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For $\theta \to 0$, $\quad re^{i\theta}, re^{i(2\pi-\theta)} \to r$, but

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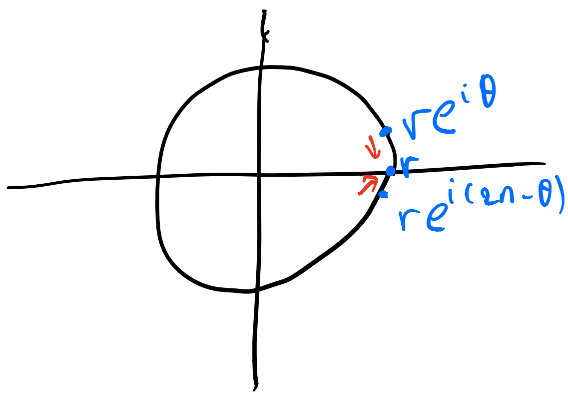

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$f(z)$ is holomorphic on $\mathbb{C} \backslash R$:

$$\frac{f(a+h) - f(a)}{h} = \frac{f(a+h) - f(a)}{f^2(a+h) - f^2(a)} = \frac{1}{f(a+h) + f(a)} \to \frac{1}{2f(a)}$$

as $h \to 0$.

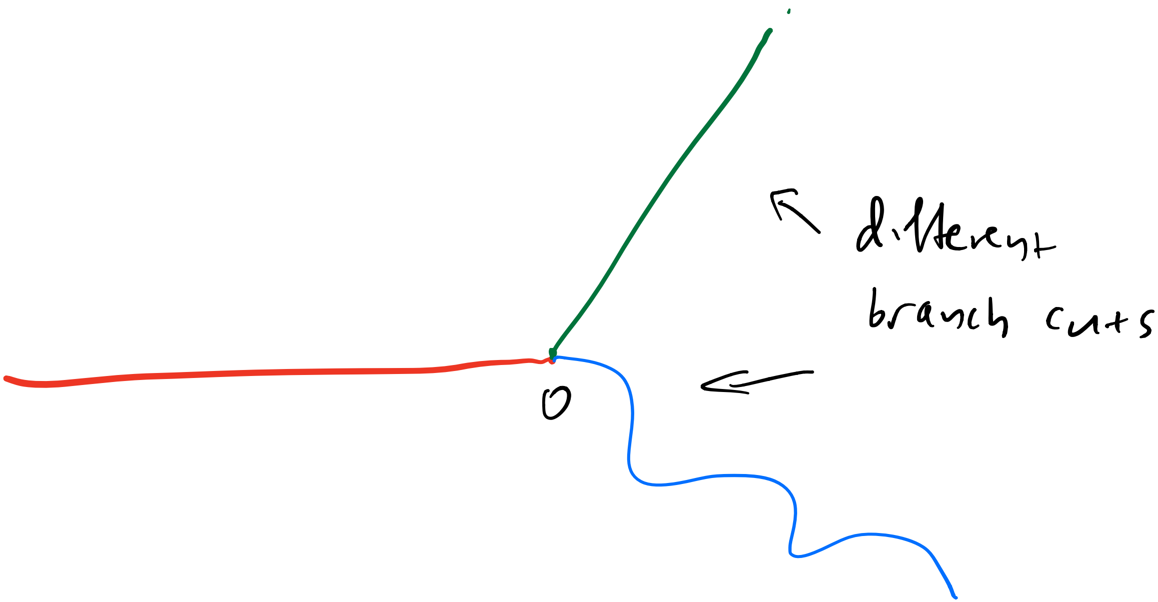
Multifunctions

The positive real axis is called a branch cut for the *multi-valued function* $z^{1/2}$.

If we set

$$g(z) = g(re^{i\theta}) = r^{1/2}e^{i(\frac{\theta}{2}+\pi)} = -r^{1/2}e^{i\theta/2}.$$

we obtain another *branch* of $z^{1/2}$ on $\mathbb{C}\backslash R$.



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Definition

A *multi-valued function* or multifunction on a subset $U \subseteq \mathbb{C}$ is a map $f\colon U \to \mathcal{P}(\mathbb{C})$ assigning to each point in U a subset of the complex numbers. A branch of f on a subset $V \subseteq U$ is a function $g\colon V \to \mathbb{C}$ such that $g(z) \in f(z)$, for all $z \in V$. If g is continuous (or holomorphic) on V we refer to it as a continuous, (respectively holomorphic) branch of f.

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Notation: $[f(z)]$ so eg $[\mathrm{Log}(z)] = \{w \in \mathbb{C} : e^w = z\}$.

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This is because it is not possible to choose a continuous branch of $[z^{1/2}]$ on any open set containing 0.

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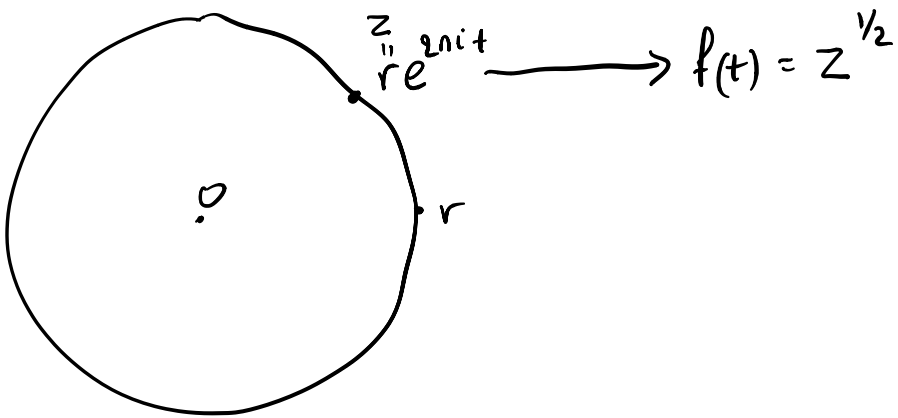

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Then $f(0) = \pm\sqrt{r}$. Say $f(0) = \sqrt{r}$.

