(t)$ is continuous at t_0 since it is differentiable there hence $\epsilon(s(t)) \to 0$ as $t \to t_0$, thus taking the limit as $t \to t_0$ we see that

$$(\gamma \circ s)'(t_0) = s'(t_0)(\gamma'(s_0) + 0) = s'(t_0)\gamma'(s(t_0)),$$

as required. □

Definition 5.4. Let $\phi\colon [a,b] \to [c,d]$ be continuously differentiable with $\phi(a) = c$ and $\phi(b) = d$, and let $\gamma\colon [c,d] \to \mathbb{C}$ be a C^1-path, then setting $\tilde{\gamma} = \gamma \circ \phi$, by Lemma 5.3 we see that $\tilde{\gamma}\colon [a,b] \to \mathbb{C}$ is again a C^1-path with the same image as γ and we say that $\tilde{\gamma}$ is a *reparametrization* of γ.

Definition 5.5. We will say two parametrized paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ are *equivalent* if there is a continuously differentiable bijective function $s\colon [a,b] \to [c,d]$ such that $s'(t) > 0$ for all $t \in [a,b]$ and $\gamma_1 = \gamma_2 \circ s$. It is straight-forward to check that equivalence is indeed an equivalence relation on parametrized paths, and we will call the equivalence classes *oriented curves* in the complex plane. We denote the equivalence class of γ by $[\gamma]$. The condition that $s'(t) > 0$ ensures that the path is traversed in the same direction for each of the parametrizations γ_1 and γ_2. Moreover γ_1 is piecewise C^1 if and only if γ_2 is.

Recall we saw before (in the context of a general metric space) that any path $\gamma\colon [a,b] \to \mathbb{C}$ has an *opposite* path γ^- and that two paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$ can be *concatenated* to give a path $\gamma_1 \star \gamma_2$. See Definition **??** for the explicit formuals If $\gamma, \gamma_1, \gamma_2$ are piecewise C^1 then so are γ^- and $\gamma_1 \star \gamma_2$. (Indeed a piecewise C^1 path is precisely a finite concatenation of C^1 paths).

Remark 5.6. Note that if $\gamma\colon [a,b] \to \mathbb{C}$ is piecewise C^1, then by choosing a reparametrization by a function $\psi\colon [a,b] \to [a,b]$ which is strictly increasing and has vanishing derivative at the points where γ fails to be C^1, we can replace γ by $\tilde{\gamma} = \gamma \circ \psi$ to obtain a C^1 path with the same image. For this reason, some texts insist that C^1 paths have everywhere non-vanishing derivative. In this course we will not insist on this. Indeed sometimes it is convenient to consider a *constant* path, that is a path $\gamma\colon [a,b] \to \mathbb{C}$ such that $\gamma(t) = z_0$ for all $t \in [a,b]$ (and hence $\gamma'(t) = 0$ for all $t \in [a,b]$).

Example 5.7. The most basic example of a closed curve is a circle: If $z_0 \in \mathbb{C}$ and $r > 0$ then the path $z(t) = z_0 + re^{2\pi it}$ (for $t \in [0,1]$) is the simple closed path with *positive orientation* encircling z_0 with radius r. The path $\tilde{z}(t) = z_0 + re^{-2\pi it}$ is the simple closed path encircling z_0 with radius r and *negative orientation*.

Another useful path is a line segment: if $a, b \in \mathbb{C}$ then we denote by $\gamma_{[a,b]}\colon [0,1] \to \mathbb{C}$ the path given by $t \mapsto a + t(b-a) = (1-t)a + tb$ traverses the line segment from a to b. We denote the corresponding oriented curve by $[a,b]$ (which is consistent with the notation for an interval in the real line). One of the simplest classes of closed paths are triangles: given three points a, b, c, we define the triangle, or triangular path, associated to them, to be the concatenation of the associated line segments, that is $T_{a,b,c} = \gamma_{a,b} \star \gamma_{b,c} \star \gamma_{c,a}$.

5.2. **Integration along a path.** To define the integral of a complex-valued function along a path, we first need to define the integral of functions

$F: [a, b] \to \mathbb{C}$ on a closed interval $[a, b]$ taking values in $\mathbb{C}$. Last year in Analysis III the Riemann integral was defined for a function on a closed interval $[a, b]$ taking values in $\mathbb{R}$, but it is easy to extend this to functions taking values in $\mathbb{C}$: Indeed we may write $F(t) = G(t) + iH(t)$ where G, H are functions on $[a, b]$ taking real values. Then we say that F is Riemann integrable if both G and H are, and we define:

$$\int_a^b F(t)dt = \int_a^b G(t)dt + i\int_a^b H(t)dt$$

It is easy to check that the integral is then complex linear, that is, if F_1, F_2 are complex-valued Riemann integrable functions on $[a, b]$, and $\alpha, \beta \in \mathbb{C}$, then $\alpha F_1 + \beta F_2$ is Riemann integrable and

$$\int_a^b (\alpha.F_1 + \beta.F_2)dt = \alpha.\int_a^b F_1 dt + \beta.\int_a^b F_2 dt.$$

Note that if F is continuous, then its real and imaginary parts are also continuous, and so in particular Riemann integrable[3]. The class of Riemann integrable (real or complex valued) functions on a closed interval is however slightly larger than the class of continuous functions, and this will be useful to us at certain points. In particular, we have the following:

Lemma 5.8. *Let $[a, b]$ be a closed interval and $S \subset [a, b]$ a finite set. If f is a bounded continuous function (taking real or complex values) on $[a, b]\backslash S$ then it is Riemann integrable on $[a, b]$.*

Proof. The case of complex-valued functions follows from the real case by taking real and imaginary parts. For the case of a function $f : [a, b]\backslash S \to \mathbb{R}$, let $a = x_0 < x_1 < x_2 < \ldots < x_k = b$ be any partition of $[a, b]$ which includes the elements of S. Then on each open interval (x_i, x_{i+1}) the function f is bounded and continuous, and hence integrable. We may therefore set

$$\int_a^b f(t)dt = \int_{x_0}^{x_1} f(t)dt + \int_{x_1}^{x_2} f(t)dt + \ldots + \int_{x_{k-1}}^{x_k} f(t)dt$$

The standard additivity properties of the integral then show that $\int_a^b f(t)dt$ is independent of any choices. □

Remark 5.9. Note that normally when one speaks of a function f being integrable on an interval $[a, b]$ one assumes that f is defined on all of $[a, b]$. However, if we change the value of a Riemann integrable function f at a finite set of points, then the resulting function is still Riemann integrable and its integral is the same. Thus if one prefers the function f in the previous lemma to be defined on all of $[a, b]$ one can define f to take any values at all on the finite set S.

[3]It is clear this definition extends to give a notion of the integral of a function $f : [a, b] \to \mathbb{R}^n$ – we say f is integrable if each of its components is, and then define the integral to be the vector given by the integrals of each component function.

It is easy to check that the Riemann integral of complex-valued functions is complex linear. We also note a version of the triangle inequality for complex-valued functions:

Lemma 5.10. *Suppose that $F\colon [a,b] \to \mathbb{C}$ is a complex-valued function. Then we have*

$$\Big|\int_a^b F(t)dt\Big| \leq \int_a^b |F(t)|dt.$$

Proof. First note that if $F(t) = x(t)+iy(t)$ then $|F(t)| = \sqrt{x^2+y^2}$ so that if F is integrable $|F(t)|$ is also[4]. We may write $\int_a^b F(t)dt = re^{i\theta}$, where $r \in [0,\infty)$ and $\theta \in [0,2\pi)$. Now taking the components of F in the direction of $e^{i\theta}$ and $e^{i(\theta+\pi/2)} = ie^{i\theta}$, we may write $F(t) = u(t)e^{i\theta} + iv(t)e^{i\theta}$. Then by our choice of θ we have $\int_a^b F(t)dt = e^{i\theta}\int_a^b u(t)dt$, and so

$$\Big|\int_a^b F(t)dt\Big| = \Big|\int_a^b u(t)dt\Big| \leq \int_a^b |u(t)|dt \leq \int_a^b |F(t)|dt,$$

where in the first inequality we used the triangle inequality for the Riemann integral of real-valued functions. □

We are now ready to define the integral of a function $f\colon \mathbb{C} \to \mathbb{C}$ along a piecewise-C^1 curve.

Definition 5.11. If $\gamma\colon [a,b] \to \mathbb{C}$ is a piecewise-C^1 path and $f\colon \mathbb{C} \to \mathbb{C}$, then we define the integral of f along γ to be

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

In order for this integral to exist in the sense we have defined, we have seen that it suffices for the functions $f(\gamma(t))$ and $\gamma'(t)$ to be bounded and continuous at all but finitely many t. Our definition of a piecewise C^1-path ensures that $\gamma'(t)$ is bounded and continuous away from finitely many points (the boundedness follows from the existence of the left and right hand limits at points of discontinuity of $\gamma'(t)$). For most of our applications, the function f will be continuous on the whole image γ^* of γ, but it will occasionally be useful to weaken this to allow $f(\gamma(t))$ finitely many (bounded) discontinuities.

Lemma 5.12. *If $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 path and $\tilde{\gamma}\colon [c,d] \to \mathbb{C}$ is an equivalent path, then for any continuous function $f\colon \mathbb{C} \to \mathbb{C}$ we have*

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

In particular, the integral only depends on the oriented curve $[\gamma]$.

---

[4]The simplest way to see this is to use that fact that if ϕ is continuous and f is Riemann integrable, then $\phi \circ f$ is Riemann integrable.

Proof. Since $\tilde{\gamma}$ is equivalent to γ there is a continuously differentiable function $s\colon [c,d] \to [a,b]$ with $s(c) = a$, $s(d) = b$ and $s'(t) > 0$ for all $t \in [c,d]$. Suppose first that γ is C^1. Then by the chain rule we have

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\gamma(s(t)))(\gamma \circ s)'(t)dt \\ &= \int_c^d f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\ &= \int_a^b f(\gamma(s))\gamma'(s)ds \\ &= \int_\gamma f(z)dz.\end{aligned}$$

where in the second last equality we used the change of variables formula. If $a = x_0 < x_1 < \ldots < x_n = b$ is a decomposition of $[a,b]$ into subintervals such that γ is C^1 on $[x_i, x_{i+1}]$ for $1 \leq i \leq n-1$ then since s is a continuous increasing bijection, we have a corresponding decomposition of $[c,d]$ given by the points $s^{-1}(x_0) < \ldots < s^{-1}(x_n)$, and we have

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\ &= \sum_{i=0}^{n-1} \int_{s^{-1}(x_i)}^{s^{-1}(x_{i+1})} f(\gamma(s(t))\gamma'(s(t))s'(t)dt \\ &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(\gamma(x))\gamma'(x)dx \\ &= \int_a^b f(\gamma(x))\gamma'(x)dx = \int_\gamma f(z)dz.\end{aligned}$$

where the third equality follows from the case of C^1 paths established above. □

Definition 5.13. If $\gamma\colon [a,b] \to \mathbb{C}$ is a C^1 path then we define the *length* of γ to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)|dt.$$

Using the chain rule as we did to show that the integrals of a function $f\colon \mathbb{C} \to \mathbb{C}$ along equivalent paths are equal, one can check that the length of a parametrized path is also constant on equivalence classes of paths, so in fact the above defines a length function for oriented curves. The definition extends in the obvious way to give a notion of length for piecewise C^1-paths. More generally, one can define the integral *with respect to arc-length* of a function $f\colon U \to \mathbb{C}$ such that $\gamma^* \subseteq U$ to be

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt.$$

This integral is invariant with respect to C^1 reparametrizations $s\colon [c,d] \to [a,b]$ if we require $s'(t) \neq 0$ for all $t \in [c,d]$ (the condition $s'(t) > 0$ is not necessary because of this integral takes the modulus of $\gamma'(t)$). In particular $\ell(\gamma) = \ell(\gamma^-)$.

The integration of functions along piecewise smooth paths has many of the properties that the integral of real-valued functions along an interval possess. We record some of the most standard of these:

Proposition 5.14. *Let $f, g\colon U \to \mathbb{C}$ be continuous functions on an open subset $U \subseteq \mathbb{C}$ and $\gamma, \eta\colon [a,b] \to \mathbb{C}$ be piecewise-C^1 paths whose images lie in U. Then we have the following:*

(1) *(Linearity): For $\alpha, \beta \in \mathbb{C}$,*

$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$

(2) *If γ^- denotes the opposite path to γ then*

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

(3) *(Additivity): If $\gamma \star \eta$ is the concatenation of the paths γ, η in U, we have*

$$\int_{\gamma\star\eta} f(z)dz = \int_\gamma f(z)dz + \int_\eta f(z)dz.$$

(4) *(Estimation Lemma.) We have*

$$|\int_\gamma f(z)dz| \leq \sup_{z\in\gamma^*} |f(z)|.\ell(\gamma).$$

Proof. Since f, g are continous, and γ, η are piecewise C^1, all the integrals in the statement are well-defined: the functions $f(\gamma(t))\gamma'(t)$, $f(\eta(t))\eta'(t)$, $g(\gamma(t))\gamma'(t)$ and $g(\eta(t))\eta'(t)$ are all Riemann integrable. It is easy to see that one can reduce these claims to the case where γ is smooth. The first claim is immediate from the linearity of the Riemann integral, while the second claim follows from the definitions and the fact that $(\gamma^-)'(t) = -\gamma'(a+b-t)$. The third follows immediately for the corresponding additivity property of Riemann integrable functions.

For the fourth part, first note that $\gamma([a,b])$ is compact in $\mathbb{C}$ since it is the image of the compact set $[a,b]$ under a continuous map. It follows that the function $|f|$ is bounded on this set so that $\sup_{z\in\gamma([a,b])} |f(z)|$ exists. Thus we have

$$\begin{aligned}
\Big|\int_\gamma f(z)dz\Big| &= \Big|\int_a^b f(\gamma(t))\gamma'(t)dt\Big| \\
&\leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\
&\leq \sup_{z\in\gamma^*} |f(z)| \int_a^b |\gamma'(t)|dt \\
&= \sup_{z\in\gamma^*} |f(z)|.\ell(\gamma).
\end{aligned}$$

where for the first inequality we use the triangle inequality for complex-valued functions as in Lemma 5.10 and the positivity of the Riemann integral for the second inequality. □

Remark 5.15. We give part (4) of the above proposition a name (the "estimation lemma") because it will be very useful later in the course. We will give one important application of it now:

Proposition 5.16. *Let $f_n\colon U \to \mathbb{C}$ be a sequence of continuous functions on an open subset U of the complex plane. Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a path whose image is contained in U. If (f_n) converges uniformly to a function f on the image of γ then*

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz.$$

Proof. We have

$$\begin{aligned}
\left|\int_\gamma f(z)dz - \int_\gamma f_n(z)dz\right| &= \left|\int_\gamma (f(z) - f_n(z))dz\right| \\
&\leq \sup_{z\in\gamma^*}\{|f(z) - f_n(z)|\}.\ell(\gamma),
\end{aligned}$$

by the estimation lemma. Since we are assuming that f_n tends to f uniformly on γ^* we have $\sup\{|f(z) - f_n(z)| : z \in \gamma^*\} \to 0$ as $n \to \infty$ which implies the result. □

Definition 5.17. Let $U \subseteq \mathbb{C}$ be an open set and let $f\colon U \to \mathbb{C}$ be a continuous function. If there exists a differentiable function $F\colon U \to \mathbb{C}$ with $F'(z) = f(z)$ then we say F is a *primitive* for f on U.

The fundamental theorem of calculus has the following important consequence[5]:

Theorem 5.18. *(Fundamental theorem of Calculus): Let $U \subseteq \mathbb{C}$ be a open and let $f\colon U \to \mathbb{C}$ be a continuous function. If $F\colon U \to \mathbb{C}$ is a primitive for f and $\gamma\colon [a,b] \to U$ is a piecewise C^1 path in U then we have*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

[5]You should compare this to the existence of a potential in vector calculus.

In particular the integral of such a function f around any closed path is zero.

Proof. First suppose that γ is C^1. Then we have

$$\begin{aligned}\int_\gamma f(z)dz = \int_\gamma F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \frac{d}{dt}(F\circ\gamma)(t)dt \\ &= F(\gamma(b)) - F(\gamma(a)),\end{aligned}$$

where in second line we used a version of the chain rule[6] and in the last line we used the Fundamental theorem of Calculus from Prelims analysis on the real and imaginary parts of $F\circ\gamma$.

If γ is only[7] piecewise C^1, then take a partition $a = a_0 < a_1 < \ldots < a_k = b$ such that γ is C^1 on $[a_i, a_{i+1}]$ for each $i \in \{0, 1, \ldots, k-1\}$. Then we obtain a telescoping sum:

$$\begin{aligned}\int_\gamma f(z) &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \sum_{i=0}^{k-1}\int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt \\ &= \sum_{i=0}^{k-1}(F(\gamma(a_{i+1})) - F(\gamma(a_i))) \\ &= F(\gamma(b)) - F(\gamma(a)),\end{aligned}$$

Finally, since γ is closed precisely when $\gamma(a) = \gamma(b)$ it follows immediately that the integral of f along a closed path is zero. □

Remark 5.19. If $f(z)$ has finitely many point of discontinuity $S \subset U$ but is bounded near them, and $\gamma(t) \in S$ for only finitely many t, then provided F is continuous and $F' = f$ on $U\backslash S$, the same proof shows that the fundamental theorem still holds – one just needs to take a partition of $[a, b]$ to take account of those singularities along with the singularities of $\gamma'(t)$.

Theorem 5.18 already has an important consequence:

Corollary 5.20. *Let U be a domain and let $f\colon U \to \mathbb{C}$ be a function with $f'(z) = 0$ for all $z \in U$. Then f is constant.*

Proof. Pick $z_0 \in U$. Since U is path-connected, if $w \in U$, we may find[8] a piecewise C^1-path $\gamma\colon [0, 1] \to U$ such that $\gamma(a) = z_0$ and $\gamma(b) = w$. Then by

[6] See the appendix for a discussion of this – we need a version of the chain rule for a composition of real-differentiable functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathbb{R} \to \mathbb{R}^2$.

[7] The reason we must be careful about this case is that the Fundamental Theorem of Calculus only holds when the integrand is continuous.

[8] Check that you see that if U is an open subset of $\mathbb{C}$ which is path-connected then any two points can be joined by a piecewise C^1-path.

Theorem 5.18 we see that

$$f(w) - f(z_0) = \int_\gamma f'(z)dz = 0,$$

so that f is constant as required. □

The following theorem is a kind of converse to the fundamental theorem:

Theorem 5.21. *If U is a domain (i.e. it is open and path connected) and $f: U \to \mathbb{C}$ is a continuous function such that for any closed path in U we have $\int_\gamma f(z)dz = 0$, then f has a primitive.*

Proof. Fix z_0 in U, and for any $z \in U$ set

$$F(z) = \int_\gamma f(z)dz.$$

where $\gamma\colon [a,b] \to U$ with $\gamma(a) = z_0$ and $\gamma(b) = z$.

We claim that $F(z)$ is independent of the choice of γ. Indeed if γ_1, γ_2 are two such paths, let $\gamma = \gamma_1 \star \gamma_2^-$ be the path obtained by concatenating γ_1 and the opposite γ_2^- of γ_2 (that is, γ traverses the path γ_1 and then goes backward along γ_2). Then γ is a closed path and so, using Proposition 5.14 we have

$$0 = \int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2^-} f(z)dz,$$

hence since $\int_{\gamma_2^-} f(z)dz = -\int_{\gamma_2} f(z)dz$ we see that $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.

Next we claim that F is differentiable with $F'(z) = f(z)$. To see this, fix $w \in U$ and $\epsilon > 0$ such that $B(w,\epsilon) \subseteq U$ and choose a path $\gamma\colon [a,b] \to U$ from z_0 to w. If $z_1 \in B(w,\epsilon) \subseteq U$, then the concatenation of γ with the straight-line path $s\colon [0,1] \to U$ given by $s(t) = w + t(z-w)$from w to z is a path γ_1 from z_0 to z. It follows that

$$\begin{aligned}
F(z_1) - F(w) &= \int_{\gamma_1} f(z)dz - \int_\gamma f(z)dz \\
&= (\int_\gamma f(z)dz + \int_s f(z)dz) - \int_\gamma f(z)dz \\
&= \int_s f(z)dz.
\end{aligned}$$

But then we have for $z_1 \neq w$

$$\begin{aligned}
\left|\frac{F(z_1) - F(w)}{z_1 - w} - f(w)\right| &= \left|\frac{1}{z_1 - w}\left(\int_0^1 f(w + t(z_1 - w)(z_1 - w)dt\right) - f(w)\right| \\
&= \left|\int_0^1 (f(w + t(z_1 - w)) - f(w))dt\right| \\
&\leq \sup_{t\in[0,1]} |f(w + t(z_1 - w)) - f(w)| \\
&\to 0 \text{ as } z_1 \to w
\end{aligned}$$

as f is continuous at w. Thus F is differentiable at w with derivative $F'(w) = f(w)$ as claimed. □

Remark 5.22. Note that any two primitives for a function f differ by a constant: This follows immediately from Corollary 5.20, since if F_1 and F_2 are two primitives, their difference $(F_1 - F_2)$ has zero derivative.

6. Winding numbers

The previous section on the fundamental theorem of calculus in the complex plane shows that not every holomorphic function can have a primitive. The most fundamental example of this is the function $f(z) = 1/z$ on the domain $\mathbb{C}^\times$.

Example 6.1. Let $f\colon \mathbb{C}^\times \to \mathbb{C}^\times$ be the function $f(z) = 1/z$. Then f does not have a primitive on $\mathbb{C}^\times$. Indeed if $\gamma\colon [0,1] \to \mathbb{C}$ is the path $\gamma(t) = \exp(2\pi i t)$ then

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 \frac{1}{\exp(2\pi i t)}.(2\pi i \exp(2\pi i t))dt = 2\pi i.$$

Since the path γ is closed, this integral would have to be zero if $f(z)$ has a primitive in an open set containing γ^*, thus $f(z)$ has no primitive on $\mathbb{C}^\times$ as claimed.

Note that $1/z$ *does* have a primitive on any domain in $\mathbb{C}^\times$ where we can chose a branch of the argument function (or equivalently a branch of $[\text{Log}(z)]$): Indeed if $l(z)$ is a branch of $[\text{Log}(z)]$ on a domain $D \subset \mathbb{C}^\times$ then since $\exp(l(z)) = z$ the chain rule shows that $\exp(l(z)).l'(z) = 1$ and hence $l'(z) = 1/z$.

In the present section we investigate the change in argument as we move along a path. It will turn out to be a basic ingredient in computing integrals around closed paths. In more detail, suppose that $\gamma\colon [0,1] \to \mathbb{C}$ is a closed path which does not pass through 0. We would like to give a rigorous definition of the number of times γ "goes around the origin". Roughly speaking, this will be the change in argument $\arg(\gamma(t))$, and therein lies the difficulty, since $\arg(z)$ cannot be defined continuously on all of $\mathbb{C}\backslash\{0\}$. The next Proposition shows that we *can* however always define the argument as a continuous function *of the parameter* $t \in [0,1]$:

Proposition 6.2. *Let $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ be a path. Then there is continuous function $a\colon [0,1] \to \mathbb{R}$ such that*

$$\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}.$$

Moreover, if a and b are two such functions, then there exists $n \in \mathbb{Z}$ such that $a(t) = b(t) + n$ for all $t \in [0,1]$.

Proof. By replacing $\gamma(t)$ with $\gamma(t)/|\gamma(t)|$ we may assume that $|\gamma(t)| = 1$ for all t. Since γ is continuous on a compact set, it is uniformly continuous, so that there is a $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < \sqrt{3}$ for any s, t with $|s - t| < \delta$.

Choose an integer $n > 0$ such that $n > 1/\delta$ so that on each subinterval $[i/n, (i+1)/n]$ we have $|\gamma(s) - \gamma(t)| < \sqrt{3}/2$. Now on any half-plane in $\mathbb{C}$ we may certainly define a holomorphic branch of $[\mathrm{Log}(z)]$ (simply pick a branch cut along a ray in the opposite half-plane) and hence a continuous argument function, and if $|z_1| = |z_2| = 1$ and $|z_1 - z_2| < \sqrt{3}$, then the angle between z_1 and z_2 is at most $\pi/3$. It follows there exists a continuous functions $a_i \colon [j/n, (j+1)/n] \to \mathbb{R}$ such that $\gamma(t) = e^{2\pi i a_j(t)}$ for $t \in [j/n, (j+1)/n]$ (since $\gamma([j/n, (j+1)/n])$ must lie in an arc of length at most $2\pi/3$). Now since $e^{2\pi i a_j(j/n)} = e^{2\pi i a_{j-1}(j/n)}$ $a_{j-1}(j/n)$ and $a_i(j/n)$ differ by an integer. Thus we can successively adjust the a_j for $j > 1$ by an integer (as if $\gamma(t) = e^{2\pi i a_j(t)}$ then $\gamma(t) = e^{2\pi i(a(t)+n)}$ for any $n \in \mathbb{Z}$) to obtain a continuous function $a \colon [0,1] \to \mathbb{C}$ such that $\gamma(t) = e^{2\pi i a(t)}$ as required. Finally, the uniqueness statement follows because $e^{2\pi i(a(t)-b(t))} = 1$, hence $a(t) - b(t) \in \mathbb{Z}$, and since $[0,1]$ is connected it follows $a(t) - b(t)$ is constant as required. □

Definition 6.3. If $\gamma \colon [0,1] \to \mathbb{C}\backslash\{0\}$ is a closed path and $\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}$ as in the previous lemma, then since $\gamma(0) = \gamma(1)$ we must have $a(1) - a(0) \in \mathbb{Z}$. This integer is called the *winding number* $I(\gamma, 0)$ of γ around 0. It is uniquely determined by the path γ because the function a is unique up to an integer. By translation, if γ is any closed path and z_0 is not in the image of γ, we may define the winding number $I(\gamma, z_0)$ of γ about z_0 in the same fashion. Explicitly, if γ is a closed path with $z_0 \notin \gamma^*$ then let $t \colon \mathbb{C} \to \mathbb{C}$ be given by $t(z) = z - z_0$ and define $I(\gamma, z_0) = I(t \circ \gamma, 0)$.

Remark 6.4. Note that if $\gamma \colon [0,1] \to U$ where $0 \notin U$ and there exists a holomorphic branch $L \colon U \to \mathbb{C}$ of $[\mathrm{Log}(z)]$ on U, then $I(\gamma, 0) = 0$. Indeed in this case we may define $a(t) = \Im(L(\gamma(t)))$, and since $\gamma(0) = \gamma(1)$ it follows $a(1) - a(0) = 0$ as claimed. Note also that the definition of the winding number only requires the closed path γ to be continuous, not piecewise C^1. Of course as usual, we will mostly only be interested in piecewise C^1 paths, as these are the ones along which we can integrate functions.

We now see that the winding number has a natural interpretation in term of path integrals: Note that if γ is piecewise C^1 then the function $a(t)$ is also piecewise C^1, since any branch of the logarithm function is in fact differentiable where it is defined, and $a(t)$ is locally given as $\Im(\log(\gamma(t))$ for a suitable branch.

Lemma 6.5. *Let γ be a piecewise C^1 closed path and $z_0 \in \mathbb{C}$ a point not in the image of γ. Then the winding number $I(\gamma, z_0)$ of γ around z_0 is given by*

$$I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Proof. If $\gamma \colon [0,1] \to \mathbb{C}$ we may write $\gamma(t) = z_0 + r(t)e^{2\pi i a(t)}$ (where $r(t) = |\gamma(t) - z_0| > 0$ is continuous and the existence of $a(t)$ is guaranteed by

Proposition 6.2). Then we have

$$\begin{aligned}\int_\gamma \frac{dz}{z-z_0} &= \int_0^1 \frac{1}{r(t)e^{2\pi i a(t)}}.\left(r'(t)+2\pi i r(t)a'(t)\right)e^{2\pi i a(t)}dt \\ &= \int_0^1 r'(t)/r(t) + 2\pi i a'(t)dt = [\log(r(t)) + 2\pi i a(t)]_0^1 \\ &= 2\pi i(a(1)-a(0)),\end{aligned}$$

since $r(1) = r(0) = |\gamma(0) - z_0|$. □

The next Proposition will be useful not only for the study of winding numbers. We first need a definition:

Definition 6.6. If $f\colon U \to \mathbb{C}$ is a function on an open subset U of $\mathbb{C}$, then we say that f is *analytic* on U if for every $z_0 \in U$ there is an $r > 0$ with $B(z_0, r) \subseteq U$ such that there is a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ with radius of convergence at least r and $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$. An analytic function is holomorphic, as any power series is (infinitely) complex differentiable.

Proposition 6.7. *Let U be an open set in $\mathbb{C}$ and let $\gamma\colon [0,1] \to U$ be a closed path. If $f(z)$ is a continuous function on γ^* then the function*

$$I_f(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz,$$

is analytic in w.

In particular, if $f(z) = 1$ this shows that the function $w \mapsto I(\gamma, w)$ is a continuous function on $\mathbb{C}\backslash\gamma^*$, and hence, since it is integer-valued, it is constant on the connected components of $\mathbb{C}\backslash\gamma^*$.

Proof. We wish to show that for each $z_0 \notin \gamma^*$ we can find a disk $B(z_0, \epsilon)$ within which $I_f(\gamma, w)$ is given by a power series in $(w - z_0)$. Translating if necessary we may assume $z_0 = 0$.

Now since $\mathbb{C}\backslash\gamma^*$ is open, there is some $r > 0$ such that $B(0, 2r) \cap \gamma^* = \emptyset$. We claim that $I_f(\gamma, w)$ is holomorphic in $B(0.r)$. Indeed if $w \in B(0, r)$ and $z \in \gamma^*$ it follows that $|w/z| < 1/2$. Moreover, since γ^* is compact, $M = \sup\{|f(z)| : z \in \gamma^*\}$ is finite, and hence

$$|f(z).w^n/z^{n+1}| = |f(z)||z|^{-1}|w/z|^n < \frac{M}{2r}(1/2)^n, \quad \forall z \in \gamma^*.$$

It follows from the Weierstrass M-test that the series

$$\sum_{n=0}^{\infty} \frac{f(z).w^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{f(z)}{z}(w/z)^n = \frac{f(z)}{z}(1-w/z)^{-1} = \frac{f(z)}{z-w}$$

viewed as a function of z, converges uniformly on γ^* to $f(z)/(z-w)$. Thus for all $w \in B(0, r)$ we have

$$I_f(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w} = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz\right)w^n,$$

hence $I_f(\gamma, w)$ is given by a power series in $B(0,r)$ (and hence is also holomorphic there) as required. Finally, if $f = 1$, then since $I_1(\gamma, z) = I(\gamma, z)$ is integer-valued, it follows it must be constant on any connected component of $\mathbb{C}\backslash\gamma^*$ as required. □

Remark 6.8. Note that since the coefficients of a power series centred at a point z_0 are given by its derivatives at that point, the proof above actually also gives formulae for the derivatives of $g(w) = I_f(\gamma, w)$ at z_0:

$$g^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-z_0)^{n+1}}.$$

Remark 6.9. If γ is a closed path then γ^* is compact and hence bounded. Thus there is an $R > 0$ such that the connected set $\mathbb{C}\backslash B(0,R) \cap \gamma^* = \emptyset$. It follows that $\mathbb{C}\backslash\gamma^*$ has exactly one unbounded connected component. Since

$$|\int_\gamma \frac{d\zeta}{\zeta - z}| \leq \ell(\gamma).\sup_{\zeta\in\gamma^*}|1/(\zeta - z)| \to 0$$

as $z \to \infty$ it follows that $I(\gamma, z) = 0$ on the unbounded component of $\mathbb{C}\backslash\gamma^*$.

Definition 6.10. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a closed path. We say that a point z is in the *inside*[9] of γ if $z \notin \gamma^*$ and $I(\gamma, z) \neq 0$. The previous remark shows that the inside of γ is a union of bounded connected components of $\mathbb{C}\backslash\gamma^*$. (We don't, however, know that the inside of γ is necessarily non-empty.)

Example 6.11. Suppose that $\gamma_1\colon [-\pi, \pi] \to \mathbb{C}$ is given by $\gamma_1 = 1 + e^{it}$ and $\gamma_2\colon [0, 2\pi] \to \mathbb{C}$ is given by $\gamma_2(t) = -1 + e^{-it}$. Then if $\gamma = \gamma_1 \star \gamma_2$, γ traverses a figure-of-eight and it is easy to check that the inside of γ is $B(1,1) \cup B(-1,1)$ where $I(\gamma, z) = 1$ for $z \in B(1,1)$ while $I(\gamma, z) = -1$ for $z \in B(-1,1)$.

Remark 6.12. It is a theorem, known as the *Jordan Curve Theorem*, that if $\gamma\colon [0,1] \to \mathbb{C}$ is a simple closed curve, so that $\gamma(t) = \gamma(s)$ if and only if $s = t$ or $s, t \in \{0,1\}$, then $\mathbb{C}\backslash\gamma^*$ is the union of precisely one bounded and one unbounded component, and on the bounded component $I(\gamma, z)$ is either 1 or -1. If $I(\gamma, z) = 1$ for z on the inside of γ we say γ is postively oriented and we say it is negatively oriented if $I(\gamma, z) = -1$ for z on the inside.

7. CAUCHY'S THEOREM

The key insight into the study of holomorphic functions is Cauchy's theorem, which (somewhat informally) states that if $f\colon U \to \mathbb{C}$ is holomorphic and γ is a path in U whose interior lies entirely in U then $\int_\gamma f(z)dz = 0$. It will follow from this and Theorem 5.21 that, at least locally, every holomorphic function has a primitive. The strategy to prove Cauchy's theorem goes as follows: first show it for the simplest closed contours – triangles. Then use this to deduce the existence of a primitive (at least for certain kinds of

[9]The term *interior* of γ might be more natural, but we have already used this in the first part of the course to mean something quite different.

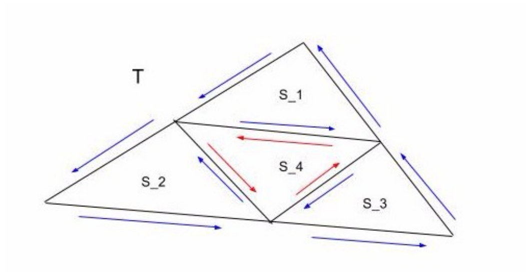

FIGURE 2. Subdivision of a triangle

sufficiently nice open sets U which are called "star-like") and then use Theorem 5.18 to deduce the result for arbitrary paths in such open subsets. We will discuss more general versions of the theorem later, after we have applied Cauchy's theorem for star-like domains to obtain important theorems on the nature of holomorphic functions. First we recall the definition of a triangular path:

Definition 7.1. A *triangle* or *triangular path* T is a path of the form $\gamma_1 \star \gamma_2 \star \gamma_3$ where $\gamma_1(t) = a + t(b-a)$, $\gamma_2(t) = b + t(c-b)$ and $\gamma_3(t) = c + t(a-c)$ where $t \in [0,1]$ and $a, b, c \in \mathbb{C}$. (Note that if $\{a, b, c\}$ are collinear, then T is a degenerate triangle.) That is, T traverses the boundary of the triangle with vertices $a, b, c \in \mathbb{C}$. The solid triangle $\mathcal{T}$ bounded by T is the region

$$\mathcal{T} = \{t_1 a + t_2 b + t_3 c : t_i \in [0,1], \sum_{i=1}^{3} t_i = 1\},$$

with the points in the interior of $\mathcal{T}$ corresponding to the points with $t_i > 0$ for each $i \in \{1, 2, 3\}$. We will denote by $[a, b]$ the line segment $\{a + t(b-a) : t \in [0,1]\}$, the side of T joining vertex a to vertex b. When we need to specify the vertices a, b, c of a triangle T, we will write $T_{a,b,c}$.

Theorem 7.2. *(Cauchy's theorem for a triangle): Suppose that $U \subseteq \mathbb{C}$ is an open subset and let $T \subseteq U$ be a triangle whose interior is entirely contained in U. Then if $f : U \to \mathbb{C}$ is holomorphic we have*

$$\int_T f(z)dz = 0$$

Proof. The proof proceeds using a version of the "divide and conquer" strategy one uses to prove the Bolzano-Weierstrass theorem. Suppose for the sake of contradiction that $\int_T f(z)dz \neq 0$, and let $I = |\int_T f(z)dz| > 0$. We build a sequence of smaller and smaller triangles T^n around which the integral of f is not too small, as follows: Let $T^0 = T$, and suppose that we have constructed T^i for $0 \leq i < k$. Then take the triangle T^{k-1} and join the midpoints of the edges to form four smaller triangles, which we will denote S_i ($1 \leq i \leq 4$).

Then we have $\int_{T^{k-1}} f(z)dz = \sum_{i=1}^{4} \int_{S_i} f(z)dz$, since the integrals around the interior edges cancel (see Figure 2). In particular, we must have

$$I_k = |\int_{T^{k-1}} f(z)dz| \leq \sum_{i=1}^{4} |\int_{S_i} f(z)dz|,$$

so that for some i we must have $|\int_{S_i} f(z)dz| \geq I_{k-1}/4$. Set T^k to be this triangle S_i. Then by induction we see that $\ell(T^k) = 2^{-k}\ell(T)$ while $I_k \geq 4^{-k}I$.

Now let $\mathcal{T}$ be the solid triangle with boundary T and similarly let $\mathcal{T}^k$ be the solid triangle with boundary T^k. Then we see that $\text{diam}(\mathcal{T}^k) = 2^{-k}\text{diam}(\mathcal{T}) \to 0$, and the sets $\mathcal{T}^k$ are clearly nested. It follows from Lemma **??** that there is a unique point z_0 which lies in every $\mathcal{T}^k$. Now by assumption f is holomorphic at z_0, so we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z),$$

where $\psi(z) \to 0 = \psi(z_0)$ as $z \to z_0$. Note that ψ is continuous and hence integrable on all of U. Now since the linear function $z \mapsto f'(z_0)z + f(z_0)$ clearly has a primitive it follows from Theorem 5.18

$$\int_{T^k} f(z)dz = \int_{T^k} (z - z_0)\psi(z)dz$$

Now since z_0 lies in $\mathcal{T}^k$ and z is on the boundary T^k of $\mathcal{T}^k$, we see that $|z - z_0| \leq \text{diam}(\mathcal{T}^k) = 2^{-k}\text{diam}(T)$. Thus if we set $\eta_k = \sup_{z \in T^k} |\psi(z)|$, it follows by the estimation lemma that

$$\begin{aligned} I_k = \Big|\int_{T^k} (z - z_0)\psi(z)dz\Big| &\leq \eta_k.\text{diam}(T^k)\ell(T^k) \\ &= 4^{-k}\eta_k.\text{diam}(T).\ell(T). \end{aligned}$$

But since $\psi(z) \to 0$ as $z \to z_0$, it follows $\eta_k \to 0$ as $k \to \infty$, and hence $4^k I_k \to 0$ as $k \to \infty$. On the other hand, by construction we have $4^k I_k \geq I > 0$, thus we have a contradiction as required. □

Definition 7.3. Let X be a subset in $\mathbb{C}$. We say that X is *convex* if for each $z, w \in U$ the line segment between z and w is contained in X. We say that X is *star-like* if there is a point $z_0 \in X$ such that for every $w \in X$ the line segment $[z_0, w]$ joining z_0 and w lies in X. We will say that X is star-like with respect to z_0 in this case. Thus a convex subset is thus starlike with respect to every point it contains.

Example 7.4. A disk (open or closed) is convex, as is a solid triangle or rectangle. On the other hand a cross, such as $\{0\} \times [-1, 1] \cup [-1, 1] \times \{0\}$ is star-like with respect to the origin, but is not convex.

Theorem 7.5. *(Cauchy's theorem for a star-like domain): Let U be a star-like domain. Then if $f: U \to \mathbb{C}$ is holomorphic and $\gamma: [a, b] \to U$ is a closed path in*

U we have

$$\int_\gamma f(z)dz = 0.$$

Proof. The proof proceeds similarly to the proof of Theorem 5.21: by Theorem 5.18 it suffices to show that f has a primitive in U. To show this, let $z_0 \in U$ be a point for which the line segment from z_0 to every $z \in U$ lies in U. Let $\gamma_z = z_0 + t(z - z_0)$ be a parametrization of this curve, and define

$$F(z) = \int_{\gamma_z} f(\zeta)d\zeta.$$

We claim that F is a primitive for f on U. Indeed pick $\epsilon > 0$ such that $B(z,\epsilon) \subseteq U$. Then if $w \in B(z,\epsilon)$ note that the triangle T with vertices z_0, z, w lies entirely in U by the assumption that U is star-like with respect to z_0. It follows from Theorem 7.2 that $\int_T f(\zeta)d\zeta = 0$, and hence if $\eta(t) = w + t(z - w)$ is the straight-line path going from w to z (so that T is the concatenation of γ_w, η and γ_z^-) we have

$$\begin{aligned}
\Big|\frac{F(z) - F(w)}{z - w} - f(z)\Big| &= \Big|\int_\eta \frac{f(\zeta)}{z - w}d\zeta - f(z)\Big| \\
&= \Big|\int_0^1 f(w + t(z - w))dt - f(z)\Big| \\
&= \Big|\int_0^1 (f(w + t(z - w)) - f(z)dt\Big| \\
&\leq \sup_{t\in[0,1]} |f(w + t(z - w)) - f(z)|,
\end{aligned}$$

which, since f is continuous at w, tends to zero as $w \to z$ so that $F'(z) = f(z)$ as required.

□

Note that our proof of Cauchy's theorem for a star-like domain D proceeded by showing that any holomorphic function on D has a primitive, and hence by the fundamental theorem of calculus its integral around a closed path is zero. This motivates the following definition:

Definition 7.6. We say that a domain $D \subseteq \mathbb{C}$ is *primitive*[10] if any holomorphic function $f\colon D \to \mathbb{C}$ has a primitive in D.

Thus, for example, our proof of Theorem 7.5 shows that all star-like domains are primitive. The following Lemma shows however that we can build many primitive domains which are not star-like.

Lemma 7.7. *Suppose that D_1 and D_2 are primitive domains and $D_1 \cap D_2$ is connected. Then $D_1 \cup D_2$ is primitive.*

[10] This is *not* standard terminology. The reason for this will become clear later.

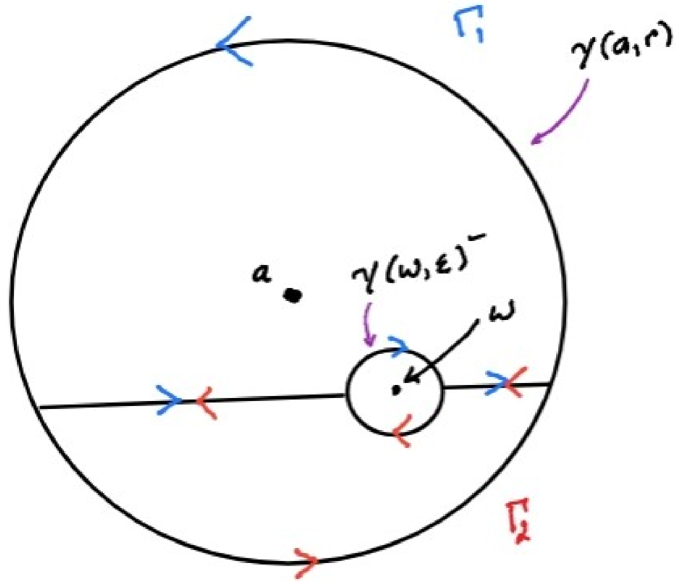

FIGURE 3. Contours for the proof of Theorem 7.8.

Proof. Let $f\colon D_1 \cup D_2 \to \mathbb{C}$ be a holomorphic function. Then $f_{|D_1}$ is a holomorphic function on D_1, and thus it has a primitive $F_1\colon D_1 \to \mathbb{C}$. Similarly $f_{|D_2}$ has a primitive, F_2 say. But then $F_1 - F_2$ has zero derivative on $D_1 \cap D_2$, and since by assumption $D_1 \cap D_2$ is connected (and thus path-connected) it follows $F_1 - F_2$ is constant, c say, on $D_1 \cap D_2$. But then if $F\colon D_1 \cup D_2 \to \mathbb{C}$ is a defined to be F_1 on D_1 and $F_2 + c$ on D_2 it follows that F is a primitive for f on $D_1 \cup D_2$ as required. □

7.1. **Cauchy's Integral Formula.** We are now almost ready to prove one of the most important consequences of Cauchy's theorem – the integral formula. This formula will have incredibly powerful consequences.

Theorem 7.8. *(Cauchy's Integral Formula.) Suppose that $f\colon U \to \mathbb{C}$ is a holomorphic function on an open set U which contains the disc $\bar{B}(a,r)$. Then for all $w \in B(a,r)$ we have*

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz,$$

where γ is the path $t \mapsto a + re^{2\pi i t}$.

Proof. Fix $w \in B(a,r)$. We use the contours Γ_1 and Γ_2 as shown in Figure 3 (where Γ_1 follows the direction of the blue arrows, and Γ_2 the directions of the red arrows). These paths join the circular contours $\gamma(a,r)$ and $\gamma(w,\epsilon)^-$

where ϵ is small enough to lie in the interior of $B(a,r)$. By the additivity properties of path integrals, the contributions of the line segments cancel so that

$$\int_{\Gamma_1} \frac{f(z)}{z-w}dz + \int_{\Gamma_2} \frac{f(z)}{z-w}dz = \int_{\gamma(a,r)} \frac{f(z)}{z-w}dz - \int_{\gamma(w,\epsilon)} \frac{f(z)}{z-w}dz.$$

On the other hand, each of Γ_1, Γ_2 lies in a primitive domain in which $f/(z-w)$ is holomorphic – indeed by the quotient rule, $f(z)/(z-w)$ is holomophic on $U\backslash\{w\}$ – so each of the integrals on the left-hand side vanish, and hence

$$\frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)}{z-w}dz.$$

Thus we can replace the integral over the circle $\gamma(a,r)$ with an integral over an arbtitrary small circle centred at w itself. But for such a small circle,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)}{z-w}dz &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)-f(w)}{z-w}dz + \frac{f(w)}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{dz}{z-w}. \\ &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)-f(w)}{z-w}dz + f(w)I(\gamma(w,\epsilon),w) \\ &= \frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z)-f(w)}{z-w}dz + f(w)\end{aligned}$$

But since f is complex differentiable at $z=w$, the term $(f(z)-f(w))/(z-w)$ is bounded as $\epsilon \to 0$, so that by the estimation lemma its integral over $\gamma(w,\epsilon)$ tends to zero. Thus as $\epsilon \to 0$ the path integral around $\gamma(w,\epsilon)$ tends to $f(w)$. But since it is also equal to $(2\pi i)^{-1}\int_{\gamma(a,r)} f(z)/(z-w)dz$, which is independent of ϵ, we conclude that it must in fact be equal to $f(w)$. The result follows.