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$$s = \sup\{t \in [0, 1] : f(t) = \sqrt{r}e^{\pi it}\}$$

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So $s = 1$. But then $f(0) = \sqrt{r} \neq f(1) = \sqrt{r}e^{\pi i} = -\sqrt{r}$, however $re^{2\pi i\cdot 0} = re^{2\pi i\cdot 1}$

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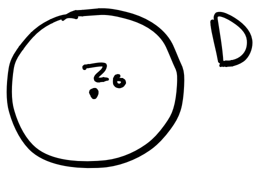

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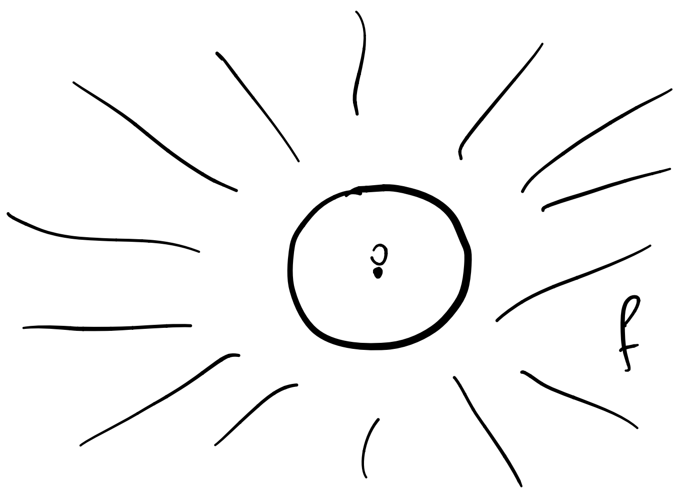
When $\mathbb{C}\backslash U$ is bounded, we say that f does not have a branch point at ∞ if there is a holomorphic branch of f defined on $\mathbb{C}\backslash B(0, R) \subseteq U$ for some $R > 0$. Otherwise we say that ∞ is a branch point of f.



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For example $0, \infty$ are the branch points of $[z^{1/2}]$.

More examples

The Logarithm

$[\text{Log}(z)] = \{\log(|z|) + i(\theta + 2n\pi) : n \in \mathbb{Z}\}$ where $z = |z|e^{i\theta}$.

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$$L_n(z) = \mathrm{L}(z) + 2in\pi$$

The branch points of $[\mathrm{Log}(z)]$ are 0 and ∞, as it is not possible to make a continuous choice of logarithm on any circle $S(0, r)$.

We note that $L(z)$ is also holomorphic. Indeed for small $h \neq 0$, $L(a+h) \neq L(a)$ and

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We note that the same argument applies to any continuous branch of the logarithm.

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Note $(z_1 z_2)^\alpha \neq z_1^\alpha z_2^\alpha$ in general!

Binomial theorem for complex powers

$$[(1+z)^\alpha] = \{\exp(\alpha \cdot w) : w \in \mathbb{C}, \exp(w) = 1+z\}.$$

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$$\left| \binom{\alpha}{k} \Big/ \binom{\alpha}{k+1} \right| = \left| \frac{k+1}{\alpha - k} \right| \xrightarrow[k \to \infty]{} 1$$

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$s'(z) = \sum k \binom{\alpha}{k} z^{k-1} = \sum (\alpha-k+1)\binom{\alpha}{k-1} z^{k-1}$, $\quad z s'(z) = \sum (k-1)\binom{\alpha}{k-1} z^{k-1}$

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Differentiating term by term: $(1+z)s'(z) = \alpha \cdot s(z)$.
Now $f(z)$ is defined on all of $B(0,1)$. We claim that $f(z) = s(z)$ on $B(0,1)$.

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$$g'(z) = (s'(z) - \alpha s(z) L'((1+z)) \exp(-\alpha \cdot L(1+z)) = 0$$

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We used here the following:

Fact. If for a holomorphic function g, $g'(z) = 0$ for all z in a connected open set, then it is constant. We have already proven this when the open set is $\mathbb{C}$ and we will prove it soon in general.

The Argument

$[\arg(z)] := \{\theta \in \mathbb{R} : z = |z|e^{i\theta}\}$ defined on $\mathbb{C} \setminus \{0\}$. Clearly if $z = |z|e^{i\theta}$ then $\arg(z)$ is equal to the set $\{\theta + 2n\pi : n \in \mathbb{Z}\}$.

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We note that $f(e^{i0}) = g(0) = 2n\pi$. Since f is continuous there is some $\delta > 0$ such that $f(e^{it}) = g(t)$ for all $t \in [0, \delta)$.

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This is an open and closed subset of $[0, 2\pi)$, so, since $[0, 2\pi)$ is connected, $A = [0, 2\pi)$, which proves our claim.

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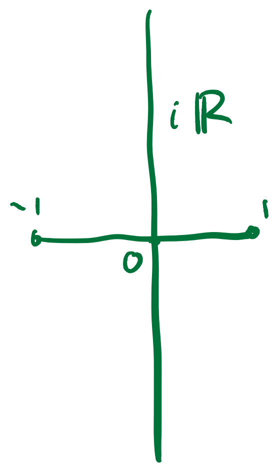
The argument multifunction is closely related to the logarithm. There is a continuous branch of $[\mathrm{Log}(z)]$ on a set U if and only if there is continuous branch of $[\arg(z)]$ on U. Indeed if $f(z)$ is a continuous branch of $[\arg(z)]$ on U we may define a continuous branch of $[\mathrm{Log}(z)]$ by $g(z)=log|z|+if(z)$, and conversely given $g(z)$ we may define $f(z)=\Im(g(z))$.