□

Remark 7.9. The same result holds for any oriented curve γ once we weight the left-hand side by the winding number[11] of a path around the point $w \notin \gamma^*$, provided that f is holomorphic on the inside of γ.

7.2. **Applications of the Integral Formula.**

Remark 7.10. Note that Cauchy's integral formula can be interpreted as saying the value of $f(w)$ for w inside the circle is obtained as the "convolution" of f and the function $1/(z-w)$ on the boundary circle. Since the function $1/(z-w)$ is infinitely differentiable one can use this to show that f itself is infinitely differentiable as we will shortly show. If you take the Integral Transforms, you will see convolution play a crucial role in the theory of transforms. In particular, the convolution of two functions often inherits the "good" properties of either. We next show that in fact the formula implies a strong version of Taylor's Theorem.

[11] Which, as we used in the proof above, is 1 in the case of a point inside a positively oriented circular path.

Corollary 7.11. *If $f\colon U \to \mathbb{C}$ is holomorphic on an open set U, then for any $z_0 \in U$, $f(z)$ is equal to its Taylor series at z_0 and the Taylor series converges on any open disk centred at z_0 lying in U. Moreover the derivatives of f at z_0 are given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)}{(z-z_0)^{n+1}}dz. \tag{7.1}$$

For any $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ with $z_0 \in B(a,r)$.

Proof. This follows immediately from the Integral formula, the proof of Proposition 6.7, and Remark 6.8. The integral formulae of Equation 7.1 for the derivatives of f are also referred to as *Cauchy's Integral Formulae*. □

Definition 7.12. Recall that a function which is locally given by a power series is said to be *analytic*. We have thus shown that any holomorphic function is actually analytic, and from now on we may use the terms interchangeably (as you may notice is common practice in many textbooks).

One famous application of the Integral formula is known as Liouville's theorem, which will give an easy proof of the Fundamental Theorem of Algebra[12]. We say that a function $f\colon \mathbb{C} \to \mathbb{C}$ is *entire* if it is complex differentiable on the whole complex plane.

Theorem 7.13. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. If f is bounded then it is constant.*

Proof. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $\gamma_R(t) = Re^{2\pi i t}$ be the circular path centred at the origin with radius R. Then for $R > |w|$ the integral formula shows

$$\begin{aligned}
|f(w) - f(0)| &= \Big|\frac{1}{2\pi i}\int_{\gamma_R} f(z)\big(\frac{1}{z-w} - \frac{1}{z}\big)dz\Big| \\
&= \frac{1}{2\pi}\Big|\int_{\gamma_R} \frac{w.f(z)}{z(z-w)}dz\Big| \\
&\leq \frac{2\pi R}{2\pi} \sup_{z:|z|=R}\Big|\frac{w.f(z)}{z(z-w)}\Big| \\
&\leq R.\frac{M|w|}{R.(R-|w|)} = \frac{M|w|}{R-|w|},
\end{aligned}$$

Thus letting $R \to \infty$ we see that $|f(w) - f(0)| = 0$, so that f is constant an required.

□

Theorem 7.14. *Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ is a non-constant polynomial where $a_k \in \mathbb{C}$ and $a_n \neq 0$. Then there is a $z_0 \in \mathbb{C}$ for which $p(z_0) = 0$.*

[12]Which, when it comes down to it, isn't really a theorem in algebra. The most "algebraic" proof of that I know uses Galois theory, which you can learn about in Part B.

Proof. By rescaling p we may assume that $a_n = 1$. If $p(z) \neq 0$ for all $z \in \mathbb{C}$ it follows that $f(z) = 1/p(z)$ is an entire function (since p is clearly entire). We claim that f is bounded. Indeed since it is continuous it is bounded on any disc $\bar{B}(0, R)$, so it suffices to show that $|f(z)| \to 0$ as $z \to \infty$, that is, to show that $|p(z)| \to \infty$ as $z \to \infty$. But we have

$$|p(z)| = |z^n + \sum_{k=0}^{n-1} a_k z^k| = |z^n|\{|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^n - k}|\} \geq |z^n|.(1 - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}}).$$

Since $\frac{1}{|z|^m} \to 0$ as $|z| \to \infty$ for any $m \geq 1$ it follows that for sufficiently large $|z|$, say $|z| \geq R$, we will have $1 - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} \geq 1/2$. Thus for $|z| \geq R$ we have $|p(z)| \geq \frac{1}{2}|z|^n$. Since $|z|^n$ clearly tends to infinity as $|z|$ does it follows $|p(z)| \to \infty$ as required. □

Remark 7.15. The crucial point of the above proof is that one term of the polynomial – the leading term in this case– dominates the behaviour of the polynomial for large values of z. All proofs of the fundamental theorem hinge on essentially this point. Note that $p(z_0) = 0$ if and only if $p(z) = (z - z_0)q(z)$ for a polynomial $q(z)$, thus by induction on degree we see that the theorem implies that a polynomial over $\mathbb{C}$ factors into a product of degree one polynomials.

Corollary 7.16. *(Riemann's removable singularity theorem): Suppose that U is an open subset of $\mathbb{C}$ and $z_0 \in U$. If $f\colon U\backslash\{z_0\} \to \mathbb{C}$ is holomorphic and bounded near z_0, then f extends to a holomorphic function on all of U.*

Proof. Define $h(z)$ by

$$h(z) = \begin{cases} (z - z_0)^2 f(z), & z \neq 0; \\ 0, & z = z_0 \end{cases}$$

Then clearly $h(z)$ is holomorphic on $U\backslash\{z_0\}$, using the fact that f is and standard rules for complex differentiablility. On the other hand, at $z = z_0$ we see directly that

$$\frac{h(z) - h(z_0)}{z - z_0} = (z - z_0)f(z) \to 0$$

as $z \to z_0$ since f is bounded near z_0 by assumption. It follows that h is in fact holomorphic everywhere in U. But then if we chose $r > 0$ is such that $\bar{B}(z_0, r) \subset U$, then by Corollary 7.11 $h(z)$ is equal to its Taylor series centred at z_0, thus

$$h(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

But since we have $h(z_0) = h'(z_0) = 0$ we see $a_0 = a_1 = 0$, and hence $\sum_{k=0}^{\infty} a_{k+2}(z - z_0)^k$ defines a holomorphic function in $B(z_0, r)$. Since this clearly agrees with $f(z)$ on $B(z_0, r)\backslash\{0\}$, we see that by redefining $f(z_0) = a_2$, we can extend f to a holomorphic function on all of U as required. □

We end this section with a kind of converse to Cauchy's theorem:

Theorem 7.17. *(Morera's theorem) Suppose that $f\colon U \to \mathbb{C}$ is a continuous function on an open subset $U \subseteq \mathbb{C}$. If for any closed path $\gamma\colon [a,b] \to U$ we have $\int_\gamma f(z)dz = 0$, then f is holomorphic.*

Proof. By Theorem 5.21 we know that f has a primitive $F\colon U \to \mathbb{C}$. But then F is holomorphic on U and so infinitely differentiable on U, thus in particular $f = F'$ is also holomorphic. □

Remark 7.18. One can prove variants of the above theorem: If U is a star-like domain for example, then our proof of Cauchy's theorem for such domains shows that $f\colon U \to \mathbb{C}$ has a primitive (and hence will be differentiable itself) provided $\int_T f(z)dz = 0$ for every triangle in U. In fact the assumption that $\int_T f(z)dz = 0$ for all triangles whose interior lies in U suffices to imply f is holomorphic for *any* open subset U: To show f is holomorphic on U, it suffices to show that f is holomorphic on $B(a,r)$ for each open disk $B(a,r) \subset U$. But this follows from the above as disks are star-like (in fact convex). It follows that we can characterize the fact that $f\colon U \to \mathbb{C}$ is holomorphic on U by an integral condition: $f\colon U \to \mathbb{C}$ is holomorphic if and only if for all triangles T which bound a solid triangle $\mathcal{T}$ with $\mathcal{T} \subset U$, the integral $\int_T f(z)dz = 0$.

This characterization of the property of being holomorphic has some important consequences. We first need a definition:

Definition 7.19. Let U be an open subset of $\mathbb{C}$. If (f_n) is a sequence of functions defined on U, we say $f_n \to f$ *uniformly on compacts* if for every compact subset K of U, the sequence $(f_{n|K})$ converges uniformly to $f_{|K}$. Note that in this case f is continuous if the f_n are: Indeed to see that f is continuous at $a \in U$, note that since U is open, there is some $r > 0$ with $B(a,r) \subseteq U$. But then $K = \bar{B}(a, r/2) \subseteq U$ and $f_n \to f$ uniformly on K, whence f is continuous on K, and so certainly it is continuous at a.

Example 7.20. Convergence of power series $f(z) = \sum_{k=0}^{\infty} a_n z^n$ is a basic example of convergence on compacts: if R is the radius of convergences of $f(z)$ the partial sums $s_n(z)$ of the power series $B(0,R)$ converge uniformly on compacts in $B(0,R)$. The convergence is *not* necessarily uniform on $B(0,R)$, as the example $f(z) = \sum_{n=0}^{\infty} z^n$ shows. Nevertheless, since $B(0,R) = \bigcup_{r<R} \bar{B}(0,r)$ is the union of its compact subsets, many of the good properties of the polynomial functions $s_n(z)$ are inherited by the power series because the convergence is uniform on compact subsets.

Proposition 7.21. *Suppose that U is a domain and the sequence of holomorphic functions $f_n\colon U \to \mathbb{C}$ converges to $f\colon U \to \mathbb{C}$ uniformly on compacts in U. Then f is holomorphic.*

Proof. Note by the above that f is continuous on U. Since the property of being holomorphic is local, it suffices to show for each $w \in U$ that there is a ball $B(w,r) \subseteq U$ within which f is holomorphic. Since U is open, for any such w we may certainly find $r > 0$ such that $B(w,r) \subseteq U$. Then as $B(w,r)$ is convex, Cauchy's theorem for a star-like domain shows that for every closed path $\gamma\colon [a,b] \to B(w,r)$ whose image lies in $B(w,r)$ we have $\int_\gamma f_n(z)dz = 0$ for all $n \in \mathbb{N}$.

But $\gamma^* = \gamma([a,b])$ is a compact subset of U, hence $f_n \to f$ uniformly on γ^*. It follows that

$$0 = \int_\gamma f_n(z)dz \to \int_\gamma f(z)dz,$$

so that the integral of f around any closed path in $B(w,r)$ is zero. But then Theorem 5.21 shows that f has a primitive F on $B(w,r)$. But we have seen that any holomorphic function is in fact infinitely differentiable, so it follows that F, and hence f is infinitely differentiable on $B(w,r)$ as required. □

8. The identity theorem, isolated zeros and singularities

The fact that any complex differentiable function is in fact analytic has some very surprising consequences – the most striking of which is perhaps captured by the "Identity theorem". This says that if f, g are two holomorphic functions defined on a domain U and we let $S = \{z \in U : f(z) = g(z)\}$ be the locus on which they are equal, then if S has a limit point in U it must actually be all of U. Thus for example if there is a disk $B(a,r) \subseteq U$ on which f and g agree (not matter how small r is), then in fact they are equal on all of U! The key to the proof of the Identity theorem is the following result on the zeros of a holomorphic function:

Proposition 8.1. *Let U be an open set and suppose that $g\colon U \to \mathbb{C}$ is holomorphic on U. Let $S = \{z \in U : g(z) = 0\}$. If $z_0 \in S$ then either z_0 is isolated in S (so that g is non-zero in some disk about z_0 except at z_0 itself) or $g = 0$ on a neighbourhood of z_0. In the former case there is a unique integer $k > 0$ and holomorphic function g_1 such that $g(z) = (z - z_0)^k g_1(z)$ where $g_1(z_0) \neq 0$.*

Proof. Pick any $z_0 \in U$ with $g(z_0) = 0$. Since g is analytic at z_0, if we pick $r > 0$ such that $\bar{B}(z_0, r) \subseteq U$, then we may write

$$g(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k,$$

for all $z \in B(z_0, r) \subseteq U$, where the coeficients c_k are given as in Theorem 7.11. Now if $c_k = 0$ for all k, it follows that $g(z) = 0$ for all $z \in B(0, r)$. Otherwise, we set $k = \min\{n \in \mathbb{N} : c_n \neq 0\}$ (where since $g(z_0) = 0$ we have $c_0 = 0$ so that $k \geq 1$). Then if we let $g_1(z) = (z - z_0)^{-k} g(z)$, clearly $g_1(z)$ is holomorphic on $U \backslash \{z_0\}$, but since in $B(z_0, r)$ we have we have $g_1(z) = \sum_{n=0}^{\infty} c_{k+n}(z - z_0)^n$, it follows if we set $g_1(z_0) = c_k \neq 0$ then g_1 becomes a holomorphic function on all of U. Since g_1 is continuous at z_0 and $g_1(z_0) \neq 0$, there is an $\epsilon > 0$ such that $g_1(z) \neq 0$ for all $z \in B(z_0, \epsilon)$. But

$(z - z_0)^k$ vanishes only at z_0, hence it follows that $g(z) = (z - z_0)^k g_1(z)$ is non-zero on $B(a, \epsilon)\backslash\{z_0\}$, so that z_0 is isolated.

Finally, to see that k is unique, suppose that $g(z) = (z - z_0)^k g_1(z) = (z-z_0)^l g_2(z)$ say with $g_1(z_0)$ and $g_2(z_0)$ both nonzero. If $k < l$ then $g(z)/(z-z_0)^k = (z - z_0)^{l-k} g_2(z)$ for all $z \neq z_0$, hence as $z \to z_0$ we have $g(z)/(z - z_0)^k \to 0$, which contradicts the assumption that $g_1(z) \neq 0$. By symmetry we also cannot have $k > l$ so $k = l$ as required. □

Remark 8.2. The integer k in the previous proposition is called the *multiplicity* of the zero of g at $z = z_0$ (or sometimes the *order of vanishing*).

Theorem 8.3. *(Identity theorem): Let U be a domain and suppose that f_1, f_2 are holomorphic functions defined on U. Then if $S = \{z \in U : f_1(z) = f_2(z)\}$ has a limit point in U, we must have $S = U$, that is $f_1(z) = f_2(z)$ for all $z \in U$.*

Proof. Let $g = f_1 - f_2$, so that $S = g^{-1}(\{0\})$. We must show that if S has a limit point then $S = U$. Since g is clearly holomorphic in U, by Proposition 8.1 we see that if $z_0 \in S$ then either z_0 is an isolated point of S or it lies in an open ball contained in S. It follows that $S = V \cup T$ where $T = \{z \in S : z \text{ is isolated}\}$ and $V = \text{int}(S)$ is open. But since g is continuous, $S = g^{-1}(\{0\})$ is closed in U, thus $V \cup T$ is closed, and so $\text{Cl}_U(V)$, the closure[13] of V in U, lies in $V \cup T$. However, by definition, no limit point of V can lie in T so that $\text{Cl}_U(V) = V$, and thus V is open and closed in U. Since U is connected, it follows that $V = \emptyset$ or $V = U$. In the former case, all the zeros of g are isolated so that $S' = T' = \emptyset$ and S has no limit points. In the latter case, $V = S = U$ as required.

□

Remark 8.4. The requirement in the theorem that S have a limit point *lying in U* is essential: If we take $U = \mathbb{C}\backslash\{0\}$ and $f_1 = \exp(1/z) - 1$ and $f_2 = 0$, then the set S is just the points where f_1 vanishes on U. Now the zeros of f_1 have a limit point at $0 \notin U$ since $f_1(1/(2\pi i n)) = 0$ for all $n \in \mathbb{N}$, but certainly f_1 is not identically zero on U!

We now wish to study singularities of holomorphic functions. We are mainly interested in isolated singularities. The key result here is Riemann's removable singularity theorem, Corollary 7.16.

Definition 8.5. Let $f: U \to \mathbb{C}$ be a function, where U is open. We say that $z_0 \in \bar{U}$ is a *regular* point of f if f is holomorphic at z_0. Otherwise we say that z_0 is *singular*.

We say that z_0 is an *isolated singularity* if f is holomorphic on $B(z_0, r)\backslash\{z_0\}$ for some $r > 0$.

Suppose that z_0 is an isolated singularity of f. If f is bounded near z_0 we say that f has a *removable singularity* at z_0, since by Corollary 7.16 it can

[13]I use the notation $\text{Cl}_U(V)$, as opposed to $\bar{V}$, to emphasize that I mean the closure of V in U, not in $\mathbb{C}$, that is, $\text{Cl}_U(V)$ is equal to the union of V with the limits points of V which lie in U.

be extended to a holomorphic function at z_0. If f is not bounded near z_0, but the function $1/f(z)$ has a removable singularity at z_0, that is, $1/f(z)$ extends to a holomorphic function on all of $B(z_0, r)$, then we say that f has a *pole* at z_0. By Proposition 8.1 we may write $(1/f)(z) = (z-z_0)^m g(z)$ where $g(z_0) \neq 0$ and $m \in \mathbb{Z}_{>0}$. (Note that the extension of $1/f$ to z_0 must vanish there, as otherwise f would be bounded near z_0.) We say that m is the *order* of the pole of f at z_0. In this case we have $f(z) = (z - z_0)^{-m}.(1/g)$ near z_0, where $1/g$ is holomorphic near z_0 since $g(z_0) \neq 0$. If $m = 1$ we say that f has a *simple pole* at z_0.

Finally, if f has an isolated singularity at z_0 which is not removable nor a pole, we say that z_0 is an *essential singularity*.

If z_0 is singular but not isolated we simply call it *non-isolated singularity*.

Lemma 8.6. *Let f be a holomorphic function with a pole of order m at z_0. Then there is an $r > 0$ such that for all $z \in B(z_0, r)\backslash\{z_0\}$ we have*

$$f(z) = \sum_{n \geq -m} c_n (z - z_0)^n$$

Proof. As we have already seen, we may write $f(z) = (z-z_0)^{-m}h(z)$ where m is the order of the pole of f at z_0 and $h(z)$ is holomorphic and non-vanishing at z_0. The claim follows since, near z_0, $h(z)$ is equal to its Taylor series at z_0, and multiplying this by $(z-z_0)^{-m}$ gives a series of the required form for $f(z)$. □

Definition 8.7. The series $\sum_{n \geq -m} c_n(z - z_0)^n$ is called the *Laurent series* for f at z_0. We will show later that if f has an isolated essential singularity it still has a Laurent series expansion, but the series is then involves infinitely many positive and negative powers of $(z - z_0)$.

A function on an open set U which has only isolated singularities all of which are poles is called a *meromorphic* function on U. (Thus, strictly speaking, it is a function only defined on the complement of the poles in U.)

Lemma 8.8. *Suppose that f has an isolated singularity at a point z_0. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole of f then $1/f(z) = (z - z_0)^k g(z)$ where $g(z_0) \neq 0$ and $k > 0$. But then for $z \neq z_0$ we have $f(z) = (z - z_0)^{-k}(1/g(z))$, and since $g(z_0) \neq 0$, $1/g(z)$ is bounded away from 0 near z_0, while $|(z - z_0)^{-k}| \to \infty$ as $z \to z_0$, so $|f(z)| \to \infty$ as $z \to z_0$ as required.

On the other hand, if $|f(z)| \to \infty$ as $z \to z_0$, then $1/f(z) \to 0$ as $z \to z_0$, so that $1/f(z)$ has a removable singularity and f has a pole at z_0. □

Remark 8.9. The previous Lemma can be rephrased to say that f has a pole at z_0 precisely when f extends to a continuous function $f\colon U \to \mathbb{C}_\infty$ with $f(z_0) = \infty$.

The case where f has an essential singularity is more complicated. We prove that near an isolated singularity the values of a holomorphic function are dense:

Theorem 8.10. *(Casorati-Weierstrass): Let U be an open subset of $\mathbb{C}$ and let $a \in U$. Suppose that $f\colon U\backslash\{a\} \to \mathbb{C}$ is a holomorphic function with an isolated essential singularity at a. Then for all $\rho > 0$ with $B(a,\rho) \subseteq U$, the set $f(B(a,\rho)\backslash\{a\})$ is dense in $\mathbb{C}$, that is, the closure of $f(B(a,\rho)\backslash\{a\})$ is all of $\mathbb{C}$.*

Proof. Suppose, for the sake of a contradiction, that there is some $\rho > 0$ such that $z_0 \in \mathbb{C}$ is not a limit point of $f(B(a,\rho)\backslash\{a\})$. Then the function $g(z) = 1/(f(z) - z_0)$ is bounded and non-vanishing on $B(a,\rho)\backslash\{a\}$, and hence by Riemann's removable singularity theorem, it extends to a holomorphic function on all of $B(a,\rho)$. But then $f(z) = z_0 + 1/g(z)$ has at most a pole at a which is a contradiction. □

Remark 8.11. In fact much more is true: Picard showed that if f has an isolated essential singularity at z_0 then in any open disk about z_0 the function f takes every complex value infinitely often with at most one exception. The example of the function $f(z) = \exp(1/z)$, which has an essential singularity at $z = 0$ shows that this result is best possible, since $f(z) \neq 0$ for all $z \neq 0$.

8.1. **Principal parts.**

Definition 8.12. Recall that by Lemma 8.6 if a function f has a pole of order k at z_0 then near z_0 we may write

$$f(z) = \sum_{n\geq -k} c_n(z-z_0)^n.$$

The function $\sum_{n=-k}^{-1} c_n(z-z_0)^n$ is called the *principal part* of f at z_0, and we will denote it by $P_{z_0}(f)$. It is a rational function which is holomorphic on $\mathbb{C}\backslash\{z_0\}$. Note that $f - P_{z_0}(f)$ is holomorphic at z_0 (and also holomorphic wherever f is). The *residue* of f at z_0 is defined to be the coefficient c_{-1} and denoted $\text{Res}_{z_0}(f)$.

The reason for introducing these definitions is the following: Suppose that $f\colon U \to \mathbb{C}_\infty$ is a meromorphic function with poles at a finite set $S \subseteq U$. Then for each $z_0 \in S$ we have the principal part $P_{z_0}(f)$ of f at z_0, a rational function which is holomorphic everywhere on $\mathbb{C}\backslash\{z_0\}$. The difference

$$g(z) = f(z) - \sum_{z_0\in S} P_{z_0}(f),$$

is holomorphic on all of U (away from S the is clear because each term is, at $z_0 \in S$ the terms $P_s(f)$ for $s \in S\backslash\{z_0\}$ are all holomorphic, while $f(z) - P_{z_0}(f)$ is holomorphic at z_0 by the definition of $P_{z_0}(f)$). Thus if U is starlike and $\gamma\colon [0,1] \to U$ is any closed path in U with $\gamma^* \cap S = \emptyset$, we have

$$\int_\gamma f(z)dz = \int_\gamma g(z)dz + \sum_{z_0 \in S} \int_\gamma P_{z_0}(f)dz = \sum_{z_0 \in S} \int_\gamma P_{z_0}(f)dz.$$

The most important term in the principal part $P_{z_0}(f)$ is the term $c_{-1}/(z - z_0)$. This is because every other term has a primitive on $\mathbb{C}\backslash\{z_0\}$, hence by the Fundamental Theorem of Calculus it is the only part which contributes to the integral of $P_{z_0}(f)$ around the closed path γ. Combining these observations we see that

$$\int_\gamma f(z)dz = \sum_{z_0 \in S} \text{Res}_{z_0}(f) \int_\gamma \frac{dz}{z - z_0} = 2\pi i \sum_{z_0 \in S} \text{Res}_{z_0}(f).I(\gamma, z_0),$$

where $I(\gamma, z_0)$ denotes the winding number of γ about the pole z_0. This is the *residue theorem* for meromorphic functions on a starlike domain. We will shortly generalize it.

Lemma 8.13. *Suppose that f has a pole of order m at z_0, then*

$$Res_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z))$$

Proof. Since f has a pole of order m at z_0 we have $f(z) = \sum_{n \geq -m} c_n(z - z_0)^n$ for z sufficiently close to z_0. Thus

$$(z - z_0)^m f(z) = c_{-m} + c_{-m+1}(z - z_0) + \ldots + c_{-1}(z - z_0)^{m-1} + \ldots$$

and the result follows from the formula for the derivatives of a power series. □

Remark 8.14. The last lemma is perhaps most useful in the case where the pole is simple, since in that case no derivatives need to be computed. In fact there is a special case which is worth emphasizing: Suppose that $f = g/h$ is a ratio of two holomorphic functions defined on a domain $U \subseteq \mathbb{C}$, where h is non-constant. Then f is meromorphic with poles at the zeros[14] of h. In particular, if h has a simple zero at z_0 and g is non-vanishing there, then f correspondingly has a simple pole at z_0. Since the zero of h is simple at z_0, we must have $h'(z_0) \neq 0$, and hence by the previous result

$$\text{Res}_{z_0}(f) = \lim_{z \to z_0} \frac{g(z)(z - z_0)}{h(z)} = \lim_{z \to z_0} g(z). \lim_{z \to z_0} \frac{z - z_0}{h(z) - h(z_0)} = g(z_0)/h'(z_0)$$

where the last equality holds by standard Algebra of Limits results.

[14]Strictly speaking, the poles of f form a subset of the zeros of h, since if g also vanishes at a point z_0, then f may have a removable singularity at z_0.

9. Homotopies, simply-connected domains and Cauchy's theorem

A crucial point in our proof of Cauchy's theorem for a triangle was that the interior of the triangle was entirely contained in the open set on which our holomorphic function f was defined. In general however, given a closed curve, it is not always easy to say what we mean by the "interior" of the curve. In fact there is a famous theorem, known as the Jordan Curve Theorem, which resolves this problem, but to prove it would take us too far afield. Instead we will take a slightly different strategy: in fact we will take two different approaches: the first using the notion of homotopy and the second using the winding number. For the homotopy approach, rather than focusing only on closed curves and their "interiors" we consider arbitrary curves and study what it means to deform one to another.

Definition 9.1. Suppose that U is an open set in $\mathbb{C}$ and $a, b \in U$ and that $\eta\colon [0,1] \to U$ and $\gamma\colon [0,1] \to U$ are paths in U such that $\gamma(0) = \eta(0) = a$ and $\gamma(1) = \eta(1) = b$. We say that γ and η are *homotopic* in U if there is a continuous function $h\colon [0,1] \times [0,1] \to U$ such that

$$h(0,s) = a, \quad h(1,s) = b$$
$$h(t,0) = \gamma(t), \quad h(t,1) = \eta(t).$$

One should think of h as a family of paths in U indexed by the second variable s which continuously deform γ into η.

A special case of the above definition is when $a = b$ and γ and η are closed paths. In this case there is a constant path $c_a\colon [0,1] \to U$ going from a to $b = a$ which is simply given by $c_a(t) = a$ for all $t \in [0,1]$. We say a closed path starting and ending at a point $a \in U$ is *null homotopic* if it is homotopic to the constant path c_a. One can show that the relation "γ is homotopic to η" is an equivalence relation, so that any path γ between a and b belongs to a unique equivalence class, known as its homotopy class.

Definition 9.2. Suppose that U is a domain in $\mathbb{C}$. We say that U is *simply connected* if for every $a, b \in U$, any two paths from a to b are homotopic in U.

Lemma 9.3. *Let U be a convex open set in $\mathbb{C}$. Then U is simply connected. Moreover if U_1 and U_2 are homeomorphic, then U_1 is simply connected if and only if U_2 is.*

Proof. Suppose that $\gamma\colon [0,1] \to U$ and $\eta\colon [0,1] \to U$ are paths starting and ending at a and b respectively for some $a, b \in U$. Then for $(s,t) \in [0,1] \times [0,1]$ let

$$h(t,s) = (1-s)\gamma(t) + s\eta(t)$$

It is clear that h is continuous and one readily checks that h gives the required homotopy. For the moreover part, if $f\colon U_1 \to U_2$ is a homeomorphism then it is clear that f induces a bijection between continuous paths in U_1 to those in U_2 and also homotopies in U_1 to those in U_2, so the claim follows. □

Remark 9.4. (*Non-examinable*) In fact, with a bit more work, one can show that any starlike domain D is also simply-connected. The key is to show that a domain is simply-connected if all closed paths starting and ending at a given point $z_0 \in D$ are null-homotopic. If D is star-like with respect to $z_0 \in D$, then if $\gamma\colon [0,1] \to D$ is a closed path with $\gamma(0) = \gamma(1) = z_0$, it follows $h(s,t) = z_0 + s(\gamma(t) - z_0)$ gives a homotopy between γ and the constant path c_{z_0}.

Thus we see that we already know many examples of simply connected domains in the plane, such as disks, ellipsoids, half-planes. The second part of the above lemma also allows us to produce non-convex examples:

Example 9.5. Consider the domain

$$D_{\eta,\epsilon} = \{z \in \mathbb{C} : z = re^{i\theta} : \eta < r < 1, 0 < \theta < 2\pi(1-\epsilon)\},$$

where $0 < \eta, \epsilon < 1/10$ say, then $D_{\eta,\epsilon}$ is clearly not convex, but it is the image of the convex set $(0,1) \times (0, 1-\epsilon)$ under the map $(r,\theta) \mapsto re^{2\pi i\theta}$. Since this map has a continuous (and even differentiable) inverse, it follows $D_{\eta,\epsilon}$ is simply-connected. When η and ϵ are small, the boundary of this set, oriented anti-clockwise, is a version of what is called a *key-hole contour*.

We are now ready to state our extension of Cauchy's theorem. The proof is given in the Appendices.

Theorem 9.6. *Let U be a domain in $\mathbb{C}$ and $a, b \in U$. Suppose that γ and η are paths from a to b which are homotopic in U and $f\colon U \to \mathbb{C}$ is a holomorphic function. Then*

$$\int_\gamma f(z)dz = \int_\eta f(z)dz.$$

Remark 9.7. Notice that this theorem is really more general than the previous versions of Cauchy's theorem we have seen – in the case where a holomorphic function $f\colon U \to \mathbb{C}$ has a primitive the conclusion of the previous theorem is of course obvious from the Fundamental theorem of Calculus[15], and our previous formulations of Cauchy's theorem were proved by producing a primitive for f on U. One significance of the homotopy form of Cauchy's theorem is that it applies to domains U even when there is no primitive for f on U.

Theorem 9.8. *Suppose that U is a simply-connected domain, let $a, b \in U$, and let $f\colon U \to \mathbb{C}$ be a holomorphic function on U. Then if γ_1, γ_2 are paths from a to b we have*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

[15] Indeed the hypothesis that the paths γ and η are homotopic is irrelevant when f has a primitive on U.

In particular, if γ *is a closed oriented curve we have* $\int_\gamma f(z)dz = 0$*, and hence any holomorphic function on* U *has a primitive.*

Proof. Since U is simply-connected, any two paths from from a to b are homotopic, so we can apply Theorem 9.6. For the last part, in a simply-connected domain any closed path $\gamma\colon [0,1] \to U$, with $\gamma(0) = \gamma(1) = a$ say, is homotopic to the constant path $c_a(t) = a$, and hence $\int_\gamma f(z)dz = \int_{c_a} f(z)dz = 0$. The final assertion then follows from the Theorem 5.21. □

Example 9.9. If $U \subseteq \mathbb{C}\backslash\{0\}$ is simply-connected, the previous theorem shows that there is a holomorphic branch of $[\mathrm{Log}(z)]$ defined on all of U (since any primitive for $f(z) = 1/z$ will be such a branch).

Remark 9.10. Recall that in Definition 7.6 we called a domain D in the complex plane *primitive* if every holomorphic function $f\colon D \to \mathbb{C}$ on it had a primitive. Theorem 9.8 shows that any simply-connected domain is primitive. In fact the converse is also true – any primitive domain is necessarily simply-connected. Thus the term "primitive domain" is in fact another name for a simply-connected domain.

The definition of winding number allows us to give another version of Cauchy's integral formula (sometimes called the *winding number* or *homology* form of Cauchy's theorem).

Theorem 9.11. *Let* $f\colon U \to \mathbb{C}$ *be a holomorphic function and let* $\gamma\colon [0,1] \to U$ *be a closed path whose inside lies entirely in* U*, that is* $I(\gamma, z) = 0$ *for all* $z \notin U$*. Then we have, for all* $z \in U\backslash\gamma^*$*,*

$$\int_\gamma f(\zeta)d\zeta = 0; \quad \int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = 2\pi i I(\gamma, z)f(z).$$

Moreover, if U *is simply-connected and* $\gamma\colon [a,b] \to U$ *is any closed path, then* $I(\gamma, z) = 0$ *for any* $z \notin U$*, so the above identities hold for all closed paths in such* U*.*

Remark 9.12. The "moreover" statement in fact just uses the fact that a simply-connected domain is primitive: if D is a domain and $w \notin D$, then the function $1/(z-w)$ is holomorphic on all of D, and hence has a primitive on D. It follows $I(\gamma, w) = 0$ for any path γ with $\gamma^* \subseteq D$.

Remark 9.13. This version of Cauchy's theorem has a natural extension: instead of integrating over a single closed path, one can integrate over formal sums of closed paths, which are known as *cycles*: if $a \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_k$ are closed paths and $a_1, \dots, a_k$ are complex numbers (we will usually only consider the case where they are integers) then we define the integral around the formal sum $\Gamma = \sum_{i=1}^k a_i\gamma_i$ of a function f to be

$$\int_\Gamma f(z)dz = \sum_{i=1}^k a_i \int_{\gamma_i} f(z)dz.$$

Since the winding number can be expressed as an integral, this also gives a natural defintion of the winding number for such Γ: explicitly $I(\Gamma, z) = \sum_{i=1}^{k} a_i I(\gamma_i, z)$. If we write $\Gamma^* = \gamma_1^* \cup \ldots \cup \gamma_k^*$ then $I(\Gamma, z)$ is defined for all $z \notin \Gamma^*$. The winding number version Cauchy's theorem then holds (with the same proof) for cycles in an open set U, where we define the inside of a cycle to be the set of $z \in \mathbb{C}$ for which $I(\Gamma, z) \neq 0$.

Note that if z is inside Γ then it must be the case that z is inside some γ_i, but the converse is not necessarily the case: it may be that z lies inside some of the γ_i but does not lie inside Γ. One natural way in which cycles arise are as the boundaries of an open subsets of the plane: if Ω is an domain in the plane, then $\partial\Omega$, the boundary of Ω is often a *union* of curves rather than a single curve[16]. For example if $r < R$ then $\Omega = B(0, R)\backslash\bar{B}(0, r)$ has a boundary which is a union of two concentric circles. If these circles are oriented correctly, then the "inside" of the cycle Γ which they form is precisely Ω (see the discussion of Laurent series below for more details). Thus the origin, although inside each of the circles $\gamma(0, r)$ and $\gamma(0, R)$, is not inside Γ. The cycles version of Cauchy's theorem is thus closest to Green's theorem in multivariable calculus.

As a first application of this new form of Cauchy's theorem, we establish the *Laurent expansion* of a function which is holomorphic in an annulus. This is a generalization of Taylor's theorem, and we already saw it in the special case of a function with a pole singularity.

Definition 9.14. Let $0 \leq r < R$ be real numbers and let $z_0 \in \mathbb{C}$. An open *annulus* is a set

$$A = A(r, R, z_0) = B(z_0, R)\backslash\bar{B}(z_0, r) = \{z \in \mathbb{C} : r < |z - z_0| < R\}.$$

If we write (for $s > 0$) $\gamma(z_0, s)$ for the closed path $t \mapsto z_0 + se^{2\pi i t}$ then notice that the inside of the cycle $\Gamma_{r,R,z_0} = \gamma(z_0, R) - \gamma(z_0, r)$ is precisely A, since for any s, $I(\gamma(z_0, s), z)$ is 1 precisely if $z \in B(z_0, s)$ and 0 otherwise.

Theorem 9.15. *Suppose that $0 < r < R$ and $A = A(r, R, z_0)$ is an annulus centred at z_0. If $f : U \to \mathbb{C}$ is holomorphic on an open set U which contains $\bar{A}$, then there exist $c_n \in \mathbb{C}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad \forall z \in A.$$

Moreover, the c_n are unique and are given by the following formulae:

$$c_n = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where $s \in [r, R]$ and for any $s > 0$ we set $\gamma_s(t) = z_0 + se^{2\pi i t}$.

[16]Of course in general the boundary of an open set need not be so nice as to be a union of curves at all.

Proof. By translation we may assume that $z_0 = 0$. Since A is the inside of the cycle Γ_{r,R,z_0} it follows from the winding number form of Cauchy's integral formula that for $w \in A$ we have

$$2\pi i f(w) = \int_{\gamma_R} \frac{f(z)}{z-w} dz - \int_{\gamma_r} \frac{f(z)}{z-w} dz$$

But now the result follows in the same way as we showed holomorphic functions were analytic: if we fix w, then, for $|w| < |z|$ we have $\frac{1}{z-w} = \sum_{n=0}^{\infty} w^n/z^{n+1}$, converging uniformly in z in $|z| > |w| + \epsilon$ for any $\epsilon > 0$. It follows that

$$\int_{\gamma_R} \frac{f(z)}{z-w} dz = \int_{\gamma_R} \sum_{n=0}^{\infty} \frac{f(z)w^n}{z^{n+1}} dz = \sum_{n\geq 0} \left(\int_{\gamma_R} \frac{f(z)}{z^{n+1}} dz \right) w^n.$$

for all $w \in A$. Similarly since for $|z| < |w|$ we have[17] $\frac{1}{w-z} = \sum_{n\geq 0} z^n/w^{n+1} = \sum_{n=-1}^{-\infty} w^n/z^{n+1}$, again converging uniformly on $|z|$ when $|z| < |w| - \epsilon$ for $\epsilon > 0$, we see that

$$\int_{\gamma_r} \frac{f(z)}{w-z} dz = \int_{\gamma_r} \sum_{n=-1}^{-\infty} f(z)w^n/z^{n+1} dz = \sum_{n=-1}^{-\infty} (\int_{\gamma_r} \frac{f(z)}{z^{n+1}} dz) w^n.$$

Thus taking $(c_n)_{n\in\mathbb{Z}}$ as in the statement of the theorem, we see that

$$f(w) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z-w} dz - \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z-w} dz = \sum_{n\in\mathbb{Z}} c_n z^n,$$

as required. To see that the c_n are unique, one checks using uniform convergence that if $\sum_{n\in\mathbb{Z}} d_n z^n$ is any series expansion for $f(z)$ on A, then the d_n must be given by the integral formulae above.

Finally, to see that the c_n can be computed using any circular contour γ_s, note that if $r \leq s_1 < s_2 \leq R$ then $f/(z-z_0)^{n+1}$ is holomorphic on the inside of $\Gamma = \gamma_{s_2} - \gamma_{s_1}$, hence by the homology form of Cauchy's theorem $0 = \int_\Gamma f(z)/(z-z_0)^{n+1} dz = \int_{\gamma_{s_2}} f(z)/(z-z_0)^{n+1} dz - \int_{\gamma_{s_1}} f(z)/(z-z_0)^{n+1} dz$. $\square$

Remark 9.16. Note that the above proof shows that the integral $\int_{\gamma_R} \frac{f(z)}{z-w} dz$ defines a holomorphic function of w in $B(z_0, R)$, while $\int_{\gamma_r} \frac{f(z)}{z-w} dz$ defines a holomorphic function of w on $\mathbb{C}\backslash B(z_0, r)$. Thus we have actually expressed $f(w)$ on A as the difference of two functions which are holomorphic on $B(z_0, R)$ and $\mathbb{C}\backslash \bar{B}(z_0, r)$ respectively.

Corollary 9.17. *If $f \colon U \to \mathbb{C}$ is a holomorphic function on an open set U containing an annulus $A = A(r, R, z_0)$ then f has a Laurent expansion on A. In particular, if f has an isolated singularity at z_0, then it has a Laurent expansion on a punctured disc $B(z_0, r)\backslash\{z_0\}$ for sufficiently small $r > 0$.*

[17] Note the sign change.

Proof. This follows from the previous Theorem and the fact that for any $0 \leq r \leq R$ we have

$$A(r, R, z_0) = \bigcup_{r<r_1<R_1<R} \overline{A(r_1, R_1, z_0)}.$$

The final sentence follows from the fact that $B(z_0, r)\backslash\{z_0\} = A(0, r, z_0)$. □

Definition 9.18. Let $f\colon U\backslash S \to \mathbb{C}$ be a function which is holomorphic on a domain U except at a discrete set $S \subseteq U$. Then for any $a \in S$ Corollary 9.17 shows that for $r > 0$ sufficiently small, we have

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n, \quad \forall z \in B(a, r)\backslash\{a\}.$$

We define

$$P_a(f) = \sum_{n=-1}^{-\infty} c_n(z-a)^n,$$

to be the *principal part* of f at a. This generalizes the previous definition we gave for the principal part of a meromorphic function. Note that the proof of Theorem 9.17 shows that the series $P_a(f)$ is uniformly convergent on $\mathbb{C}\backslash B(a, r)$ for all $r > 0$, and hence defines a holomorphic function on $\mathbb{C}\backslash\{a\}$.

10. The argument principle

Lemma 10.1. *Suppose that $f\colon U \to \mathbb{C}$ is a meromorphic and has a zero of order k or a pole of order k at $z_0 \in U$. Then $f'(z)/f(z)$ has a simple pole at z_0 with residue k or $-k$ respectively.*

Proof. If $f(z)$ has a zero of order k we have $f(z) = (z-z_0)^k g(z)$ where $g(z)$ is holomorphic near z_0 and $g(z_0) \neq 0$. It follows that

$$f'(z)/f(z) = \frac{k}{z-z_0} + g'(z)/g(z),$$

and since $g(z) \neq 0$ near z_0 it follows $g'(z)/g(z)$ is holomorphic near z_0, so that the result follows. The case where f has a pole at z_0 is similar. □

Remark 10.2. Note that if U is an open set on which one can define a holomorphic branch L of $[\text{Log}(z)]$ then $g(z) = L(f(z))$ has $g'(z) = f'(z)/f(z)$. Thus integrating $f'(z)/f(z)$ along a path γ will measure the change in argument around the origin of the path $f(\gamma(t))$. The residue theorem allows us to relate this to the number of zeros and poles of f inside γ, as the next theorem shows:

Theorem 10.3. *(Argument principle): Suppose that U is an open set and $f\colon U \to \mathbb{C}$ is a meromorphic function on U. If $\overline{B(a, r)} \subseteq U$ and N is the number of* *zeros (counted with multiplicity) and P is the number of poles (again counted with multiplicity) of f inside $B(a,r)$ and f has neither on $\partial B(a,r)$ then*

$$N - P = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz,$$

where $\gamma(t) = a + re^{2\pi it}$ is a path with image $\partial B(a,r)$. Moreover this is the winding number of the path $\Gamma = f \circ \gamma$ about the origin.

Proof. It is easy to check that $I(\gamma, z)$ is 1 if $|z-a| \leq 1$ and is 0 otherwise. Since Lemma 10.1 shows that $f'(z)/f(z)$ has simple poles at the zeros and poles of f with residues the corresponding orders the result immediately from Theorem 11.1.

For the last part, note that the winding number of $\Gamma(t) = f(\gamma(t))$ about zero is just

$$\int_{f\circ\gamma} dw/w = \int_0^1 \frac{1}{f(\gamma(t))}f'(\gamma(t))\gamma'(t)dt = \int_\gamma \frac{f'(z)}{f(z)}dz$$

□

Remark 10.4. The argument principle also holds, with the same proof, to any closed path γ on which f is continuous and non-vanishing, provided it has winding number $+1$ around its inside. Thus for example it applies to triangles, or paths built from an arc of a circle and the line segments joining the end-points to the centre of the circle, provided they are correctly oriented.

The argument principle is very useful – we use it here to establish some important results.

Theorem 10.5. *(Rouché's theorem): Suppose that f and g are holomorphic functions on an open set U in $\mathbb{C}$ and $\bar{B}(a,r) \subset U$. If $|f(z)| > |g(z)|$ for all $z \in \partial B(a,r)$ then f and $f+g$ have the same change in argument around $\partial B(a,r)$, and hence the same number of zeros in $B(a,r)$ (counted with multiplicities).*

Proof. Let $\gamma(t) = a + re^{2\pi it}$ be a parametrization of the boundary circle of $B(a,r)$. We need to show that $(f+g)/f = 1 + g/f$ has the same number of zeros as poles (Note that $f(z) \neq 0$ on $\partial B(a,r)$ since $|f(z)| > |g(z)|$.) But by the argument principle, this number is the winding number of $\Gamma(t) = h(\gamma(t))$ about zero, where $h(z) = 1 + g(z)/f(z)$. Since, by assumption, for $z \in \gamma^*$ we have $|g(z)| < |f(z)|$ and so $|g(z)/f(z)| < 1$, the image of Γ lies entirely in $B(1,1)$ and thus in the half-plane $\{z : \Re(z) > 0\}$. Hence picking a branch of Log defined on this half-plane, we see that the integral

$$\int_\Gamma \frac{dz}{z} = \text{Log}(h(\gamma(1)) - \text{Log}(h(\gamma(0)) = 0$$

as required.

□

Remark 10.6. Rouche's theorem can be useful in counting the number of zeros of a function f – one tries to find an approximation to f whose zeros are easier to count and then by Rouche's theorem obtain information about the zeros of f. Just as for the argument principle above, it also holds for closed paths which having winding number 1 about their inside.

Example 10.7. Suppose that $P(z) = z^4 + 5z + 2$. Then on the circle $|z| = 2$, we have $|z|^4 = 16 > 5.2 + 2 \geq |5z + 2|$, so that if $g(z) = 5z + 2$ we see that $P - g = z^4$ and P have the same number of roots in $B(0, 2)$. It follows by Rouche's theorem that the four roots of $P(z)$ all have modulus less than 2. On the other hand, if we take $|z| = 1$, then $|5z + 2| \geq 5 - 2 = 3 > |z^4| = 1$, hence $P(z)$ and $5z + 2$ have the same number of roots in $B(0, 1)$. It follows $P(z)$ has one root of modulus less than 1, and 3 of modulus between 1 and 2.