More interesting example: $f(z) = [(z^2 - 1)^{1/2}]$. If we see it as composition of $z^2 - 1$ with $\sqrt{z}$:
The branch cut of the principal branch of $\sqrt{z}$ is $(-\infty, 0]$ so we need a branch cut

$$\{z : z^2 - 1 \in (-\infty, 0]\} = [-1, 1] \cup i\mathbb{R}$$



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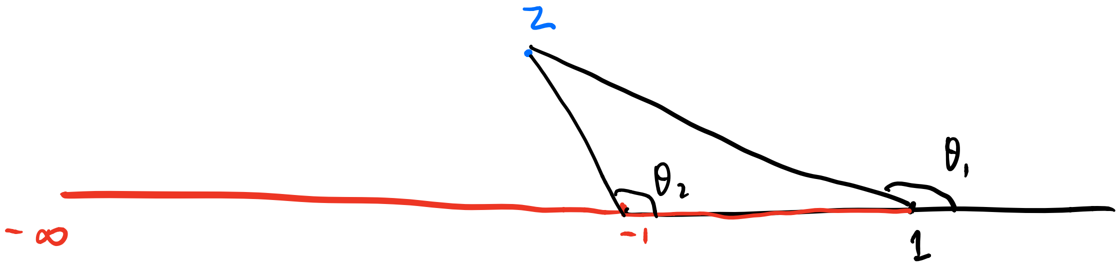
If we rewrite $f(z) = [\sqrt{z-1}\sqrt{z+1}]$, then we can take as branch cut
$(-\infty, 1] \cup (-\infty, -1] = (-\infty, 1]$ and a branch $f_2(z)$.

if $z = 1 + re^{i\theta_1} = -1 + se^{i\theta_2}$ where $\theta_1, \theta_2 \in (-\pi, \pi]$ then $f_2(z) = \sqrt{rs} \cdot e^{i(\theta_1+\theta_2)/2}$.



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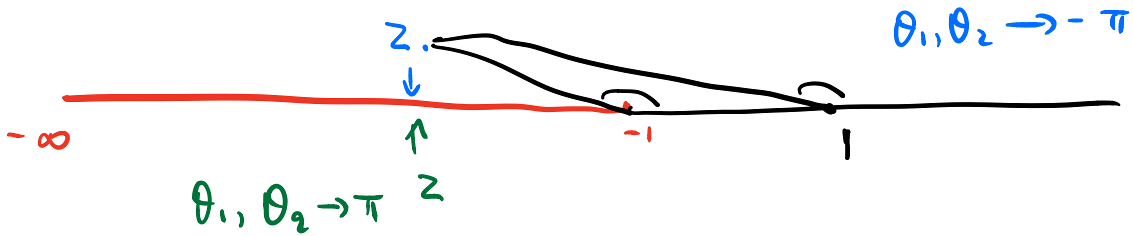

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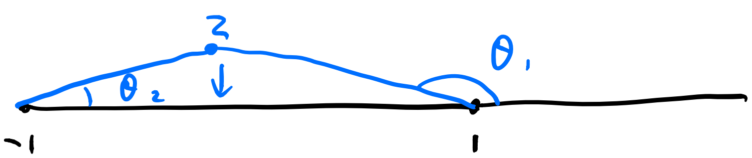

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So in fact we can take just $[-1, 1]$ as branch cut!

Riemann surfaces

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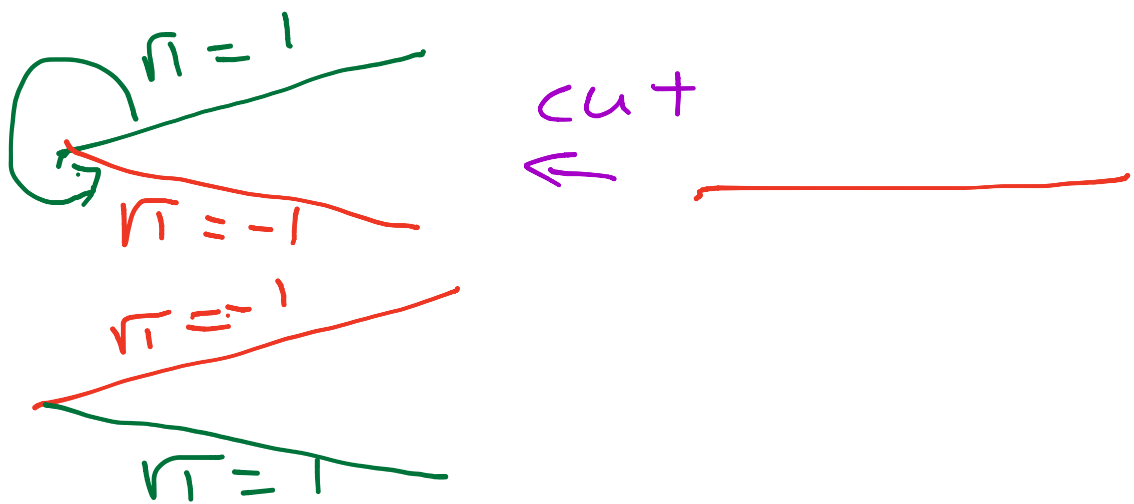
Consider $[z^{1/2}]$. We can 'join' the two branches of $[z^{1/2}]$ to obtain a function from a Riemann surface to $\mathbb{C}$.



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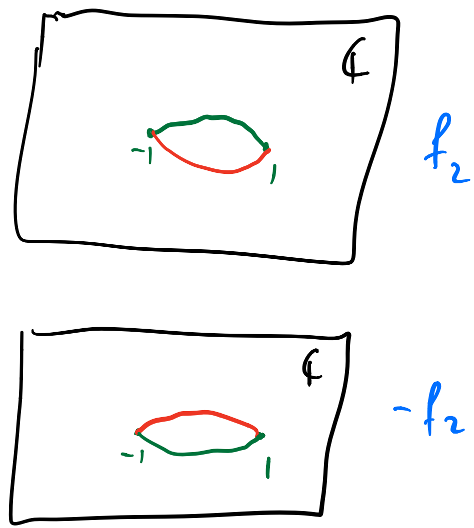

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For such z we have $w = \pm i\sqrt{1 - z^2}$, and Σ is obtained by 'gluing' together the two copies Σ_+ and Σ_- of the cut plane $\mathbb{C}\backslash[-1, 1]$ along the cut locus $[-1, 1]$.