Theorem 10.8. *(Open mapping theorem): Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and non-constant on a domain U. Then for any open set $V \subset U$ the set $f(V)$ is also open.*

Proof. Suppose that $w_0 \in f(V)$, say $f(z_0) = w_0$. Then $g(z) = f(z) - w_0$ has a zero at z_0 which, since f is nonconstant, is isolated. Thus we may find an $r > 0$ such that $g(z) \neq 0$ on $\bar{B}(z_0, r)\backslash\{z_0\} \subset U$ and in particular since $\partial B(z_0, r)$ is compact, we have $|g(z)| \geq \delta > 0$ on $\partial B(z_0, r)$. But then if $|w - w_0| < \delta$ it follows $|w - w_0| < |g(z)|$ on $\partial B(z_0, r)$, hence by Rouche's theorem, since $g(z)$ has a zero in $B(z_0, r)$ it follows $h(z) = g(z) + (w_0 - w) = f(z) - w$ does also, that is, $f(z)$ takes the value w in $B(z_0, r)$. Thus $B(w_0, \delta) \subseteq f(B(z_0, r))$ and hence $f(U)$ is open as required. □

Remark 10.9. Note that the proof actually establishes a bit more than the statement of the theorem: if $w_0 = f(z_0)$ then the multiplicity d of the zero of the function $f(z) - w_0$ at z_0 is called the *degree* of f at z_0. The proof shows that locally the function f is d-to-1, counting multiplicities, that is, there are $r, \epsilon \in \mathbb{R}_{>0}$ such that for every $w \in B(w_0, \epsilon)$ the equation $f(z) = w$ has d solutions counted with multiplicity in the disk $B(z_0, r)$.

Theorem 10.10. *(Inverse function theorem): Suppose that $f\colon U \to \mathbb{C}$ is injective and holomorphic and that $f'(z) \neq 0$ for all $z \in U$. If $g\colon f(U) \to U$ is the inverse of f, then g is holomorphic with $g'(w) = 1/f'(g(w))$.*

Proof. By the open mapping theorem, the function g is continuous, indeed if V is open in $f(U)$ then $g^{-1}(V) = f(V)$ is open by that theorem. To see that g is holomorphic, fix $w_0 \in f(U)$ and let $z_0 = g(w_0)$. Note that since g and f are continuous, if $w \to w_0$ then $f(w) \to z_0$. Writing $z = f(w)$ we have

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = 1/f'(z_0)$$

as required. □

Remark 10.11. Note that the non-trivial part of the proof of the above theorem is the fact that g is continuous! In fact the condition that $f'(z) \neq 0$ follows from the fact that f is bijective – this can be seen using the degree of f: if $f'(z_0) = 0$ and f is nonconstant, we must have $f(z) - f(z_0) = (z - z_0)^k g(z)$ where $g(z_0) \neq 0$ and $k \geq 1$. Since we can chose a holomorphic branch of $g^{1/k}$ near z_0 it follows that $f(z)$ is locally k-to-1 near z_0, which contradicts the injectivity of f. For details see the Appendices. Notice that this is in contrast with the case of a single real variable, as the example $f(x) = x^3$ shows. Once again, complex analysis is "nicer" than real analysis!

11. The Residue theorem

We can now prove one of the most useful theorems of the course – it is extremely powerful as a method for computing integrals, as you will see this course and many others.

Theorem 11.1. *(Residue theorem): Suppose that U is an open set in $\mathbb{C}$ and γ is a path whose inside is contained in U, so that for all $z \notin U$ we have $I(\gamma, z) = 0$. Then if $S \subset U$ is a finite set such that $S \cap \gamma^* = \emptyset$ and f is a holomorphic function on $U \backslash S$ we have*

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in S} I(\gamma, a) Res_a(f)$$

Proof. For each $a \in S$ let $P_a(f)(z) = \sum_{n=-1}^{-\infty} c_n(a)(z-a)^n$ be the principal part of f at a, a holomorphic function on $\mathbb{C} \backslash \{a\}$. Then by definition of $P_a(f)$, the difference $f - P_a(f)$ is holomorphic at $a \in S$, and thus $g(z) = f(z) - \sum_{a \in S} P_a(f)$ is holomorphic on all of U. But then by Theorem 9.11 we see that $\int_\gamma g(z)dz = 0$, so that

$$\int_\gamma f(z)dz = \sum_{a \in S} \int_\gamma P_a(f)(z)dz$$

But by the proof of Theorem 9.17, the series $P_a(f)$ converges uniformly on γ^* so that

$$\begin{aligned} \int_\gamma P_a(f)dz &= \int_\gamma \sum_{n=-1}^{-\infty} c_n(a)(z-a)^n = \sum_{n=1}^{\infty} \int_\gamma \frac{c_{-n}(a)dz}{(z-a)^n} \\ &= \int_\gamma \frac{c_{-1}(a)dz}{z-a} = I(\gamma, a)\text{Res}_a(f), \end{aligned}$$

since for $n > 1$ the function $(z-a)^{-n}$ has a primitive on $\mathbb{C} \backslash \{a\}$. The result follows.

□

Remark 11.2. In practice, in applications of the residue theorem, the winding numbers $I(\gamma, a)$ will be simple to compute in terms of the argument of $(z - a)$ – in fact most often they will be 0 or ± 1 as we will usually apply the theorem to integrals around simple closed curves.

11.1. **Residue Calculus.** The Residue theorem gives us a very powerful technique for computing many kinds of integrals. In this section we give a number of examples of its application.

Example 11.3. Consider the integral $\int_0^{2\pi} \frac{dt}{1+3\cos^2(t)}$. If we let γ be the path $t \mapsto e^{it}$ and let $z = e^{it}$ then $\cos(t) = \Re(z) = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(z+1/z)$. Thus we have

$$\frac{1}{1+3\cos^2(t)} = \frac{1}{1+3/4(z+1/z)^2} = \frac{1}{1+\frac{3}{4}z^2+\frac{3}{2}+\frac{3}{4}z^{-2}} = \frac{4z^2}{3+10z^2+3z^4},$$

Finally, since $dz = izdt$ it follows

$$\int_0^{2\pi} \frac{dt}{1+3\cos^2(t)} = \int_\gamma \frac{-4iz}{3+10z^2+3z^4} dz.$$

Thus we have turned our real integral into a contour integral, and to evaluate the contour integral we just need to calculate the residues of the meromorphic function $g(z) = \frac{-4iz}{3+10z^2+3z^4}$ at the poles it has inside the unit circle. Now the poles of $g(z)$ are the zeros of the polynomial $p(z) = 3+10z^2+3z^4$, which are at $z^2 \in \{-3, -1/3\}$. Thus the poles inside the unit circle are at $\pm i/\sqrt{3}$. In particular, since p has degree 4 and has four roots, they must all be simple zeros, and so g has simple poles at these points. The residue at a simple pole z_0 can be calculated as the limit $\lim_{z\to z_0}(z-z_0)g(z)$, thus we see (compare with Remark 8.14) that

$$\begin{aligned}\mathrm{Res}_{z=\pm i/\sqrt{3}}(g(z)) &= \lim_{z\to\pm i/\sqrt{3}} \frac{-4iz(z-\pm i/\sqrt{3})}{3+10z^2+3z^4} = (\pm 4/\sqrt{3}).\frac{1}{p'(\pm i/\sqrt{3})} \\ &= (\pm 4/\sqrt{3}).\frac{1}{20(\pm i/\sqrt{3})+12(\pm i/\sqrt{3})^3} = 1/4i.\end{aligned}$$

It now follows from the Residue theorem that

$$\int_0^{2\pi} \frac{dt}{1+3\cos^2(t)} = 2\pi i\big(\mathrm{Res}_{z=i/\sqrt{3}}((g(z)) + \mathrm{Res}_{z=-i/\sqrt{3}}(g(z))\big) = \pi.$$

Remark 11.4. Often we are interested in integrating along a path which is not closed or even finite, for example, we might wish to understand the integral of a function on the positive real axis. The residue theorem can still be a powerful tool in calculating these integrals, provided we complete the path to a closed one in such a way that we can control the extra contribution to the integral along the part of the path we add.

Example 11.5. If we have a function f which we wish to integrate over the whole real line (so we have to treat it as an improper Riemann integral) then we may consider the contours Γ_R given as the concatenation of the paths $\gamma_1\colon [-R, R] \to \mathbb{C}$ and $\gamma_2\colon [0,1] \to \mathbb{C}$ where

$$\gamma_1(t) = -R+t; \quad \gamma_2(t) = Re^{i\pi t}.$$

(so that $\Gamma_R = \gamma_2 \star \gamma_1$ traces out the boundary of a half-disk). In many cases one can show that $\int_{\gamma_2} f(z)dz$ tends to 0 as $R \to \infty$, and by calculating the residues inside the contours Γ_R deduce the integral of f on $(-\infty, \infty)$. To see this strategy in action, consider the integral

$$\int_0^\infty \frac{dx}{1+x^2+x^4}.$$

It is easy to check that this integral exists as an improper Riemann integral, and since the integrand is even, it is equal to

$$\frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{1+x^2+x^4}dx.$$

If $f(z) = 1/(1+z^2+z^4)$, then $\int_{\Gamma_R} f(z)dz$ is equal to $2\pi i$ times the sum of the residues inside the path Γ_R. The function $f(z) = 1/(1+z^2+z^4)$ has poles at $z^2 = \pm e^{2\pi i/3}$ and hence at $\{e^{\pi i/3}, e^{2\pi i/3}, e^{4\pi i/3}, e^{5\pi i/3}\}$. They are all simple poles and of these only $\{\omega, \omega^2\}$ are in the upper-half plane, where $\omega = e^{i\pi/3}$. Thus by the residue theorem, for all $R > 1$ we have

$$\int_{\Gamma_R} f(z)dz = 2\pi i\big(\mathrm{Res}_\omega(f(z)) + \mathrm{Res}_{\omega^2}(f(z))\big),$$

and we may calculate the residues using the limit formula as above (and the fact that it evaluates to the reciprocal of the derivative of $1+z^2+z^4$): Indeed since $\omega^3 = -1$ we have $\mathrm{Res}_\omega(f(z)) = \frac{1}{2\omega+4\omega^3} = \frac{1}{2\omega-4}$, while $\mathrm{Res}_{\omega^2}(f(z)) = \frac{1}{2\omega^2+4\omega^6} = \frac{1}{4+2\omega^2}$. Thus we obtain:

$$\begin{aligned}\int_{\Gamma_R} f(z)dz &= 2\pi i\big(\frac{1}{2\omega-4} + \frac{1}{2\omega^2+4}\big) \\ &= \pi i\big(\frac{1}{\omega-2} + \frac{1}{\omega^2+2}\big) \\ &= \pi i\big(\frac{\omega^2+\omega}{2(\omega-\omega^2)-5}\big) = -\sqrt{3}\pi/(-3) = \pi/\sqrt{3},\end{aligned}$$

(where we used the fact that $\omega^2+\omega = i\sqrt{3}$ and $\omega - \omega^2 = 1$). Now clearly

$$\int_{\Gamma_R} f(z)dz = \int_{-R}^{R}\frac{dt}{1+t^2+t^4} + \int_{\gamma_2} f(z)dz,$$

and by the estimation lemma we have

$$\big|\int_{\gamma_2} f(z)dz\big| \leq \sup_{z\in\gamma_2^*}|f(z)|.\ell(\gamma_2) \leq \frac{\pi R}{R^4-R^2-1} \to 0,$$

as $R \to \infty$, it follows that

$$\pi/\sqrt{3} = \lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = \int_{-\infty}^{\infty}\frac{dt}{1+t^2+t^4}.$$

11.2. **Jordan's Lemma and applications.** The following lemma is a basic fact on *convexity*. Note that if x, y are vectors in any vector space then the set $\{tx + (1-t)y : t \in [0,1]\}$ describes the line segment between x and y.

Lemma 11.6. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then if $[a,b]$ is an interval on which $g''(x) < 0$, the function g is concave on $[a,b]$, that is, for $x < y \in [a,b]$ we have*

$$g(tx + (1-t)y) \geq tg(x) + (1-t)g(y), \quad t \in [0,1].$$

Thus informally speaking, chords between points on the graph of g lie below the graph itself.

Proof. Given $x, y \in [a,b]$ and $t \in [0,1]$ let $\xi = tx + (1-t)y$, a point in the interval between x and y. Now the slope of the chord between $(x, g(x))$ and $(\xi, g(\xi))$ is, by the Mean Value Theorem, equal to $g'(s_1)$ where s_1 lies between x and ξ, while the slope of the chord between $(\xi, g(\xi))$ and $(y, g(y))$ is equal to $g'(s_2)$ for s_2 between ξ and y. If $g(\xi) < tg(x) + (1-t)g(y)$ it follows that $g'(s_1) < 0$ and $g'(s_2) > 0$. Thus by the mean value theorem for $g'(x)$ applied to the points s_1 and s_2 it follows there is an $s \in (s_1, s_2)$ with $g''(s) = (g'(s_2) - g'(s_1))/(s_2 - s_1) > 0$, contradicting the assumption that $g''(x)$ is negative on (a,b). □

The following lemma is an easy application of this convexity result.

Lemma 11.7. *(Jordan's Lemma): Let $f\colon \mathbb{H} \to \mathbb{C}_\infty$ be a meromorphic function on the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. Suppose that $f(z) \to 0$ as $z \to \infty$ in $\mathbb{H}$. Then if $\gamma_R(t) = Re^{it}$ for $t \in [0,\pi]$ we have*

$$\int_{\gamma_R} f(z)e^{i\alpha z}dz \to 0$$

as $R \to \infty$ for all $\alpha \in \mathbb{R}_{>0}$.

Proof. Suppose that $\epsilon > 0$ is given. Then by assumption we may find an S such that for $|z| > S$ we have $|f(z)| < \epsilon$. Thus if $R > S$ and $z = \gamma_R(t)$, it follows that

$$|f(z)e^{i\alpha z}| = \leq \epsilon e^{-\alpha R \sin(t)}.$$

But now applying Lemma 11.6 to the function $g(t) = \sin(t)$ with $x = 0$ and $y = \pi/2$ we see that $\sin(t) \geq \frac{2}{\pi}t$ for $t \in [0, \pi/2]$. Similarly we have $\sin(\pi - t) \geq 2(\pi - t)/\pi$ for $t \in [\pi/2, \pi]$. Thus we have

$$|f(z)e^{i\alpha z}| \leq \begin{cases} \epsilon.e^{-2\alpha R t/\pi}, & t \in [0,\pi/2] \\ \epsilon.e^{-2\alpha R(\pi - t)/\pi} & t \in [\pi/2, \pi] \end{cases}$$

But then it follows that

$$\Big|\int_{\gamma_R} f(z)e^{i\alpha z}dz\Big| \leq 2\int_0^{\pi/2} \epsilon R.e^{-2\alpha R t/\pi}dt = \epsilon.\pi\frac{1 - e^{-\alpha R}}{\alpha} < \epsilon.\pi/\alpha,$$

Thus since $\pi/\alpha > 0$ is independent of R, it follows that $\int_{\gamma_R} f(z)e^{i\alpha z}dz \to 0$ as $R \to \infty$ as required. □

Remark 11.8. If η_R is an arc of a semicircle in the upper half plane, say $\eta_R(t) = Re^{it}$ for $0 \leq t \leq 2\pi/3$, then the same proof shows that

$$\int_{\eta_R} f(z)e^{i\alpha z}dz \to 0 \quad \text{as} \quad R \to \infty.$$

This is sometimes useful when integrating around the boudary of a sector of disk (that is a set of the form $\{re^{i\theta} : 0 \leq r \leq R, \theta \in [\theta_1, \theta_2]\}$).

It is also useful to note that if $\alpha < 0$ then the integral of $f(z)e^{i\alpha z}$ around a semicircle in the *lower* half plane tends to zero as the radius of the semicircle tends to infinity provided $|f(z)| \to 0$ as $|z| \to \infty$ in the lower half plane. This follows immediately from the above applied to $f(-z)$.

Example 11.9. Consider the integral $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx$. This is an improper integral of an even function, thus it exists if and only if the limit of $\int_{-R}^{R} \frac{\sin(x)}{x}dx$ exists as $R \to \infty$. To compute this consider the integral along the closed curve η_R given by the concatenation $\eta_R = \nu_R \star \gamma_R$, where $\nu_R \colon [-R, R] \to \mathbb{R}$ given by $\nu_R(t) = t$ and $\gamma_R(t) = Re^{it}$ (where $t \in [0, \pi]$). Now if we let $f(z) = \frac{e^{iz}-1}{z}$, then f has a removable singularity at $z = 0$ (as is easily seen by considering the power series expansion of e^{iz}) and so is an entire function. Thus we have $\int_{\eta_R} f(z)dz = 0$ for all $R > 0$. Thus we have

$$0 = \int_{\eta_R} f(z)dz = \int_{-R}^{R} f(t)dt + \int_{\gamma_R} \frac{e^{iz}}{z}dz - \int_{\gamma_R} \frac{dz}{z}.$$

Now Jordan's lemma ensures that the second term on the right tends to zero as $R \to \infty$, while the third term integrates to $\int_0^{\pi} \frac{iRe^{it}}{Re^{it}}dt = i\pi$. It follows that $\int_{-R}^{R} f(t)dt$ tends to $i\pi$ as $R \to \infty$. and hence taking imaginary parts we conclude the improper integral $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx$ is equal to π.

Remark 11.10. The function $f(z) = \frac{e^{iz}-1}{z}$ might not have been the first meromorphic function one could have thought of when presented with the previous improper integral. A more natural candidate might have been $g(z) = \frac{e^{iz}}{z}$. There is an obvious problem with this choice however, which is that it has a pole on the contour we wish to integrate around. In the case where the pole is simple (as it is for e^{iz}/z) there is standard procedure for modifying the contour: one indents it by a small circular arc around the pole. Explicitly, we replace the ν_R with $\nu_R^- \star \gamma_\epsilon \star \nu_R^+$ where $\nu_R^{\pm}(t) = t$ and $t \in [-R, -\epsilon]$ for ν_R^-, and $t \in [\epsilon, R]$ for ν_R^+ (and as above $\gamma_\epsilon(t) = \epsilon e^{i(\pi - t)}$ for $t \in [0, \pi]$). Since $\frac{\sin(x)}{x}$ is bounded at $x = 0$ the sum

$$\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}dx + \int_{\epsilon}^{R} \frac{\sin(x)}{x}dx \to \int_{-R}^{R} \frac{\sin(x)}{x}dx,$$

as $\epsilon \to 0$, while the integral along γ_ϵ can be computed explicitly: by the Taylor expansion of e^{iz} we see that $\text{Res}_{z=0} \frac{e^{iz}}{z} = 1$, so that $e^{iz} - 1/z$ is bounded near 0. It follows that as $\epsilon \to 0$ we have $\int_{\gamma_\epsilon}(e^{iz}/z - 1/z)dz \to 0$. On the other hand $\int_{\gamma_\epsilon} dz/z = \int_{-\pi}^{0}(-\epsilon i e^{i(\pi-t)})/(e^{i(\pi-t)}dt = -i\pi$, so that we see

$$\int_{\gamma_\epsilon} \frac{e^{iz}}{z}dz \to -i\pi$$

as $\epsilon \to 0$.

Combining all of this we conclude that if $\Gamma_\epsilon = \nu_R^- \star \gamma_\epsilon \star \nu_R^+ \star \gamma_R$ then

$$\begin{aligned} 0 = \int_{\Gamma_\epsilon} f(z)dz &= \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}dx + \int_{\gamma_\epsilon} \frac{e^{iz}}{z}dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x}dx + \int_{\gamma_R} \frac{e^{iz}}{z}dz. \\ &= 2i\int_\epsilon^R \frac{\sin(x)}{x} + \int_{\gamma_\epsilon} \frac{e^{iz}}{z} + \int_{\gamma_R} \frac{e^{iz}}{z}dz \\ &\to 2i\int_0^R \frac{\sin(x)}{x}dx - i\pi + \int_{\gamma_R} \frac{e^{iz}}{z}dz. \end{aligned}$$

as $\epsilon \to 0$. Then letting $R \to \infty$, it follows from Jordans Lemma that the third term tends to zero so we see that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx = 2\int_0^\infty \frac{\sin(x)}{x}dx = \pi$$

as required.

We record a general version of the calculation we made for the contribution of the indentation to a contour in the following Lemma.

Lemma 11.11. *Let $f\colon U \to \mathbb{C}$ be a meromorphic function with a simple pole at $a \in U$ and let $\gamma_\epsilon\colon [\alpha,\beta] \to \mathbb{C}$ be the path $\gamma_\epsilon(t) = a + \epsilon e^{it}$, then*

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)dz = Res_a(f).(\beta-\alpha)i.$$

Proof. Since f has a simple pole at a, we may write

$$f(z) = \frac{c}{z-a} + g(z)$$

where $g(z)$ is holomorphic near z and $c = \mathrm{Res}_a(f)$ (indeed $c/(z-a)$ is just the principal part of f at a). But now as g is holomorphic at a, it is continuous at a, and so bounded. Let $M, r > 0$ be such that $|g(z)| < M$ for all $z \in B(a,r)$. Then if $0 < \epsilon < r$ we have

$$\Big|\int_{\gamma_\epsilon} g(z)dz\Big| \leq \ell(\gamma_\epsilon)M = (\beta-\alpha)\epsilon.M,$$

which clearly tends to zero as $\epsilon \to 0$. On the other hand, we have

$$\int_{\gamma_\epsilon} \frac{c}{z-a}dz = \int_\alpha^\beta \frac{c}{\epsilon e^{it}} i\epsilon e^{it}dt = \int_\alpha^\beta (ic)dt = ic(\beta-\alpha).$$

Since $\int_{\gamma_\epsilon} f(z)dz = \int_{\gamma_\epsilon} c/(z-a)dz + \int_{\gamma_\epsilon} g(z)dz$ the result follows. □

11.3. **On the computation of residues and principal parts.** The previous examples will hopefully have convinced you of the power of the residue theorem. Of course for it to be useful one needs to be able to calculate the residues of functions with isolated singularities. In practice the integral formulas we have obtained for the residue are often not the best way to do this. In this section we discuss a more direct approach which is often useful when one wishes to calculate the residue of a function which is given as the ratio of two holomorphic functions.

More precisely, suppose that we have a function $F\colon U \to \mathbb{C}$ given to us as a ratio f/g of two holomorphic functions f, g on U where g is non-constant. The singularities of the function F are therefore poles which are located precisely at the (isolated) zeros of the function g, so that F is meromorphic. For convenience, we assume that we have translated the plane so as to ensure the pole of F we are interested in is at $a = 0$. Let $g(z) = \sum_{n\geq 0} c_n z^n$ be the power series for g, which will converge to $g(z)$ on any $B(0,r)$ such that $\bar{B}(0,r) \subseteq U$. Since $g(0) = 0$, and this zero is isolated, there is a $k > 0$ minimal with $c_k \neq 0$, and hence

$$g(z) = c_k z^k (1 + \sum_{n\geq 1} a_n z^n),$$

where $a_n = c_{n+k}/c_k$. Now if we let $h(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$ then $h(z)$ is holomorphic in $B(0,r)$ – since $h(z) = (g(z) - c_k z^k)/(c_k z^{k+1})$ – and moreover

$$\frac{1}{g(z)} = \frac{1}{c_k z^k}\big(1 + zh(z)\big)^{-1},$$

Now as h is continuous, it is bounded on $\bar{B}(0,r)$, say $|h(z)| < M$ for all $z \in \bar{B}(0,r)$. But then we have, for $|z| \leq \delta = \min\{r, 1/(2M)\}$,

$$\frac{1}{g(z)} = \frac{1}{c_k z^k}\big(\sum_{n=0}^{\infty}(-1)^n z^n h(z)^n\big),$$

where by the Weierstrass M-test, the above series converges uniformly on $\bar{B}(0,\delta)$. Moreover, for any n, the series $\sum_{m\geq n}(-1)^m z^m h(z)^m$ is a holomorphic function which vanishes to order at least n at $z = 0$, so that $\frac{1}{c_k z^k}\sum_{n\geq k}(-1)^n z^n h(z)^n$ is holmorphic. It follows that the principal part of the Laurent series of $1/g(z)$ is equal to the principal part of the function

$$\frac{1}{c_k z^k}\sum_{n=1}^{k}(-1)^{k-1} z^k h(z)^k.$$

Since we know the power series for $h(z)$, this allows us to compute the principal part of $\frac{1}{g(z)}$ as claimed. Finally, the principal part $P_0(F)$ of $F = f/g$ at $z = 0$ is just the $P_0(f.P_0(g))$, the principal part of the function $f(z).P_0(g)$, which again is straight-forward to compute if we know the power series expansion of $f(z)$ at 0 (indeed we only need the first k terms of it). The best way to digest this analysis is by means of examples. We consider one next, and will examine another in the next section on summation of series.

Example 11.12. Consider $f(z) = 1/(z^2\sinh(z)^3)$. Now $\sinh(z) = (e^z - e^{-z})/2$ vanishes on $\pi i\mathbb{Z}$, and these zeros are all simple since $\frac{d}{dz}(\sinh(z)) = \cosh(z)$ has $\cosh(n\pi i) = (-1)^n \neq 0$. Thus $f(z)$ has a pole or order 5 at zero, and poles of order 3 at $\pi i n$ for each $n \in \mathbb{Z}\backslash\{0\}$. Let us calculate the principal part of f at $z = 0$ using the above technique. We will write $O(z^k)$ for the vector space of holomorphic functions which vanish to order k at 0.

$$\begin{aligned}
z^2\sinh(z)^3 &= z^2(z + \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7))^3 = z^5(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + O(z^6))^3 \\
&= z^5(1 + \frac{3z^2}{3!} + \frac{3z^4}{(3!)^2} + \frac{3z^4}{5!} + O(z^6)) \\
&= z^5(1 + \frac{z^2}{2} + \frac{13z^4}{120} + O(z^6)) \\
&= z^5\left(1 + z(\frac{z}{2} + \frac{13z^3}{120} + O(z^5))\right)
\end{aligned}$$

Thus, in the notation of the above discussion, $h(z) = \frac{z}{2} + \frac{13z^3}{120} + O(z^5)$, and so, as h vanishes to first order at $z = 0$, in order to obtain the principal part we just need to consider the first two terms in the geometric series $(1 + zh(z))^{-1} = \sum_{n=0}^{\infty}(-1)^n z^n h(z)^n$:

$$\begin{aligned}
1/z^2\sinh(z)^3 &= z^{-5}\big(1 + z(\frac{z}{2} + \frac{13z^3}{120} + O(z^5))\big)^{-1} \\
&= z^{-5}\big(1 - z(\frac{z}{2} + \frac{13z^3}{120}) + z^2\frac{z^2}{(2!)^2} + O(z^5)\big) \\
&= z^{-5}\big(1 - \frac{z^2}{2} + (\frac{1}{4} - \frac{13}{120})z^4 + O(z^5)\big) \\
&= \frac{1}{z^5} - \frac{1}{2z^3} + \frac{17}{120z} + O(z).
\end{aligned}$$

Thus the principal part of $f(z)$ at 0 is $P_0(f) = \frac{1}{z^5} - \frac{1}{2z^3} + \frac{17}{120z}$, and $\mathrm{Res}_0(f) = 17/120$.

There are other variants on the above method which we could have used: For example, by the binomial theorem for an arbitrary exponent we know that if $|z| < 1$ then $(1+z)^{-3} = \sum_{n\geq 0}\binom{-3}{n}z^n = 1 - 3z + 6z^2 + \ldots$. Arguing as above, it follows that for small enough z we have

$$\begin{aligned}\sinh(z)^{-3} &= z^{-3}.(1+\frac{z^2}{3!}+\frac{z^4}{5!}+O(z^6))^{-3}\\ &= z^{-3}\left(1+(-3)\big(\frac{z^2}{3!}+\frac{z^4}{5!}\big)+6\big(\frac{z^2}{3!}+\frac{z^4}{5!}\big)^2+O(z^6)\right)\\ &= z^{-3}\left(1-\frac{z^2}{2}+\big(\frac{-3}{5!}+\frac{6}{(3!)^2}\big)z^4+O(z^6)\right)\\ &= z^{-3}\left(1-\frac{z^2}{2}+\frac{17z^4}{120}+O(z^6)\right)\end{aligned}$$

yielding the same result for the principal part of $1/z^2\sinh(z)^3$.

11.4. **Summation of infinite series.** Residue calculus can also be a useful tool in calculating infinite sums, as we now show. For this we use the function $f(z) = \cot(\pi z)$. Note that since $\sin(\pi z)$ vanishes precisely at the integers, $f(z)$ is meromorphic with poles at each integer $n \in \mathbb{Z}$. Moreover, since f is periodic with period 1, in order to understand the poles of f it suffices to calculate the principal part of f at $z = 0$. We can use the method of the previous section to do this:

We have $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)$, so that $\sin(z)$ vanishes with multiplicity 1 at $z = 0$ and we may write $\sin(z) = z(1 - zh(z))$ where $h(z) = z/3! - z^3/5! + O(z^5)$ is holomorphic at $z = 0$. Then

$$\frac{1}{\sin(z)} = \frac{1}{z}(1-zh(z))^{-1} = \frac{1}{z}\big(1+\sum_{n\geq 1} z^n h(z)^n\big) = \frac{1}{z} + h(z) + O(z^2).$$

Multiplying by $\cos(z)$ we see that the principal part of $\cot(z)$ is the same as that of $\frac{1}{z}\cos(z)$ which, using the Taylor expansion of $\cos(z)$, is clearly $\frac{1}{z}$ again. By periodicity, it follows that $\cot(\pi z)$ has a simple pole with residue $1/\pi$ at each integer $n \in \mathbb{Z}$.

We can also use this strategyto find further terms of the Laurent series of $\cot(z)$: Since our $h(z)$ actually vanishes at $z = 0$, the terms $h(z)^n z^n$ vanish to order $2n$. It follows that we obtain all the terms of the Laurent series of $\cot(z)$ at 0 up to order 3, say, just by considering the first two terms of the series $1 + \sum_{n\geq 1} z^n h(z)^n$, that is, $1 + zh(z)$. Since $\cos(z) = 1 - z^2/2! + z^4/4!$, it follows that $\cot(z)$ has a Laurent series

$$\begin{aligned}\cot(z) &= (1-\frac{z^2}{2!}+O(z^4)).\big(\frac{1}{z}+(\frac{z}{3!}-\frac{z^3}{5!}+O(z^5))\big)\\ &= \frac{1}{z}-\frac{z}{3}+O(z^3)\end{aligned}$$

The fact that $f(z)$ has simple poles at each integer will allow us to sum infinite series with the help of the following:

Lemma 11.13. *Let $f(z) = \cot(\pi z)$ and let Γ_N denotes the square path with vertices $(N+1/2)(\pm 1 \pm i)$. There is a constant C independent of N such that $|f(z)| \leq C$ for all $z \in \Gamma_N^*$.*

Proof. We need to consider the horizontal and vertical sides of the square separately. Note that $\cot(\pi z) = (e^{i\pi z} + e^{-i\pi z})/(e^{i\pi z} - e^{-i\pi z})$. Thus on the horizontal sides of Γ_N where $z = x \pm (N+1/2)i$ and $-(N+1/2) \leq x \leq (N+1/2)$ we have

$$\begin{aligned} |\cot(\pi z)| &= \left| \frac{e^{i\pi(x \pm (N+1/2)i)} + e^{-i\pi(x \pm (N+1/2)i)}}{e^{i\pi(x \pm (N+1/2)i} - e^{-i\pi(x \pm (N+1/2)i)}} \right| \\ &\leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}} \\ &= \coth(\pi(N+1/2)). \end{aligned}$$

Now since $\coth(x)$ is a decreasing function for $x \geq 0$ it follows that on the horizontal sides of Γ_N we have $|\cot(\pi z)| \leq \coth(3\pi/2)$.

On the vertical sides we have $z = \pm(N+1/2) + iy$, where $-N - 1/2 \leq y \leq N + 1/2$. Observing that $\cot(z + N\pi) = \cot(z)$ for any integer N and that $\cot(z + \pi/2) = -\tan(z)$, we find that if $z = \pm(N+1/2) + iy$ for any $y \in \mathbb{R}$ then

$$|\cot(\pi z)| = |-\tan(iy)| = |-\tanh(y)| \leq 1.$$

Thus we may set $C = \max\{1, \coth(3\pi/2)\}$. □

We now show how this can be used to sum an infinite series:

Example 11.14. Let $g(z) = \cot(\pi z)/z^2$. By our discussion of the poles of $\cot(\pi z)$ above it follows that $g(z)$ has simple poles with residues $\frac{1}{\pi n^2}$ at each non-zero integer n and residue $-\pi/3$ at $z = 0$.

Consider now the integral of $g(z)$ around the paths Γ_N: By Lemma 11.13 we know $|g(z)| \leq C/|z|^2$ for $z \in \Gamma_N^*$, and for all $N \geq 1$. Thus by the estimation lemma we see that

$$\left(\int_{\Gamma_N} g(z)dz \right) \leq C.(4N+2)/(N+1/2)^2 \to 0,$$

as $N \to \infty$. But by the residue theorem we know that

$$\int_{\Gamma_N} g(z)dz = -\pi/3 + \sum_{\substack{n \neq 0, \\ -N \leq n \leq N}} \frac{1}{\pi n^2}.$$

It therefore follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

Remark 11.15. Notice that the contours Γ_N and the function $\cot(\pi z)$ clearly allows us to sum other infinite series in a similar way – for example if we wished to calculate the sum of the infinite series $\sum_{n\geq 1}\frac{1}{n^2+1}$ then we would consider the integrals of $g(z)=\cot(\pi z)/(1+z^2)$ over the contours Γ_N.

Remark 11.16. (*Non-examinable – for interest only!*): Note that taking $g(z)=(1/z^{2k})\cot(\pi z)$ for any positive integer k, the above strategy gives a method for computing $\sum_{n=1}^{\infty}1/n^{2k}$ (check that you see why we need to take even powers of n). The analysis for the case $k=1$ goes through in general, we just need to compute more and more of the Laurent series of $\cot(\pi z)$ the larger we take k to be.

One can show that $\zeta(s)=\sum_{n=1}^{\infty}1/n^s$ converges to a holomorphic function of s for any $s\in\mathbb{C}$ with $\Re(s)>1$ (as usual, we define $n^s=\exp(s.\log(n))$ where $\log$ is the ordinary real logarithm). As $s\to 1$ it can be checked that $\zeta(s)\to\infty$, however it can be shown that $\zeta(s)$ extends to a meromorphic function on all of $\mathbb{C}\backslash\{1\}$. The identity theorem shows that this extension is unique if it exists[18]. (This uniqueness is known as the principle of "analytic continuation".) The location of the zeros of the ζ-function is the famous Riemann hypothesis: apart from the "trivial zeros" at negative even integers, they are conjectured to all lie on the line $\Re(z)=1/2$. Its values at special points however are also of interest: Euler was the first to calculate $\zeta(2k)$ for positive integers k, but the values $\zeta(2k+1)$ (for k a positive integer) remain mysterious – it was only shown in 1978 by Roger Apéry that $\zeta(3)$ is irrational for example. Our analysis above is sufficient to determine $\zeta(2k)$ once one succeeds in computing explicitly the Laurent series for $\cot(\pi z)$ or equivalently the Taylor series of $z\cot(\pi z)=iz+2iz/(e^{2iz}-1)$.

11.5. **Keyhole contours.** There are many ingenious paths which can be used to calculate integrals via residue theory. One common contour is known (for obvious reasons) as a *keyhole contour*. It is constructed from two circular paths of radius ϵ and R, where we let R become arbitrarily large, and ϵ arbitrarily small, and we join the two circles by line segments with a narrow neck in between. Explicitly, if $0<\epsilon<R$ are given, pick a $\delta>0$ small, and set $\eta_+(t)=t+i\delta$, $\eta_-(t)=(R-t)-i\delta$, where in each case t runs over the closed intervals with endpoints such that the endpoints of $\eta_\pm$ lie on the circles of radius ϵ and R about the origin. Let γ_R be the positively oriented path on the circle of radius R joining the endpoints of η_+ and η_- on that circle (thus traversing the "long" arc of the circle between the two points) and similarly let γ_ϵ the path on the circle of radius ϵ which is negatively oriented and joins the endpoints of $\gamma_\pm$ on the circle of radius ϵ. Then we set $\Gamma_{R,\epsilon}=\eta_+\star\gamma_R\star\eta_-\star\gamma_\epsilon$ (see Figure 4). The keyhole contour can sometimes be useful to evaluate real integrals where the integrand is multivalued as a function on the complex plane, as the next example shows:

[18] It is this uniqueness and the fact that one can readily compute that $\zeta(-1)=-1/12$ that results in the rather outrageous formula $\sum_{n=1}^{\infty}n=-1/12$.

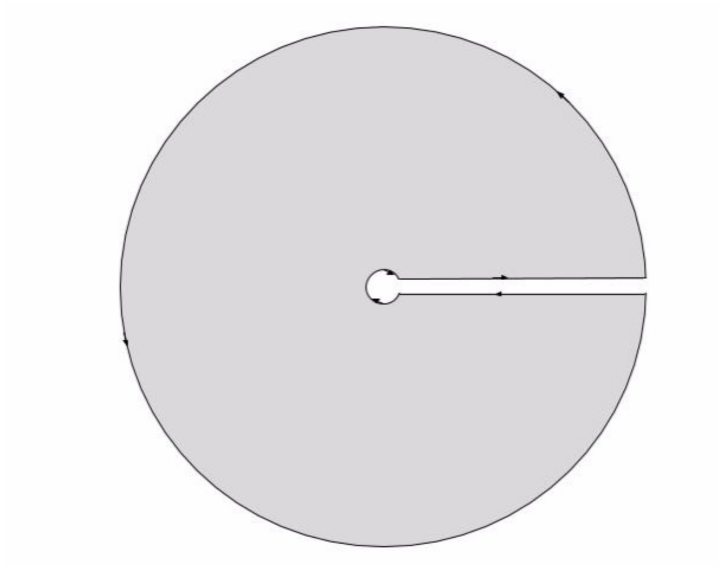

FIGURE 4. A keyhole contour.

Example 11.17. Consider the integral $\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$. Let $f(z) = z^{1/2}/(1+z^2)$, where we use the branch of the square root function which is continuous on $\mathbb{C}\backslash\mathbb{R}_{>0}$, that is, if $z = re^{it}$ with $t \in [0, 2\pi)$ then $z^{1/2} = r^{1/2}e^{it/2}$.

We use the keyhole contour $\Gamma_{R,\epsilon}$. On the circle of radius R, we have $|f(z)| \leq R^{1/2}/(R^2-1)$, so by the estimation lemma, this contribution to the integral of f over $\Gamma_{R,\epsilon}$ tends to zero as $R \to \infty$. Similarly, $|f(z)|$ is bounded by $\epsilon^{1/2}/(1-\epsilon^2)$ on the circle of radius ϵ, thus again by the estimation lemma this contribution to the integral of f over $\Gamma_{R,\epsilon}$ tends to zero as $\epsilon \to 0$. Finally, the discontinuity of our branch of $z^{1/2}$ on $\mathbb{R}_{>0}$ ensures that the contributions of the two line segments of the contour do not cancel but rather both tend to $\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$ as δ and ϵ tend to zero.

To compute $\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$ we evaluate the integral $\int_{\Gamma_{R,\epsilon}} f(z)dz$ using the residue theorem: The function $f(z)$ clearly has simple poles at $z = \pm i$, and their residues are $\frac{1}{2}e^{-\pi i/4}$ and $\frac{1}{2}e^{5\pi i/4}$ respectively. It follows that

$$\int_{\Gamma_{R,\epsilon}} f(z)dz = 2\pi i\left(\frac{1}{2}e^{-\pi i/4} + \frac{1}{2}e^{5\pi i/4}\right) = \pi\sqrt{2}.$$

Taking the limit as $R \to \infty$ and $\epsilon \to 0$ we see that $2\int_0^\infty \frac{x^{1/2}}{1+x^2}dx = \pi\sqrt{2}$, so that

$$\int_0^\infty \frac{x^{1/2}dx}{1+x^2} = \frac{\pi}{\sqrt{2}}.$$

12. Conformal transformations

Another important feature of the stereographic projection map is that it is *conformal*, meaning that it preserves angles. The following definition helps us to formalize what this means:

Definition 12.1. If $\gamma\colon [-1,1] \to \mathbb{C}$ is a C^1 path which has $\gamma'(t) \neq 0$ for all t, then we say that the line $\{\gamma(t) + s\gamma'(t) : s \in \mathbb{R}\}$ is the *tangent line* to γ at $\gamma(t)$, and the vector $\gamma'(t)$ is a tangent vector at $\gamma(t) \in \mathbb{C}$.

Remark 12.2. Note that this definition gives us a notion of tangent vectors at points on subsets of $\mathbb{R}^n$, since the notion of a C^1 path extends readily to paths in $\mathbb{R}^n$ (we just require all n component functions are continuously differentiable). In particular, if $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$ as above, a C^1 path on $\mathbb{S}$ is simply a path $\gamma\colon [a,b] \to \mathbb{R}^3$ whose image lies in $\mathbb{S}$. It is easy to check that the tangent vectors at a point $p \in \mathbb{S}$ all lie in the plane perpendicular to p – simply differentiate the identity $f(\gamma(t)) = 1$ where $f(x,y,z) = x^2 + y^2 + z^2$ using the chain rule.

We can now state what we mean by a conformal map:

Definition 12.3. Let U be an open subset of $\mathbb{C}$ and suppose that $T\colon U \to \mathbb{C}$ (or $\mathbb{S}$) is continuously differentiable in the real sense (so all its partial derivatives exist and are continuous). If $\gamma_1, \gamma_2\colon [-1,1] \to U$ are two paths with $z_0 = \gamma_1(0) = \gamma_2(0)$ then $\gamma_1'(0)$ and $\gamma_2'(0)$ are two tangent vectors at z_0, and we may consider the angle between them (formally speaking this is the difference of their arguments). By our assumption on T, the compositions $T\circ\gamma_1$ and $T\circ\gamma_2$ are C^1-paths through $T(z_0)$, thus we obtain a pair of tangent vectors at $T(z_0)$. We say that T is *conformal* at z_0 if for every pair of C^1 paths γ_1, γ_2 through z_0, the angle between their tangent vectors at z_0 is equal to the angle between the tangent vectors at $T(z_0)$ given by the C^1 paths $T \circ \gamma_1$ and $T \circ \gamma_2$. We say that T is conformal on U if it is conformal at every $z \in U$.

One of the main reasons we focus on conformal maps here is because holomorphic functions give us a way of producing many examples of them, as the following result shows.

Proposition 12.4. *Let $f\colon U \to \mathbb{C}$ be a holomorphic map and let $z_0 \in U$ be such that $f'(z_0) \neq 0$. Then f is conformal at z_0. In particular, if $f\colon U \to \mathbb{C}$ is has nonvanishing derivative on all of U, it is conformal on all of U (and locally a biholomorphism).*

Proof. We need to show that f preserves angles at z_0. Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$. We show that a version of the chain rule applies to these compositions. For $i = 1, 2$ we have

$$\eta_i'(0) = \lim_{h\to 0} \frac{f(\gamma_i(h)) - f(\gamma(0))}{h} = \lim_{h\to 0} \frac{f(\gamma_i(h)) - f(z_0)}{\gamma_i(h) - z_0} \cdot \frac{\gamma_i(h) - z_0}{h}$$

Clearly for small h, $\gamma_i(h) \neq z_0$ as $\gamma_i'(0) \neq 0$ and $\lim_{h\to 0} \dfrac{f(\gamma_i(h)) - f(z_0)}{\gamma_i(h) - z_0} = f'(z_0)$. So if we set $f'(z_0) = \rho e^{i\theta}$ we have

$$\eta_i'(0) = f'(z_0)\gamma_i'(0) = \rho e^{i\theta}\gamma_i'(0), \quad i = 1, 2.$$

Hence if ϕ_1 and ϕ_2 are the arguments of $\gamma_1'(0)$ and $\gamma_2'(0)$, then the arguments of $\eta_1'(0)$ and $\eta_2'(0)$ are $\phi_1+\theta$ and $\phi_2+\theta$ respectively. It follows that the difference between the two pairs of arguments, that is, the angles between the curves at z_0 and $f(z_0)$, are the same.

For the final part, note that if $f'(z_0) \neq 0$ then by the definition of the degree of vanishing, the function $f(z)$ is locally biholomorphic (see the proof of the inverse function theorem). □

Example 12.5. The function $f(z) = z^2$ has $f'(z)$ nonzero everywhere except the origin. It follows f is a conformal map from $\mathbb{C}^\times$ to itself. Note that the condition that $f'(z)$ is non-zero is necessary – if we consider the function $f(z) = z^2$ at $z = 0$, $f'(z) = 2z$ which vanishes precisely at $z = 0$, and it is easy to check that at the origin f in fact doubles the angles between tangent vectors.

Lemma 12.6. *The sterographic projection map $S\colon \mathbb{C} \to \mathbb{S}$ is conformal.*

Proof. Let z_0 be a point in $\mathbb{C}$, and suppose that $\gamma_1(t) = z_0 + tv_1$ and $\gamma_2(t) = z_0 + tv_2$ are two paths[19] having tangents v_1 and v_2 at $z_0 = \gamma_1(0) = \gamma_2(0)$. Then the lines L_1 and L_2 they describe, together with the point N, determine planes H_1 and H_2 in $\mathbb{R}^3$, and moreover the image of the lines under stereographic projection is the intersection of these planes with $\mathbb{S}$. Since the intersection of $\mathbb{S}$ with any plane is either empty or a circle, it follows that the paths γ_1 and γ_2 get sent to two circles C_1 and C_2 passing through $P = S(z_0)$ and N. Now by symmetry, these circles meet at the same angle at N as they do at P. Now the tangent lines of C_1 and C_2 at N are just the intersections of H_1 and H_2 with the plane tangent to $\mathbb{S}$ at N. But this means the angle between them will be the same as that between the intersection of H_1 and H_2 with the complex plane, since it is parallel to the tangent plane of $\mathbb{S}$ at N. Thus the angles between C_1 and C_2 at P and L_1 and L_2 at z_0 coincide as required. □

Although it follows easily from what we have already done, it is worth high-lighting the following:

Lemma 12.7. *Möbius transformations are conformal.*

Proof. As we have already shown, any holomorphic map is conformal wherever its derivative is nonzero.

For a Möbius map we have $f(z) = \frac{az+b}{cz+d}$ and

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0,$$

[19] with domain $[-1, 1]$ say – or even the whole real line, except that it is non-compact.

for all $z \neq -d/c$, thus f is conformal at each $z \in \mathbb{C}\backslash\{-d/c\}$. □

Remark 12.8. (off syllabus) We may see Möbius maps as maps from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ and then they are defined for any $z \in \mathbb{C}_\infty$. So, using the identification of $\mathbb{S}$ with $\mathbb{C}_\infty$ by the stereographic projection map S, we see them as maps from $\mathbb{S}$ to $\mathbb{S}$ where $\mathbb{S}$ is the unit sphere. It turns out that Möbius maps are then conformal for every $z \in \mathbb{S}$. Indeed we have seen that any Möbius transformation can be written as a composition of dilations, translations and an inversion. So it suffices to show that each of these maps is conformal. As we showed when we analysed the Example 2.6, under the identification $S : \mathbb{C}_\infty \to \mathbb{S}$, $1/z$ corresponds to the map $(t, u, v) \mapsto (t, -u, -v)$, which is a rotation by π about the x-axis, so clearly it is conformal.