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So we have a well defined 'square root' function $f : \Sigma \to \mathbb{C}$ given by $(z, w) \mapsto w$.

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if $F\colon [a,b] \to \mathbb{C}$, $F(t) = G(t) + iH(t)$, we say that F is *integrable* if G, H are integrable and define

$$\int_a^b F(t)dt = \int_a^b G(t)dt + i\int_a^b H(t)dt$$

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$\left|\int_a^b F(t)dt\right| = \left|\int_a^b Re(e^{-i\theta} F(t))dt\right| \leq \int_a^b |F(t)|dt$
since $|Re(z)| \leq |z|$.

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A *path* is a continuous function $\gamma \colon [a, b] \to \mathbb{C}$. A path is *closed* if $\gamma(a) = \gamma(b)$. If γ is a path, we will write γ^* for its image,

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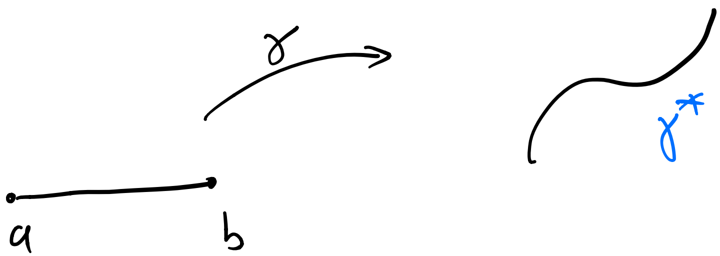

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A path $\gamma \colon [a, b] \to \mathbb{C}$ is *differentiable* if its real and imaginary parts are differentiable. Equivalently, γ is differentiable at $t_0 \in [a, b]$ if

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists. Notation: $\gamma'(t_0)$. (If $t = a$ or b then we take the one-sided limit.) A path is C^1 if it is differentiable and its derivative $\gamma'(t)$ is continuous.

EXAMPLES:

1. Line segment: $t \mapsto a + t(b - a) = (1 - t)a + tb, \;\; t \in [0, 1]$,
2. circle: $z(t) = z_0 + re^{2\pi it}$, $t \in [0, 1]$ a closed path.

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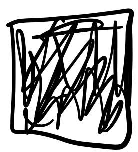
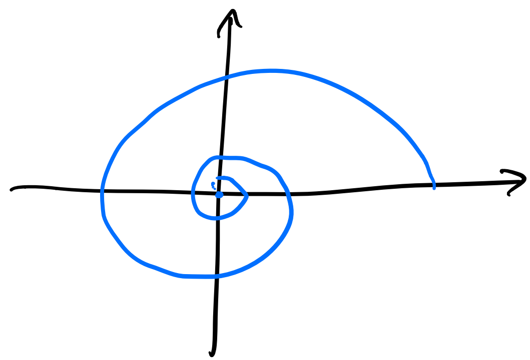
Peano curves, spirals.

$\alpha : I \longrightarrow$ 

$\gamma(t) = t^3 e^{2\pi i/t}, \quad t \in [0,1]$



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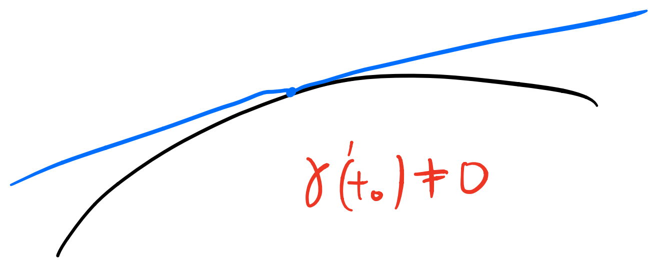

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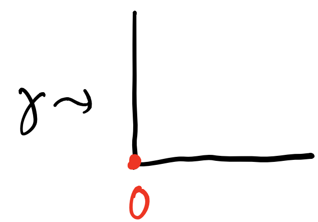
However a C^1 path might not have a tangent at every point, eg $\gamma \colon [-1,1] \to \mathbb{C}$

$$\gamma(t) = \begin{cases} t^2 & -1 \le t \le 0 \\ it^2 & 0 \le t \le 1. \end{cases}$$



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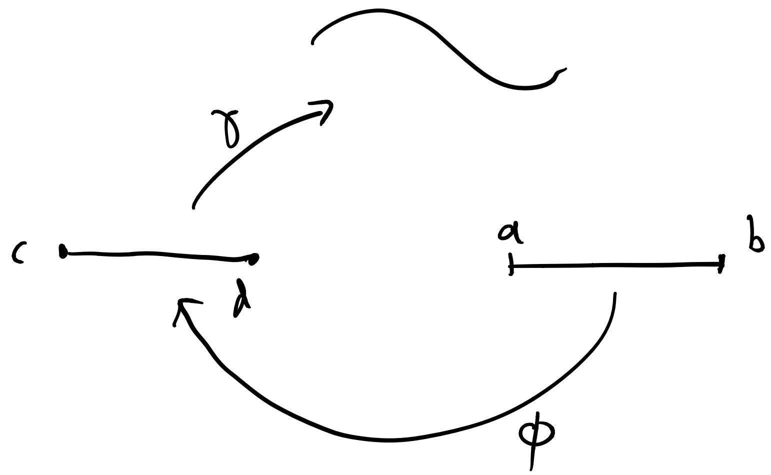

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$\gamma_1 \colon [a, b] \to \mathbb{C}$ and $\gamma_2 \colon [c, d] \to \mathbb{C}$ are equivalent if there is a continuously differentiable bijective function $s \colon [a, b] \to [c, d]$ such that $s'(t) > 0$ for all $t \in [a, b]$ and $\gamma_1 = \gamma_2 \circ s$.
Equivalence classes: *oriented curves* in the complex plane.
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Definition

If $\gamma \colon [a,b] \to \mathbb{C}$ is a C^1 path then we define the length of γ to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)| dt.$$

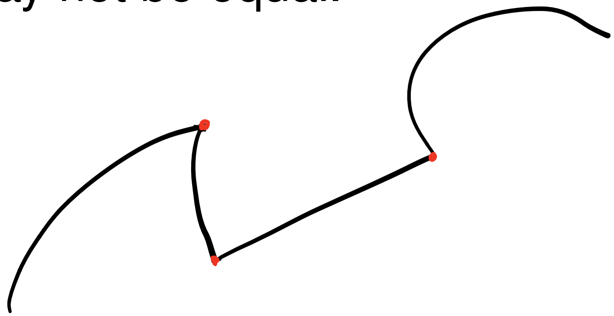
Using the chain rule one sees that the length of a parametrized path is also constant on equivalence classes of paths.

Definition

We will say a path $\gamma \colon [a, b] \to \mathbb{C}$ is piecewise C^1 if it is continuous on $[a, b]$ and the interval $[a, b]$ can be divided into subintervals on each of which γ is C^1.

So there are $a = a_0 < a_1 < \ldots < a_m = b$ such that $\gamma_{|[a_i, a_{i+1}]}$ is C^1.

Note: the left-hand and right-hand derivatives of γ at a_i $(1 \leq i \leq m - 1)$ may not be equal.



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As we have see in metric spaces two paths $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$ can be *concatenated* to give a path $\gamma_1 \star \gamma_2$. If $\gamma, \gamma_1, \gamma_2$ are piecewise C^1 then so are γ^- and $\gamma_1 \star \gamma_2$.

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A piecewise C^1 path is precisely a finite concatenation of C^1 paths.

We may define equivalence classes, reparametrisations, length as before for *piecewise* C^1 paths.

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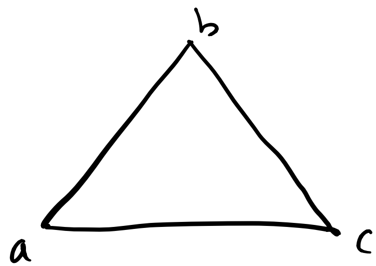
$T_{a,b,c} = \gamma_{a,b} \star \gamma_{b,c} \star \gamma_{c,a}$ where $\gamma_{x,y}$ is the line segment joining x, y.



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Recall the definition of Riemann integrable functions. We have the following:

Lemma

Let $[a, b]$ be a closed interval and $S \subset [a, b]$ a finite set. If f is a bounded continuous function (taking real or complex values) on $[a, b] \setminus S$ then it is Riemann integrable on $[a, b]$.

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Let $a = x_0 < x_1 < x_2 < \ldots < x_k = b$ be any partition of $[a, b]$ which includes the elements of S.
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Integral along a path

Definition

If $\gamma\colon [a, b] \to \mathbb{C}$ is a piecewise-C^1 path and $f\colon \mathbb{C} \to \mathbb{C}$, then we define the integral of f along γ to be

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We note that if γ is a concatenation of the C^1 paths $\gamma_1, ..., \gamma_n$ then $\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + ... + \int_{\gamma_n} f(z)dz$.

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Lemma

If $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 path and $\tilde{\gamma} : [c, d] \to \mathbb{C}$ is an equivalent path, then for any continuous function $f : \mathbb{C} \to \mathbb{C}$ we have

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

So the integral only depends on the oriented curve $[\gamma]$.

Proof.

Since $\tilde{\gamma} \sim \gamma$ there is $s \colon [c, d] \to [a, b]$ with $s(c) = a$, $s(d) = b$ and $s'(t) > 0$, $\tilde{\gamma} = \gamma \circ s$. Suppose first that γ is C^1. Then by the chain rule we have

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If $a = x_0 < x_1 < \ldots < x_n = b$ such that γ is C^1 on $[x_i, x_{i+1}]$ we have a corresponding decomposition of $[c, d]$ given by the points $s^{-1}(x_0) < \ldots < s^{-1}(x_n)$, and

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We define also the integral *with respect to arc-length* of a function $f\colon U \to \mathbb{C}$ such that $\gamma^* \subseteq U$ to be

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This integral is invariant with respect to C^1 reparametrizations $s\colon [c,d] \to [a,b]$ if we require $s'(t) \neq 0$ for all $t \in [c,d]$. Note that in this case

$$\int_\gamma f(z)|dz| = \int_{\gamma^-} f(z)|dz|.$$

Properties of the integral

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Let $f, g\colon U \to \mathbb{C}$ be continuous functions on an open subset $U \subseteq \mathbb{C}$ and $\gamma, \eta\colon [a, b] \to \mathbb{C}$ be piecewise-C^1 paths whose images lie in U. Then we have the following:

1. (*Linearity*): For $\alpha, \beta \in \mathbb{C}$,

$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$

2. If γ^- denotes the opposite path to γ then

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

3. (*Additivity*): If $\gamma \star \eta$ is the concatenation of the paths γ, η in U, we have

$$\int_{\gamma\star\eta} f(z)dz = \int_\gamma f(z)dz + \int_\eta f(z)dz.$$

4. (Estimation Lemma.) We have

length of γ

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma^*} |f(z)|.\ell(\gamma).$$

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Proposition

Let $f_n \colon U \to \mathbb{C}$ be a sequence of continuous functions. Suppose that $\gamma \colon [a, b] \to U$ is a path. If (f_n) converges uniformly to a function f on the image of γ then

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz.$$

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Proof. We have

$$\begin{aligned} \left| \int_\gamma f(z)dz - \int_\gamma f_n(z)dz \right| &= \left| \int_\gamma (f(z) - f_n(z))dz \right| \\ &\leq \sup_{z \in \gamma^*} \{|f(z) - f_n(z)|\}.\ell(\gamma), \end{aligned}$$

by the estimation lemma.

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$\sup\{|f(z) - f_n(z)| : z \in \gamma^*\} \to 0$ as $n \to \infty$ which implies the result.

Example. Let's say

$$\sum_{n=1}^{\infty} a_n z^n$$

converges on $B(0, R)$. Then convergence is uniform on $B(0, r)$ for $r < R$. So if γ is a piecewise C^1 curve in $B(0, r)$ we have

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Definition

Let $U \subseteq \mathbb{C}$ be an open set and let $f\colon U \to \mathbb{C}$ be a continuous function. If there exists a differentiable function $F\colon U \to \mathbb{C}$ with $F'(z) = f(z)$ then we say F is a primitive for f on U.