The maps $z \mapsto z+a$, $z \mapsto az$ $(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$ (as the stereographic projection map S is conformal and compositions of conformal maps are conformal). We claim that if f is $z \mapsto z + a$ or $z \mapsto az$ then f is conformal at N as well.

To see this we consider the images of great circles through N. These circles correspond to lines through 0 under S and as in the lemma 12.6 we note that the angles of two such circles at N is equal to the angle of the lines at 0. But, since f is conformal as a map $\mathbb{C} \to \mathbb{C}$ the angles at 0 are preserved by f, so the angles at N are preserved as well.

Since a Möbius map is given by the four entries of a 2×2 matrix, up to simultaneous rescaling, the following result is perhaps not too surprising.

Proposition 12.9. *If z_1, z_2, z_3 and w_1, w_2, w_3 are triples of pairwise distinct complex numbers, then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for each $i = 1, 2, 3$.*

Proof. It is enough to show that, given any triple (z_1, z_2, z_3) of complex numbers, we can find a Möbius transformations which takes z_1, z_2, z_3 to $0, 1, \infty$ respectively. Indeed if f_1 is such a transformation, and f_2 takes $0, 1, \infty$ to w_1, w_2, w_3 respectively, then clearly $f_2 \circ f_1^{-1}$ is a Möbius transformation which takes z_i to w_i for each i.

Now consider

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

It is easy to check that $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$, and clearly f is a Möbius transformation as required. If any of z_1, z_2 or z_3 is ∞, then one can find a similar transformation (for example by letting $z_i \to \infty$ in the above formula). Indeed if $z_1 = \infty$ then we set $f(z) = \frac{z_2 - z_3}{z - z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z - z_1}{z - z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z - z_1}{z_2 - z_1}$.

To see the f is unique, suppose f_1 and f_2 both took z_1, z_2, z_3 to w_1, w_2, w_3. Then taking Möbius transformations g, h sending z_1, z_2, z_3 and w_1, w_2, w_3 to $0, 1, \infty$ the transformations hf_1g^{-1} and hf_2g^{-1} both take $(0, 1, \infty)$ to $(0, 1, \infty)$. But suppose $T(z) = \frac{az+b}{cz+d}$ is any Möbius transformation with $T(0) = 0$, $T(1) = 1$ and $T(\infty) = \infty$. Since T fixes ∞ it follows $c = 0$. Since $T(0) = 0$ it follows that $b/d = 0$ hence $b = 0$, thus $T(z) = a/d \cdot z$, and since $T(1) = 1$ it follows $a/d = 1$ and hence $T(z) = z$. Thus we see that

$$hf_1g^{-1} = hf_2g^{-1} = \text{id},$$

and so $f_1 = f_2$ as required. □

Example 12.10. The above lemma shows that we can use Möbius transformations as a source of conformal maps. For example, suppose we wish to find a conformal transformation which takes the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ to the unit disk $B(0, 1)$. The boundary of $\mathbb{H}$ is the real line, and we know Möbius transformations take lines to lines or circles, and in the latter case this means the point $\infty \in \mathbb{C}_\infty$ is sent to a finite complex number. Now any circle is uniquely determined by three points lying on it, and we know Möbius transformations allow us to take any three points to any other three points. Thus if we take f the Möbius map which sends $0 \mapsto -i$, and $1 \mapsto 1$, $\infty \mapsto i$ the real axis will be sent to the unit circle. Now we have

$$f(z) = \frac{iz + 1}{z + i}$$

(one can find f in a similar fashion to the proof of Proposition 12.9).

So far, we have found a Möbius transformation which takes the real line to the unit circle. Since $\mathbb{C}\backslash\mathbb{R}$ has two connected components, the upper and lower half planes, $\mathbb{H}$ and $i\mathbb{H}$, and similarly $\mathbb{C}\backslash\mathbb{S}^1$ has two connected components, $B(0, 1)$ and $\mathbb{C}\backslash\bar{B}(0, 1)$. Since a Möbius transformation is continuous, it maps connected sets to connected sets, thus to check whether $f(\mathbb{H}) = B(0, 1)$ it is enough to know which component of $\mathbb{C}\backslash\mathbb{S}^1$ a single point in $\mathbb{H}$ is sent to. But $f(i) = 0 \in B(0, 1)$, so we must have $f(\mathbb{H}) = B(0, 1)$ as required.

Note that if we had taken $g(z) = (z + i)/(iz + 1)$ for example, then g also maps $\mathbb{R}$ to the unit circle $\mathbb{S}^1$, but $g(-i) = 0$, so[20] g maps the lower half plane to $B(0, 1)$. If we had used this transformation, then it would be easy to "correct" it to get what we wanted: In fact there are (at least) two simple things one could do: First, one could note that the map $R(z) = -z$ (a rotation by π) sends the upper half plane to the lower half place, so that the composition $g \circ R$ is a Möbius transformation taking $\mathbb{H}$ to $B(0, 1)$. Alternatively, the inversion $j(z) = 1/z$ sends $\mathbb{C}\backslash\bar{B}(0, 1)$ to $B(0, 1)$, so that $j \circ g$ also sends $\mathbb{H}$ to $B(0, 1)$. Explicitly, we have

$$g \circ R(z) = \frac{z - i}{iz - 1} = \frac{-i(iz + 1)}{i(z + i)} = -f(z), \quad j \circ g(z) = \frac{iz + 1}{z + i} = f(z).$$

Note in particular that f is far from unique – indeed if f is any Möbius transformation which takes $\mathbb{H}$ to $B(0, 1)$ then composing it with any Möbius

[20]A Möbius map is a continuous function on $\mathbb{C}_\infty$, and if we remove a circle from $\mathbb{C}_\infty$ the complement is a disjoint union of two connected components, just the same as when we remove a line or a circle from the plane, thus the connectedness argument works just as well when we include the point at infinity.

transformation which preserves $B(0,1)$ will give another such map. Thus for example $e^{i\theta}.f$ will be another such transformation.

Definition 12.11. If there is a bijective conformal transformation between two domains U and V in the complex plane then we say that they are *conformally equivalent*.

Since two conformally equivalent domains are in particular homeomorphic, one can not expect that any two domains are conformally equivalent. However it turns out that this is the case if we restrict to simply-connected domains (that is, domains in which any path can be continuously deformed to any other path with the same end-points). Since it will play a distinguished role later, we will write $\mathbb{D}$ for the unit disc $B(0,1)$.

Theorem 12.12. *(Riemann's mapping theorem): Let U be an open connected and simply-connected proper subset of* $\mathbb{C}$*. Then for any* $z_0 \in U$ *there is a unique bijective conformal transformation* $f\colon U \to \mathbb{D}$ *such that* $f(z_0) = 0$, $f'(z_0) > 0$.

Remark 12.13. The proof of this theorem is beyond the scope of this course, but it is a beautiful and fundamental result. The proof in fact only uses the fact that on a simply-connected domain any holomorphic function has a primitive, and hence it in fact shows that such domains are simply-connected in the topological sense (since a conformal transformation is in particular a homeomorphism, and the disc is simply-connected). It relies crucially on *Montel's theorem* on families of holomorphic functions, see for example the text of Shakarchi and Stein[21] for an exposition of the argument.

Note that it follows immediately from Liouville's theorem that there can be no bijective conformal transformation taking $\mathbb{C}$ to $B(0,1)$, so the whole complex plane is indeed an exception. The uniqueness statement of the theorem reduces to the question of understanding the conformal transformations of the disk $\mathbb{D}$ to itself.

Of course knowing that a conformal transformation between two domains D_1 and D_2 exists still leaves the challenge of constructing one. As we will see in the next section on harmonic maps, this is an important question. In simple cases one can often do so by hand, as we now show.

In addition to Möbius transformations, it is often useful to use the exponential function and branches of the multifunction $[z^\alpha]$ (away from the origin) when constructing conformal maps. We give an example of the kind of constructions one can do:

Example 12.14. Let $D_1 = B(0,1)$ and $D_2 = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Since these domains are both convex, they are simply-connected, so Riemann's mapping theorem ensure that there is a conformal map sending D_2 to D_1. To construct such a map, note that the domain is defined by the two curves $\gamma(0,1)$ and the real axis. It can be convenient to map the two

[21] *Complex Analysis*, Princeton Lecture in Analysis II, E. M. Stein & R. Shakarchi. P.U.P.

points of intersection of these curves, ± 1 to 0 and ∞. We can readily do this with a Möbius transformation:

$$f(z) = \frac{z-1}{z+1},$$

Now since f is a Möbius transformation, it follows that $f(\mathbb{R})$ and $f(\gamma(0,1))$ are lines (since they contain ∞) passing through the origin. Indeed $f(\mathbb{R}) = \mathbb{R}$, and since f had inverse $f^{-1} = \frac{z+1}{z-1}$ it follows that the image of $\gamma(0,1)$ is $\{w \in \mathbb{C} : |w-1| = |w+1|\}$, that is, the imaginary axis. Since $f(i/2) = (-3+4i)/5$ it follows by connectedness that $f(D_1)$ is the second quadrant $Q = \{w \in \mathbb{C} : \Re(z) < 0, \Im(z) > 0\}$.

Now the squaring map $s \colon \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ maps Q bijectively to the half-plane $H = \{w \in \mathbb{C} : \Im(w) < 0\}$, and is conformal except at $z = 0$ (which is on the boundary, not in the interior, of Q). We may then use a Möbius map to take this half-plane to the unit disc: indeed in Example 12.10 we have already seen that the Möbius transformation $g(z) = \frac{z+i}{iz+1}$ takes the lower-half plane to the upper-half plane.

Putting everything together, we see that $F = g \circ s \circ f$ is a conformal transformation taking D_1 to D_2 as required. Calculating explicitly we find that

$$F(z) = i\left(\frac{z^2+2iz+1}{z^2-2iz+1}\right)$$

Remark 12.15. Note that there are couple of general principles one should keep in mind when constructing conformal transformations between two domains D_1 and D_2. Often if the boundary of D_1 has distinguished points (such as ± 1 in the above example) it is convenient to move these to "standard" points such as 0 and ∞, which one can do with a Möbius transformation. The fact that Möbius transformations are three-transitive and takes lines and circles to lines and circles and moreover act transitively on such means that we can always use Möbius transformations to match up those parts of the boundary of D_1 and D_2 given by line segments or arcs of circles. However these will not be sufficient in general: indeed in the above example, the fact that the boundary of D_1 is a union of a semicircle and a line segment, while that of D_2 is just a circle implies there is no Möbius transformation taking D_1 to D_2, as it would have to take ∂D_1 to ∂D_2, which would mean that its inverse would not take the unit circle to either a line or a circle. Branches of fractional power maps $[z^\alpha]$ are often useful as they allow us to change the angle at the points of intersection of arcs of the boundary (being conformal on the interior of the domain but not on its boundary).

12.1. **Conformal transformations and the Laplace equation.** In this section we will use the term *conformal map* or *conformal transformation* somewhat abusively to mean a holomorphic function whose derivative does not vanish on its domain of definition. (We have seen already that this implies the function is conformal in the sense of the previous section.)

Recall that a function $v\colon \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if it is twice differentiable and $\partial_x^2 v + \partial_y^2 v = 0$. Often one seeks to find solutions to this equation on a domain $U \subset \mathbb{R}^2$ where we specify the values of v on the boundary ∂U of U. This problem is known as the *Dirichlet problem*, and makes sense in any dimension (using the appropriate Laplacian). In dimension 2, complex analysis and in particular conformal maps are a powerful tool by which one can study this problem, as the following lemma show.

Lemma 12.16. *Suppose that $U \subset \mathbb{C}$ is a simply-connected open subset of $\mathbb{C}$ and $v\colon U \to \mathbb{R}$ is twice continuously differentiable and harmonic. Then there is a holomorphic function $f\colon U \to \mathbb{C}$ such that $\Re(f) = v$. In particular, any such function v is analytic.*

Proof. (*Sketch*): Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y\partial_x v = \partial_x\partial_y v,$$

so that g satisfies the Cauchy-Riemann equations. It follows from Theorem 3.9 that g is holomorphic. Now since U is simply-connected, it follows that g has a primitive $G\colon U \to \mathbb{C}$. But then it follows that if $G = a(z) + ib(z)$ we have $\partial_z G = \partial_x a - i\partial_y a = g(z) = \partial_x v - i\partial_y v$, hence the partial derivatives of a and v agree on all of U. But then if $z_0, z \in U$ and γ is a path between then, the chain rule[22] shows that

$$\begin{aligned}\int_\gamma (\partial_x v + i\partial_y v)dz &= \int_0^1 (\partial_x(v(\gamma(t)) + i\partial_y v(\gamma(t)))\gamma'(t)dt \\ &= \int_0^1 \frac{d}{dt}(v(\gamma(t)))dt = v(z) - v(z_0),\end{aligned}$$

Similarly, we see that the same path integral is also equal to $a(z) - a(z_0)$. It follows that $a(z) = v(z) + (a(z_0) - v(z_0))$, thus if we set $f(z) = G(z) - (G(z_0) - v(z_0))$ we obtain a holomorphic function on U whose real part is equal to v as required.

Since we know that any holomorphic function is analytic, it follows that v is analytic (and in particular, infinitely differentiable). □

The previous Lemma shows that, at least locally (in a disk say) harmonic functions and holomorphic functions are in correspondence – given a holomorphic function f we obtain a harmonic function by taking its real part, while if u is harmonic the previous lemma shows we can associate to it a holomorphic function f whose real part equals u (and in fact examining the proof, we see that f is actually unique up to a purely imaginary constant). Thus if we are seeking a harmonic function on an open set U whose values

[22]This uses the chain rule for a composition $g \circ f$ of real-differentiable functions $f\colon \mathbb{R} \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$, applied to the real and imaginary parts of the integrand.

are a given function g on ∂U, then it suffices to find a holomorphic function f on U such that $\Re(f) = g$ on the boundary ∂U.

Now if $H\colon U \to V$ was a bijective conformal transformation which extends to a homeomorphism $\bar{H}\colon \bar{U} \to \bar{V}$ which thus takes ∂U homeomorphically to ∂V, then if $f\colon V \to \mathbb{C}$ is holomorphic, so is $f \circ H$. Thus in particular $\Re(f \circ H)$ is a harmonic function on U. It follows that we can use conformal transformations to transport solutions of Laplace's equation from one domain to another: if we can use a conformal transformation H to take a domain U to a domain V where we already have a supply of holomorphic functions satisfying various boundary conditions, the conformal transformation H gives us a corresponding set of holomorphic (and hence harmonic) functions on U. We state this a bit more formally as follow:

Lemma 12.17. *If U and V are domains and $G\colon U \to V$ is a conformal transformation, then if $u\colon V \to \mathbb{R}$ is a harmonic function on V, the composition $u \circ G$ is harmonic on U.*

Proof. To see that $u \circ G$ is harmonic we need only check this in a disk $B(z_0, r) \subseteq U$ about any point $z_0 \in U$. If $w_0 = G(z_0)$, the continuity of G ensures we can find $\delta, \epsilon > 0$ such that $G(B(z_0, \delta)) \subseteq B(w_0, \epsilon) \subseteq V$. But now since $B(w_0, \epsilon)$ is simply-connected we know by Lemma 12.16 we can find a holomorphic function $f(z)$ with $u = \Re(f)$. But then on $B(z_0, \delta)$ we have $u \circ G = \Re(f \circ G)$, and by the chain rule $f \circ G$ is holomorphic, so that its real part is harmonic as required. □

Remark 12.18. You can also give a more direct computational proof of the above Lemma. Note also that we only need G to be holomorphic – the fact that it is a conformal equivalence is not necessary. On the other hand if we are trying to produce harmonic functions with prescribed boundary values, then we will need to use carefully chosen conformal transformations.

This strategy for studying harmonic functions might at first sight appear over-optimistic, in that the domains one can obtain from a simple open set like $B(0, 1)$ or the upper-half plane $\mathbb{H}$ might consist of only a small subset of the open sets one might be interested in. However, the Riemann mapping theorem (Theorem 12.12) show that *every* domain which is simply connected, other than the whole complex plane itself, is in fact conformally equivalent to $B(0, 1)$.

For the solution of Dirichlet's problem one needs something slightly stronger-namely that the conformal equivalence extends continuously to the boundary. One has the following theorem which is beyond the scope of this course (for a proof see the book Introduction to Complex Analysis by K. Kodaira, p. 215):

Theorem 12.19. *Let U, V be bounded domains in $\mathbb{C}$ and let $f : U \to V$ be a conformal map. If $\partial U, \partial V$ are piecewise C^1 Jordan curves the conformal map $f : U \to V$ can be extended to a homeomorphism $\bar{f} : \bar{U} \to \bar{V}$.*

For convenience, we will write $\mathbb{D}$ for the open disk $B(0,1)$ of radius 1 centred at 0.

In the course so far, the main examples of conformal transformations we have are the following:

(1) The exponential function is conformal everywhere, since it is its own derivative and it is everywhere nonzero.
(2) Möbius transformations understood as maps on the extended complex plane are everywhere conformal.
(3) Fractional exponents: In cut planes the functions $z \mapsto z^\alpha$ for $\alpha \in \mathbb{C}$ are conformal (the cut removes the origin, where the derivative may vanish).

Let us see how to use these transformations to obtain solutions of the Laplace equation. First notice that Cauchy's integral formula suggests a way to produce solutions to Laplace's equation in the disk: Suppose that u is a harmonic function defined on $B(0,r)$ for some $r > 1$. Then by Lemma 12.16 we know there is a holomorphic function $f\colon B(0,r) \to \mathbb{C}$ such that $u = \Re(f)$. By Cauchy's integral formula, if γ is a parametrization of the positively oriented unit circle, then for all $w \in B(0,1)$ we have $f(w) = \frac{1}{2\pi i}\int_\gamma f(z)/(z-w)dz$, and so

$$u(z) = \Re\big(\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w}\big).$$

Since the integrand uses only the values of f on the boundary circle, we have almost recovered the function u from its values on the boundary. (Almost, because we appear to need the values of it harmonic conjugate). The next lemma resolves this:

Lemma 12.20. *If u is harmonic on $B(0,r)$ for $r > 1$ then for all $w \in B(0,1)$ we have*

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\Re\big(\frac{e^{i\theta}+w}{e^{i\theta}-w}\big)d\theta.$$

Proof. (*Sketch.*) Take, as before, $f(z)$ holomorphic with $\Re(f) = u$ on $B(0,r)$. Then letting γ be a parametrization of the positively oriented unit circle we have

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w} - \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-\bar{w}^{-1}}$$

where the first term is $f(w)$ by the integral formula and the second term is zero because $f(z)/(z-\bar{w}^{-1})$ is holomorphic inside all of $B(0,1)$. Gathering the terms, this becomes

$$f(w) = \frac{1}{2\pi}\int_\gamma f(z)\frac{1-|w|^2}{|z-w|^2}\frac{dz}{iz} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta.$$

The advantage of this last form is that the real and imaginary parts are now easy to extract, and we see that

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) \frac{1-|w|^2}{|e^{i\theta}-w|^2} d\theta.$$

Finally for the second integral expression note that if $|z| = 1$ then

$$\frac{z+w}{z-w} = \frac{(z+w)(\bar{z}-\bar{w})}{|z-w|^2} = \frac{1-|w|^2+(\bar{z}w-z\bar{w})}{|z-w|^2}.$$

from which one readily sees the real part agrees with the corresponding factor in our first expression. □

Now the idea to solve the Dirichlet problem for the disk $B(0,1)$ is to turn this previous result on its head: Notice that it tells us the values of u inside the disk $B(0,1)$ in terms of the values of u on the boundary. Thus if we are given the boundary values, say a (periodic) function $G(e^{i\theta})$ we might reasonably hope that the integral

$$g(w) = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\theta}) \frac{1-|w|^2}{|e^{i\theta}-w|^2} d\theta,$$

would define a harmonic function with the required boundary values. Indeed it follows from the proof of the lemma that the integral is the real part of the integral

$$\frac{1}{2\pi i} \int_\gamma G(z) \frac{1}{z-w} dz,$$

which we know from Proposition 6.7 is holomorphic in w, thus $g(w)$ is certainly harmonic. It turns out that if $w \to w_0 \in \partial B(0,1)$ then provided G is continuous at w_0 then $g(w) \to G(w_0)$, hence g is in fact a harmonic function with the required boundary value.

13. APPENDIX I: SOME RESULTS FROM REAL ANALYSIS.

In this appendix we review some notions from multivariable calculus. While we give careful proofs, only the statements are examinable.

13.1. **Partial derivatives and the total derivative.**

Theorem 13.1. *Suppose that $F\colon U \to \mathbb{R}^2$ is a function defined on an open subset of $\mathbb{R}^2$, whose partial derivatives exist and are continuous on U. Then for all $z \in U$ the function F is real-differentiable, with derivative Df_z given by the matrix of partial derivative.*

Proof. Working component by component, you can check that it is in fact enough to show that a function $f\colon U \to \mathbb{R}$ with continuous partial derivatives $\partial_x f$ and $\partial_y f$ has total derivative given by $(\partial_x f, \partial_y f)$ at each $z \in U$. That is, if $z = (x, y)$ then

$$f(x+h, y+k) = f(x, y) + \partial_x f(x, y)h + \partial_y f(x, y)k + \|(h, k)\|.\epsilon(h, k),$$

where $\epsilon(h, k) \to 0$ as $(h, k) \to 0$. But now since the function $x \mapsto f(x, y)$ is differentiable at x with derivative $\partial_x f(x, y)$ we have

$$f(x+h, y) = f(x, y) + \partial_x f(x, y)h + h\epsilon_1(h)$$

where $\epsilon_1(h) \to 0$ as $h \to 0$. Now by the mean value theorem applied the function to $y \mapsto f(x+h, y)$ we have

$$f(x+h, y+k) = f(x+h, y) + \partial_y f(x+h, y+\theta_2 k)k,$$

for some $\theta_2 \in (0, 1)$. Thus using the definition of $\partial_x f(x, y)$ it follows that

$$f(x+h, y+k) = f(x, y) + \partial_x f(x, y)h + h\epsilon_1(h) + \partial_y f(x+h, y+\theta_2 k)k.$$

Thus we have

$$f(x+h, y+k) = f(x, y) + \partial_x f(x, y)h + \partial_y f(x, y)k + \|(h, k)\|\epsilon(h, k),$$

where

$$\epsilon(h, k) = \frac{h}{\sqrt{h^2+k^2}}\epsilon_1(h) + \frac{k}{\sqrt{h^2+k^2}}(\partial_y f(x+h, y+\theta_2 k) - \partial_y f(x, y)).$$

Thus since $0 \leq h/\sqrt{h^2+k^2}, k/\sqrt{h^2+k^2} \leq 1$, the fact that $\epsilon_1(h) \to 0$ as $h \to 0$ and the continuity of $\partial_y f$ at (x, y) imply that $\epsilon(h, k) \to 0$ as $(h, k) \to 0$ as required. □

Remark 13.2. Note that in fact the proof didn't use the full strength of the hypothesis of the theorem – we only actually needed the existence of the partial derivatives and the continuity of one of them at (x, y) to conclude that f is real-differentiable at (x, y).