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Theorem

(Fundamental theorem of Calculus): Let $U \subseteq \mathbb{C}$ be a open and let $f \colon U \to \mathbb{C}$ be a continuous function. If $F \colon U \to \mathbb{C}$ is a primitive for f and $\gamma \colon [a, b] \to U$ is a piecewise C^1 path in U then we have

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular the integral of such a function f around any closed path is zero.

Proof.

First suppose that γ is C^1. Then we have

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If γ is only piecewise C^1, then take a partition $a = a_0 < a_1 < \ldots < a_k = b$ such that γ is C^1 on $[a_i, a_{i+1}]$ for each $i \in \{0, 1, \ldots, k-1\}$. Then we obtain a telescoping sum:

$$\int_\gamma f(z) = \int_a^b f(\gamma(t))\gamma'(t)dt = \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt$$

$$= \sum_{i=0}^{k-1}(F(\gamma(a_{i+1})) - F(\gamma(a_i))) = F(\gamma(b)) - F(\gamma(a))$$

Finally, γ is closed iff $\gamma(a) = \gamma(b)$ so the integral of f along a closed path is zero.

Corollary

Let U be a domain and let $f \colon U \to \mathbb{C}$ be a function with $f'(z) = 0$ for all $z \in U$. Then f is constant.

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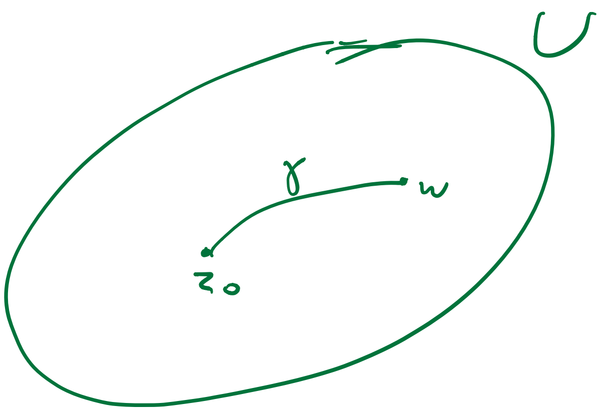
Pick $z_0 \in U$. Since U is path-connected, if $w \in U$, we may find a piecewise C^1-path $\gamma\colon [0,1] \to U$ such that $\gamma(a) = z_0$ and $\gamma(b) = w$. Then by the previous Theorem

$$f(w) - f(z_0) = \int_\gamma f'(z)dz = 0,$$

so that f is constant. □



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Theorem

If U is a domain and $f \colon U \to \mathbb{C}$ is a continuous function such that for any closed path in U we have $\int_\gamma f(z)dz = 0$, then f has a primitive.

Proof.

Fix z_0 in U, and for any $z \in U$ set $F(z) = \int_\gamma f(z)dz$.
where $\gamma\colon [a, b] \to U$ with $\gamma(a) = z_0$ and $\gamma(b) = z$.

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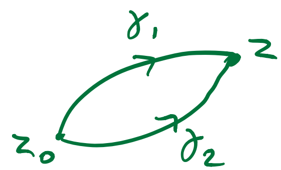

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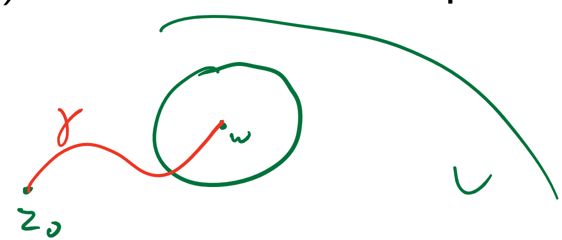

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If $z_1 \in B(w, \epsilon) \subseteq U$, then the concatenation of γ with the straight-line path $s\colon [0, 1] \to U$ given by $s(t) = w + t(z_1 - w)$ from w to z_1 is a path γ_1 from z_0 to z_1. It follows that

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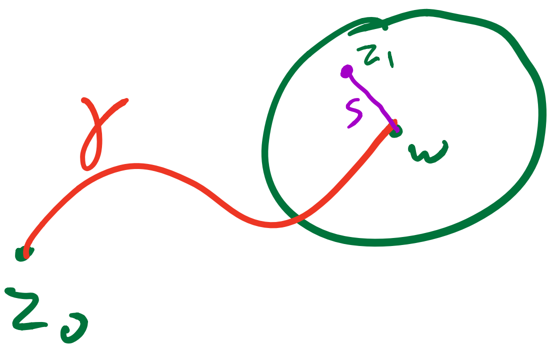

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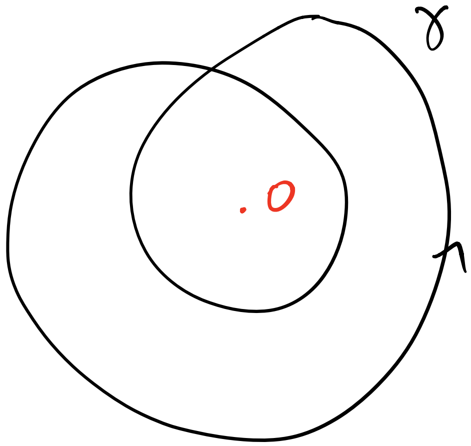
But this integral would be zero if $f(z)$ had a primitive.

Remark: $1/z$ *does* have a primitive on any domain where we can chose a branch of $[\mathrm{Log}(z)]$:
If we have $e^{L(z)} = z$ on D by the chain rule

$$\exp(L(z)) \cdot L'(z) = 1 \Rightarrow L'(z) = 1/z.$$

Let $\gamma\colon [0,1] \to \mathbb{C}$ closed path which does not pass through 0. We will give a rigorous definition of the number of times γ "goes around the origin".



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Proposition

Let $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ *be a path. Then there is continuous function* $a\colon [0,1] \to \mathbb{R}$ *such that*

$$\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}.$$

Moreover, if a and b are two such functions, then there exists $n \in \mathbb{Z}$ *such that* $a(t) = b(t) + n$ *for all* $t \in [0,1]$.

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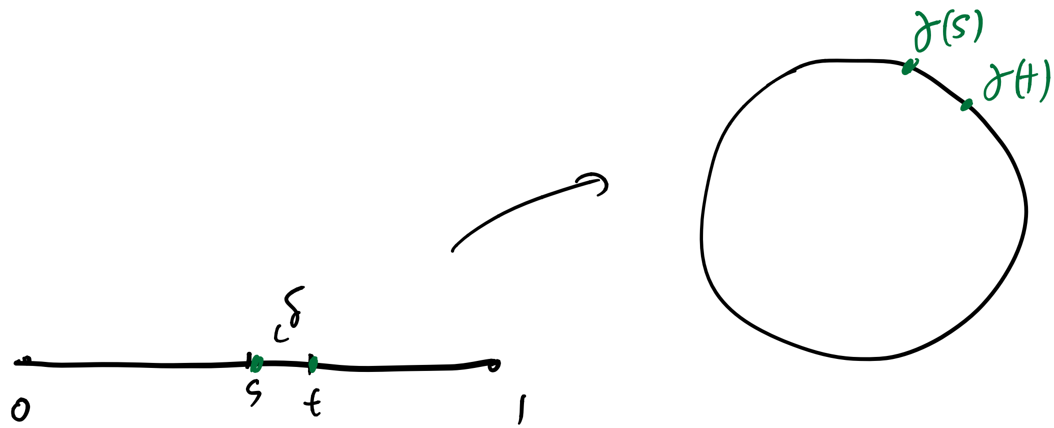

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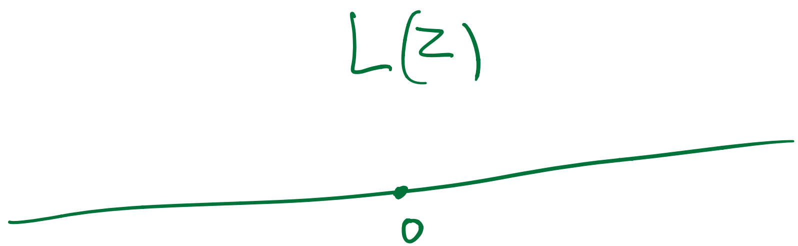

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if $|z_1| = |z_2| = 1$ and $|z_1 - z_2| < 1$, then the angle between z_1 and z_2 is less than $\pi/2$. It follows there exists continuous $a_i \colon [j/n, (j+1)/n] \to \mathbb{R}$ such that $\gamma(t) = e^{2\pi i a_j(t)}$

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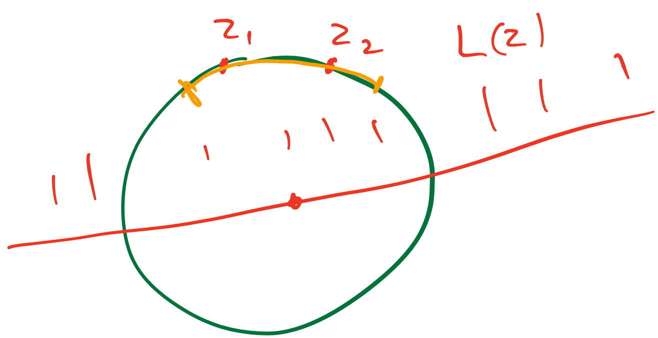

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Uniqueness: $e^{2\pi i(a(t)-b(t))} = 1$, hence $a(t) - b(t) \in \mathbb{Z}$, but $[0,1]$ is connected so $a(t) - b(t)$ is constant. $\square$

Definition

If $\gamma\colon [0,1] \to \mathbb{C}\backslash\{0\}$ is a closed path and $\gamma(t) = |\gamma(t)|e^{2\pi i a(t)}$ as in the previous lemma, then $a(1) - a(0) \in \mathbb{Z}$. This integer is called the winding number $I(\gamma, 0)$ of γ around 0.

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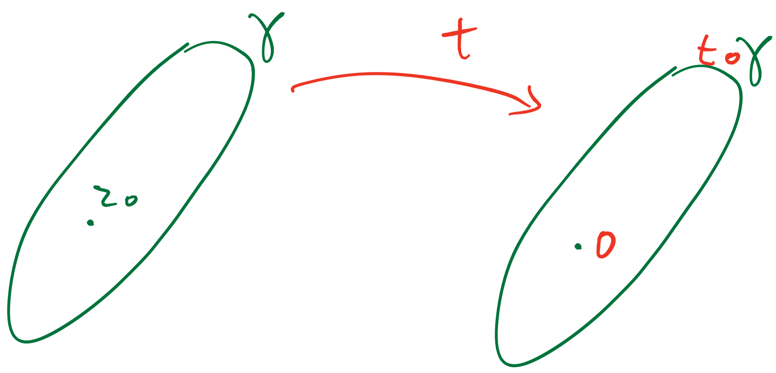

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2. if $\gamma\colon [0,1] \to U$ where $0 \notin U$ and there exists a holomorphic branch $L\colon U \to \mathbb{C}$ of $[\mathrm{Log}(z)]$ on U, then $I(\gamma, 0) = 0$. Indeed in this case we may define $a(t) = \Im(L(\gamma(t)))$, and since $\gamma(0) = \gamma(1)$ it follows $a(1) - a(0) = 0$.

The winding number for C^1 paths can be expressed using integrals:

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Let γ be a piecewise C^1 closed path and $z_0 \in \mathbb{C}$ a point not in the image of γ. Then the winding number $I(\gamma, z_0)$ of γ around z_0 is given by

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$= 2\pi i(a(1) - a(0))$, since $r(1) = r(0) = |\gamma(0) - z_0|$. $\square$

Winding numbers and analytic functions

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If $f\colon U \to \mathbb{C}$ is a function on an open subset U of $\mathbb{C}$, then we say that f is analytic on U if for every $z_0 \in \mathbb{C}$ there is an $r > 0$ with $B(z_0, r) \subseteq U$ such that there is a power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ with radius of convergence at least r and $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ An analytic function is holomorphic, as any power series is (infinitely) complex differentiable.