13.2. **The Chain Rule.** We establish a version of the chain rule which is needed for the proof that the existence of a primitive for a function $f\colon U \to \mathbb{C}$ implies that $\int_\gamma f(z)dz = 0$ for every closed curve γ in U. The proof requires one to use the fact that if $dF/dt = f$ on U then $f(\gamma(t))\gamma'(t)$ is the derivative of $F(\gamma(t))$. This is of course formally exactly what one would expect using the formula for the normal version of the chain rule, but one should be slightly careful: $F\colon \mathbb{C} \to \mathbb{C}$ is a function of a complex variable, while $\gamma\colon [a,b] \to \mathbb{C}$ is a function of real variable, so we are mixing real and complex differentiability.

That said, we have seen that a complex differentiable function is also differentiable in the real sense, with its derivative being the linear map given by multiplication by the complex number which is its complex derivative. Thus the result we need follows from a version of the chain rule for real-differentiable functions:

Lemma 13.3. *Let U be an open subset of $\mathbb{R}^2$ and let $F\colon U \to \mathbb{R}^2$ be a differentiable function. If $\gamma\colon [a,b] \to \mathbb{R}$ is a (piecewise) C^1-path with image in U, then $F(\gamma(t))$ is a differentiable function with*

$$\frac{d}{dt}(F(\gamma(t))) = DF_{\gamma(t)}(\gamma'(t))$$

Proof. Let $t_0 \in [a,b]$ and let $z_0 = \gamma(t_0) \in U$. Then by definition, there is a function $\epsilon(z)$ such that

$$F(z) = F(z_0) + DF_{z_0}(z - z_0) + |z - z_0|\epsilon(z),$$

where $\epsilon(z) \to 0 = \epsilon(z_0)$ as $z \to z_0$. But then

$$\frac{F(\gamma(t)) - F(\gamma(t_0))}{t - t_0} = DF_{z_0}(\frac{\gamma(t) - \gamma(t_0)}{t - t_0}) + \epsilon(\gamma(t)).\frac{|\gamma(t) - \gamma(t_0)|}{t - t_0}.$$

But now consider the two terms on the right-hand side of this expression: for the first term, note that a linear map is continuous, so since $(\gamma(t) - \gamma(t_0))/(t-t_0) \to \gamma'(t_0)$ as $t \to t_0$ we see that $DF_{z_0}(\frac{\gamma(t)-\gamma(t_0)}{t-t_0}) \to DF_{z_0}(\gamma'(t_0))$ as $t \to t_0$. On the other hand, for the second term, since $\frac{\gamma(t)-\gamma(t_0)}{t-t_0}$ tends to $\gamma'(t_0)$ as t tends to t_0, we see that $|\gamma(t) - \gamma(t_0)|/(t - t_0)$ is bounded as $t \to t_0$, while since $\gamma(t)$ is continuous at t_0 since it is differentiable there $\epsilon(\gamma(t)) \to \epsilon(\gamma(t_0)) = \epsilon(z_0) = 0$. It follows that the second term tends to zero, so that the left-hand side tends to $Df_{\gamma(t_0)}(\gamma'(t_0))$ as required. □

Remark 13.4. Notice that the proof above works in precisely the same way if F is a function from $\mathbb{R}^2$ to $\mathbb{R}$. Indeed a slight modification of the argument proves that if $F\colon \mathbb{R}^n \to \mathbb{R}^m$ and $G\colon \mathbb{R}^m \to \mathbb{R}^p$ then if F and G are differentiable, their composite $G \circ F$ is differentiable with derivative $DG_{F(x)} \circ DF_x$.

An easy application of the chain rule is the following constancy theorem. For the proof it is convenient to introduce some terminology:

Definition 13.5. We say a function $f\colon X \to Y$ between metric spaces is *locally constant* if for any $z \in X$ there is an $r > 0$ such that f is constant on $B(z, r)$.

Remark 13.6. Clearly a locally constant function is continuous, and moreover for such a function, the pre-image of any point in its image is an open set. Since for any continuous function the pre-image of a point is a closed set, it follows the pre-image of a point in the range of a locally-constant function is both open and closed. Thus if X is connected and f is locally constant, then f is in fact constant.

Proposition 13.7. *Suppose that $f\colon U \to \mathbb{R}^2$ is a function defined on a connected open subset of $\mathbb{R}^2$. Then if $Df_z = 0$ for all $z \in U$ the function f is constant.*

Proof. By the preceding remarks it suffices to show that f is locally constant. To see this, let $z_0 \in U$ and fix $r > 0$ such that $B(z_0, r) \subseteq U$. Then for any $z \in B(z_0, r)$ we may consider the function $F(t) = f(z_0 + t(z - z_0))$, where $t \in [0, 1]$. Note that $F = f \circ \gamma$ where $\gamma(t) = z_0 + t(z - z_0)$ is the straight line-segment from z_0 to z which lies entirely in $B(z_0, r)$ as z does. Hence applying the chain rule we have $F'(t) = Df_{z_0+t(z-z_0)}(z - z_0) = 0$ by our assumption on Df_z. It follows from the Fundamental Theorem of Calculus that

$$f(z) - f(z_0) = F(1) - F(0) = \int_0^1 F'(t)dt = 0,$$

hence f is constant on $B(z_0, r)$ as required. (The integral of the function $F'(t) = (u'(t), v'(t))$ is taken component-wise.)

□

13.3. **Symmetry of mixed partial derivatives.** We used in the proof that the real and imaginary parts of a holomorphic function are harmonic the fact that partial derivatives commute on twice continuously differentiable functions. We give a proof of this for completeness. The key to the proof will be to use difference operators:

Definition 13.8. Let $f\colon U \to \mathbb{R}$ be a function defined on an open set $U \subset \mathbb{R}^2$. Then if $s, t \in \mathbb{R}\backslash\{0\}$ let $\Delta_1^s(f), \Delta_2^t(f)$ be the function given by

$$\Delta_1^s(f)(x, y) = \frac{f(x + s, y) - f(x, y)}{s}, \quad \Delta_2^t(f)(x, y) = \frac{f(x, y + t) - f(x, y)}{t}$$

Note that if f is differentiable at (x, y) then $\partial_x f(x, y) = \lim_{s\to 0} \Delta_1^s(f)(x, y)$ and $\partial_y f(x, y) = \lim_{t\to 0} \Delta_2^t(f)(x, y)$.

It is straight-forward to check that

$$\begin{aligned}\Delta_1^2(\Delta_2^t(f))(x, y) &= \Delta_2^t(\Delta_1^s(f))(x, y)\\ &= \frac{f(x + s, y + t) - f(x + s, y) - f(x, y + t) + f(x, y)}{st}.\end{aligned}$$

That is, the two difference operators $f \mapsto \Delta_1^s(f)$ and $f \mapsto \Delta_2^t(f)$ commute with each other. We wish to use this fact to deduce that the corresponding partial differential operators also commute, but because of the limits involved, this will not be automatic, and we will need to impose the additional hypotheses that the second partial derivatives of f are continuous functions.

Since the difference operator Δ_1^s and Δ_2^t are linear, they commute with partial differentiation so that $\partial_y \Delta_1^s(f)(x,y) = \Delta_1^s(\partial_y f)(x,y)$, and similarly for ∂_x and also for Δ_2^t and ∂_x, ∂_y.

We are now ready to prove that mixed partial derivatives are equal:

Lemma 13.9. *Suppose that $f\colon U \to \mathbb{R}$ is twice continuously differentiable, so that all its second partial derivatives exist and are continuous on U. Then*

$$\partial_x \partial_y f = \partial_y \partial_x f$$

on U.

Proof. Fix $(x,y) \in U$. Since U is open, there are $\epsilon, \delta > 0$ such that $\Delta_1^s(f)$ and $\Delta_2^t(f)$ are defined on $B((x,y),\epsilon)$ for all s,t with $|s|,|t| < \delta$. Now by definition we have

$$\partial_x \partial_y f(x,y) = \partial_x(\lim_{t\to 0} \Delta_2^t(f))(x,y) = \lim_{s\to 0}\lim_{t\to 0} \Delta_1^s \Delta_2^t(f)(x,y)$$

But now using the mean value theorem for $\Delta_2^t(f)$ in the first variable, we see that

$$\Delta_1^s \Delta_2^t(f)(x,y) = \partial_x \Delta_2^t f(x+s_1,y),$$

where s_1 lies between 0 and s. But $\partial_x \Delta_2^t(f)(x+s_1,y) = \Delta_2^t \partial_x f(x+s_1,y)$, and using the mean value theorem for $\partial_x f(x+s_1,y)$ in the second variable we see that $\Delta_2^t \partial_x f(x+s_1,y) = \partial_y \partial_x f(x+s_1,y+t_1)$ where t_1 lies between 0 and t (and note that t_1 depends both on t and s_1).

But now

$$\partial_x \partial_y f(x,y) = \lim_{s\to 0}\lim_{t\to 0} \partial_y \partial_x f(x+s_1, y+t_1) = \partial_y \partial_x f(x,y),$$

by the continuity of the second partial derivatives, so we are done.

□

Example 13.10. Let $\Delta = \partial_x^2 + \partial_y^2$ be the (two-dimensional) Laplacian. Provided we are only interested in acting on twice-continuously differentiable functions $u = u(x,y)$ so that $\partial_x \partial_y(u) = \partial_y \partial_x(u)$, we can factorize Δ as

$$\Delta = (\partial_x - i\partial_y)(\partial_x + i\partial_y).$$

This factorization of the Laplacian is what allows us to relate the study of harmonic functions in the plane to complex analysis.

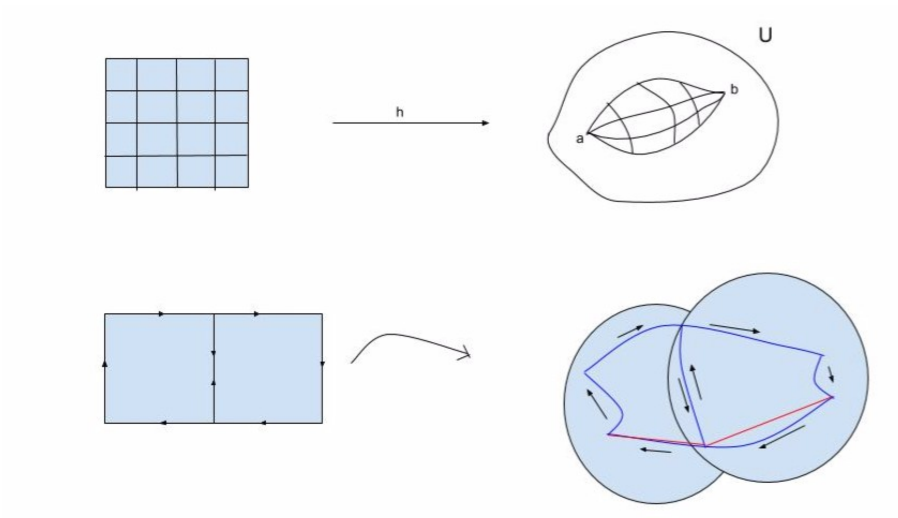

FIGURE 5. Dissecting the homotopy

14. APPENDIX II: ON THE HOMOTOPY AND HOMOLOGY VERSIONS OF CAUCHY'S THEOREM

In this appendix we give proofs of the homotopy and homology versions of Cauchy's theorem which are stated in the body of the notes. These proofs are non-examinable, but are included for the sake of completeness.

Theorem 14.1. *Let U be a domain in $\mathbb{C}$ and $a, b \in U$. Suppose that γ and η are paths from a to b which are homotopic in U and $f: U \to \mathbb{C}$ is a holomorphic function. Then*

$$\int_\gamma f(z)dz = \int_\eta f(z)dz.$$

Proof. The key to the proof of this theorem is to show that the integrals of f along two paths from a to b which "stay close to each other" are equal. We show this by covering both paths by finitely many open disks and using the existence of a primitive for f in each of the disks.

More precisely, suppose that $h: [0,1]\times[0,1]$ is a homotopy between γ and η. Let us write $K = h([0,1] \times [0,1])$ be the image of the map h, a compact subset of U. By Lemma **??** there is an $\epsilon > 0$ such that $B(z, \epsilon) \subseteq U$ for all $z \in K$.

Next we use the fact that, since $[0,1] \times [0,1]$ is compact, h is uniformly continuous. Thus we may find a $\delta > 0$ such that $|h(t_1, s_1) - h(t_2, w_2)| < \epsilon$ whenever $\|(t_1, s_1) - (t_2, s_2)\| < \delta$. Now pick $N \in \mathbb{N}$ such that $1/N < \delta$ and dissect the square $[0,1] \times [0,1]$ into N^2 small squares of side length $1/N$. For convenience, we will write $t_i = i/N$ for $i \in \{0, 1, \ldots, N\}$

For each $k \in \{1, 2, \dots, N-1\}$, let ν_k be the piecewise linear path which connects the point $h(t_j, k/N)$ to $h(t_{j+1}, k/N)$ for each $j \in \{0, 1, \dots, N)$. Explicitly, for $t \in [t_j, t_{j+1}]$, we set

$$\nu_k(t) = h(t_j, k/N)(1 - Nt - j) + h(t_{j+1}, k/N)(Nt - j)$$

We claim that

$$\int_\gamma f(z)dz = \int_{\nu_1} f(z)dz = \int_{\nu_2} f(z)dz = \dots = \int_{\nu_{N-1}} f(z)dz = \int_\eta f(z)dz$$

which will prove the theorem. In fact, we will only show that $\int_\gamma f(z)dz = \int_{\nu_1} f(z)dz$, since the other cases are almost identical.

We may assume the numbering of our squares S_i is such that $S_1, \dots, S_N$ list the bottom row of our N^2 squares from left to right. Let m_i be the centre of the square S_i and let $p_i = h(m_i)$. Then $h(S_i) \subseteq B(p_i, \epsilon)$ so that $\gamma([t_i, t_{i+1}]) \subseteq B(p_i, \epsilon)$ and $\nu_1([t_i, t_{i+1}]) \subseteq B(p_i, \epsilon)$ (since $B(p_i, \epsilon)$ is convex and by assumption contains $\nu_1(t_i)$ and $\nu_1(t_{i+1})$). Since $B(p_i, \epsilon)$ is convex, f has primitive F_i on each $B(p_i, \epsilon)$. Moreover, as primitives of f on a domain are unique up to a constant, it follows that F_i and F_{i+1} differ by a constant on $B(p_i, \epsilon) \cap B(p_{i+1}, \epsilon)$, where they are both defined. In particular, since $\gamma(t_i), \nu_1(t_i) \in B(p_i, \epsilon) \cap B(p_{i+1}, \epsilon)$, $(1 \leq i \leq N-1)$, we have

$$F_i(\gamma(t_i)) - F_{i+1}(\gamma(t_i)) = F_i(\nu_1(t_i)) - F_{i+1}(\nu_1(t_i)). \tag{14.1}$$

Now by the Fundamental Theorem we have

$$\int_{\gamma_{|[t_i, t_{i+1}]}} f(z)dz = F_i(\gamma(t_{i+1})) - F_i(\gamma_1(t_i)),$$
$$\int_{\nu_{1|[t_i, t_{i+1}]}} f(z)dz = F_i(\nu_1(t_{i+1})) - F_i(\nu_1(t_i))$$

Combining we find that:

$$\begin{aligned}\int_\gamma f(z)dz &= \sum_{i=0}^{N-1}\int_{\gamma_{|[t_i,t_{i+1}]}} f(z)dz \\ &= \sum_{i=0}^{N-1}\big(F_{i+1}(\gamma(t_{i+1})) - F_{i+1}(\gamma(t_i))\big) \\ &= F_N(\gamma(t_N)) - F_1(\gamma(0)) + \sum_{i=1}^{N-1}\big(F_i(\gamma(t_i)) - F_{i+1}(\gamma(t_i))\big) \\ &= F_N(b) - F_0(a) + \big(\sum_{i=0}^{N-1}(F_i(\nu_1(t_{i+1})) - F_{i+1}(\nu_1(t_{i+1}))\big) \\ &= \sum_{i=0}^{N-1}\big((F_{i+1}(\nu_1(t_{i+1})) - F_{i+1}(\nu_1(t_i))\big) \\ &= \sum_{i=0}^{N-1}\int_{\nu_{1|[t_i,t_{i+1}]}} f(z)dz = \int_{\nu_1} f(z)dz\end{aligned}$$

where in the fourth equality we used Equation (14.1). □

Remark 14.2. The use of the piecewise linear paths ν_k might seem unnatural – it might seem simpler to use the paths given by the homotopy, that is the paths $\gamma_k(t) = h(t, k/N)$. The reason we did not do this is because we only assume that h is continuous, so we do not know that the path γ_k is piecewise C^1 which we need in order to be able to integrate along it.

The proof of the homology form of Cauchy's theorem uses Liouville's theorem, which we proved using Cauchy's theorem for a disc.

Theorem 14.3. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma\colon [0,1] \to U$ be a closed path whose inside lies entirely in U, that is $I(\gamma, z) = 0$ for all $z \notin U$. Then we have, for all $z \in U\backslash\gamma^*$,*

$$\int_\gamma f(\zeta)d\zeta = 0; \quad \int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = 2\pi i I(\gamma, z)f(z), \quad \forall z \in U\backslash\gamma^*.$$

Moreover, if U is simply-connected and $\gamma\colon [a,b] \to U$ is any closed path, then $I(\gamma, z) = 0$ for any $z \notin U$, so the above identities hold for all closed paths in such U.

Proof. We first prove the general form of the integral formula. Note that using the integral formula for the winding number and rearranging, we wish to show that

$$F(z) = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}d\zeta = 0$$

for all $z \in U\backslash\gamma^*$. Now if $g(\zeta, z) = (f(\zeta) - f(z))/(\zeta - z)$, then since f is complex differentiable, g extends to a continuous function on $U \times U$ if we set

$g(z,z) = f'(z)$. Thus the function F is in fact defined for all $z \in U$. Moreover, if we fix ζ then, by standard properties of differentiable functions, $g(\zeta, z)$ is clearly complex differentiable as a function of z everywhere except at $z = \zeta$. But since it extends to a continuous function at ζ, it is bounded near ζ, hence by Riemann's removable singularity theorem, $z \mapsto g(\zeta, z)$ is in fact holomorphic on all of U. It follows by Theorem **??** that

$$F(z) = \int_0^1 g(\gamma(t), z)\gamma'(t)dt$$

is a holomorphic function of z.

Now let $\text{ins}(\gamma) = \{z \in \mathbb{C} : I(\gamma, z) \neq 0\}$ be the inside of γ, so by assumption we have $\text{ins}(\gamma) \subset U$, and let $V = \mathbb{C}\backslash(\gamma^* \cup \text{ins}(\gamma))$ be the complement of γ^* and its inside. If $z \in U \cap V$, that is, $z \in U$ but not inside γ or on γ^*, then

$$\begin{aligned} F(z) &= \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} - f(z)\int_\gamma \frac{d\zeta}{\zeta - z} \\ &= \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} - f(z)I(\gamma, z) \\ &= \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} = G(z) \end{aligned}$$

since $I(\gamma, z) = 0$. Now $G(z)$ is an integral which only involves the values of f on γ^* hence it is defined for all $z \notin \gamma^*$, and by Theorem **??**, $G(z)$ is holomorphic. In particular G defines a holomorphic function on V, which agrees with F on all of $U \cap V$, and thus gives an extension of F to a holomorphic function on all of $\mathbb{C}$. (Note that by the above, F and G will in general *not* agree on the inside of γ.) Indeed if we set $H(z) = F(z)$ for all $z \in U$ and $H(z) = G(z)$ for all $z \in V$ then H is a well-defined holomorphic function on all of $\mathbb{C}$. We claim that $|H| \to 0$ as $|z| \to \infty$, so that by Liouville's theorem, $H(z) = 0$, and so $F(z) = 0$ as required. But since $\text{ins}(\gamma)$ is bounded, there is an $R > 0$ such that $V \supseteq \mathbb{C}\backslash B(0, R)$, and so $H(z) = G(z)$ for $|z| > R$. But then setting $M = \sup_{\zeta \in \gamma^*} |f(\zeta)|$ we see

$$|H(z)| = \left| \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} \right| \leq \frac{\ell(\gamma).M}{|z| - R}.$$

which clearly tends to zero as $|z| \to \infty$, hence $|H(z)| \to 0$ as $|z| \to \infty$ as required.

For the second formula, simply apply the integral formula to $g(z) = (z - w)f(z)$ for any $w \notin \gamma^*$. Finally, to see that if U is simply-connected the inside of γ always lies in U, note that if $w \notin U$ then $1/(z-w)$ is holomorphic on all of U, and so $I(\gamma, w) = \int_\gamma \frac{dz}{z-w} = 0$ by the homotopy form of Cauchy's theorem. □

Remark 14.4. It is often easier to check a domain is simply-connected than it is to compute the interior of a path. Note that the above proof uses Liouville's theorem, whose proof depends on Cauchy's Integral Formula for a circular path, which was a consequence of Cauchy's theorem for a triangle, but apart from the final part of the proof on simply-connectd regions, we did not use the more sophisticated homotopy form of Cauchy's theorem. We have thus established the winding number and homotopy forms of Cauchy's theorem essentially independently of each other.

15. Appendix III: Remark on the Inverse Function Theorem

In this appendix we supply[23] *the details for the claim made in the remark after the proof of the holomorphic version of the inverse function theorem.*

There is an enhancement of the Inverse Function Theorem in the holomorphic setting, which shows that the condition $f'(z) \neq 0$ is automatic (in contrast to the case of real differentiable functions, where it is essential as one sees by considering the example of the function $f(x) = x^3$ on the real line). Indeed suppose that $f : U \to \mathbb{C}$ is a holomorphic function on an open subset $U \subset \mathbb{C}$, and that we have $z_0 \in \mathbb{U}$ such that $f'(z_0) = 0$.
Claim: In this case, f is at least 2 to 1 near z_0, and hence is not injective.

Proof of Claim: If we let $w_0 = f(z_0)$ and $g(z) = f(z) - w_0$, it follows g has a zero at z_0, and thus it is either identically zero on the connected component of U containing z_0 (in which case it is very far from being injective!) or we may write $g(z) = (z - z_0)^k h(z)$ where $h(z)$ is holomorphic on U and $h(z_0) \neq 0$. Our assumption that $f'(z_0) = 0$ implies that k, the multiplicity of the zero of g at z_0 is at least 2.

Now since $h(z_0) \neq 0$, we have $\epsilon = |h(z_0)| > 0$ and hence by the continuity of h at z_0 we may find a $\delta > 0$ such that $h(B(z_0, \delta)) \subseteq B(h(z_0), \epsilon)$. But then by taking a cut along the ray $\{-t.h(z_0) : t \in \mathbb{R}_{>0}\}$ we can define a holomorphic branch of $z \mapsto z^{1/k}$ on the whole of $B(h(z_0), \epsilon)$. Now let $\phi \colon B(z_0, \delta) \to \mathbb{C}$ be the holomorphic function given by $\phi(z) = (z - z_0).h(z)^{1/k}$ (where by our choice of δ this is well-defined) so that $\phi'(z_0) = h(z_0)^{1/k} \neq 0$. Then clearly $f(z) = w_0 + \phi(z)^k$ on $B(z_0, \delta)$. Since $\phi(z)$ is holomorphic,the open mapping theorem ensures that $\phi(B(z_0, \delta))$ is an open set, which since it contains $0 = \phi(z_0)$, contains $B(0, r)$ for some $r > 0$. But then since $z \mapsto z^k$ is k-to-1 as a map from $B(0, r)\backslash\{0\} \to B(0, r^k)\backslash\{0\}$ it follows that f takes every value in $B(w_0, r^k)\backslash\{w_0\}$ at least k times.

[23]For interest, not examination!

MATHEMATICAL INSTITUTE, OXFORD.