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Recall:

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we see this as a function of z_0.

Winding numbers and analytic functions

Definition

If $f\colon U \to \mathbb{C}$ is a function on an open subset U of $\mathbb{C}$, then we say that f is analytic on U if for every $z_0 \in \mathbb{C}$ there is an $r > 0$ with $B(z_0, r) \subseteq U$ such that there is a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ with radius of convergence at least r and $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ An analytic function is holomorphic, as any power series is (infinitely) complex differentiable.

Proposition

Let U be an open set in $\mathbb{C}$ and let $\gamma\colon [0,1] \to U$ be a closed path. If $f(z)$ is a continuous function on γ^* then the function

$$I_f(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz,$$

is analytic in w.

In particular, if $f(z) = 1$ this shows that the function $w \mapsto I(\gamma, w)$ is a continuous function on $\mathbb{C} \backslash \gamma^*$, hence constant on the connected components of $\mathbb{C} \backslash \gamma^*$.

Proof We will show that for each $z_0 \notin \gamma^*$ we can find a disk $B(z_0, \epsilon)$ within which $I_f(\gamma, w)$ is given by a power series in $(w - z_0)$. Translating if necessary we may assume $z_0 = 0$.

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The idea is that we can expand $\frac{1}{z-w} = \frac{1}{z}(1 - w/z)^{-1}$ as power series when $|w/z| < 1$, so

$$\frac{f(z)}{z-w} = \frac{f(z)}{z}(1 - w/z)^{-1} = \sum_{n=0}^{\infty} \frac{f(z)}{z}(w/z)^n = \sum_{n=0}^{\infty} \frac{f(z) \cdot w^n}{z^{n+1}}$$

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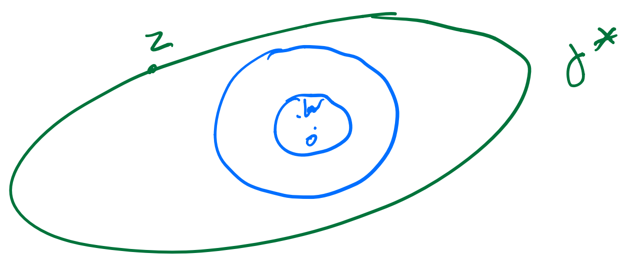
For this expansion to work we pick r so that $B(0, 2r) \cap \gamma^* = \emptyset$. We will show that the function is analytic for $w \in B(0, r)$.

$\left|\frac{w}{z}\right| < \frac{1}{2}$

as $|z| \geq 2r$



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We claim that the last series, seen as a function of z, converges uniformly on γ^*.

Recall
Weierstrass M-test: $\sum f_n(z)$ conv. unif. if $|f_n(z)| \leq M_n$ and $\sum M_n < \infty$

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We claim that the last series, seen as a function of z, converges uniformly on γ^*.

Since γ^* is compact, $M = \sup\{|f(z)| : z \in \gamma^*\}$ is finite. We apply Weierstrass M-test:

$|f(z) \cdot w^n / z^{n+1}| = |f(z)||z|^{-1}|w/z|^n < \frac{M}{2r}(1/2)^n, \quad \forall z \in \gamma^*.$

Uniform convergence implies that for all $w \in B(0, r)$ we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} dz \right) w^n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z - w} = I_f(\gamma, w).$$

hence $I_f(\gamma, w)$ is given by a power series in $B(0, r)$.

Recall If $\sum f_n \xrightarrow{\text{unif on } \gamma^*} f$ then $\int_\gamma \sum f_n \longrightarrow \int_\gamma f$

so $\int_\gamma \sum = \sum \int_\gamma$

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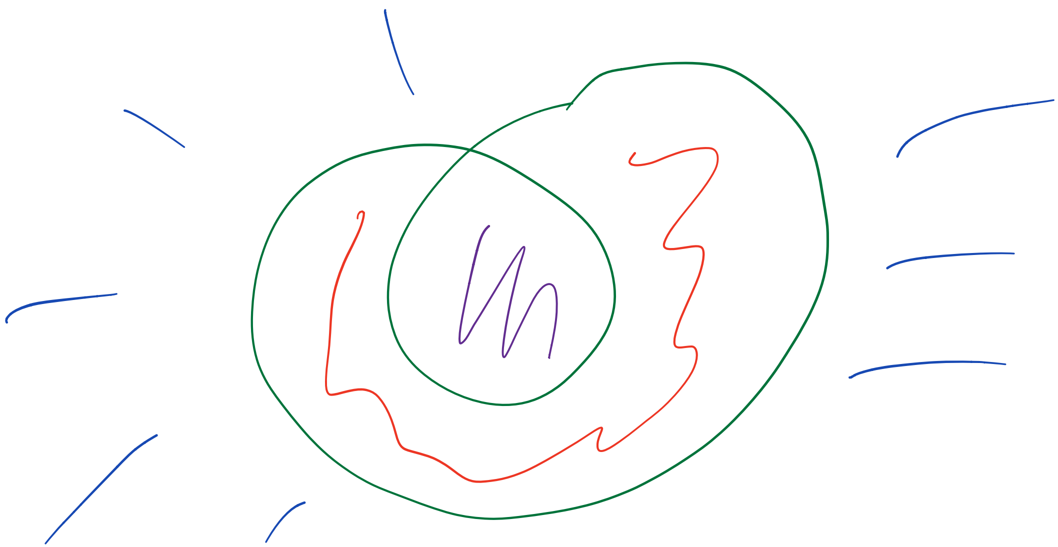
If $f = 1$, then since $I_1(\gamma, z) = I(\gamma, z)$ is integer-valued, it follows it must be constant on any connected component of $\mathbb{C} \backslash \gamma^*$.



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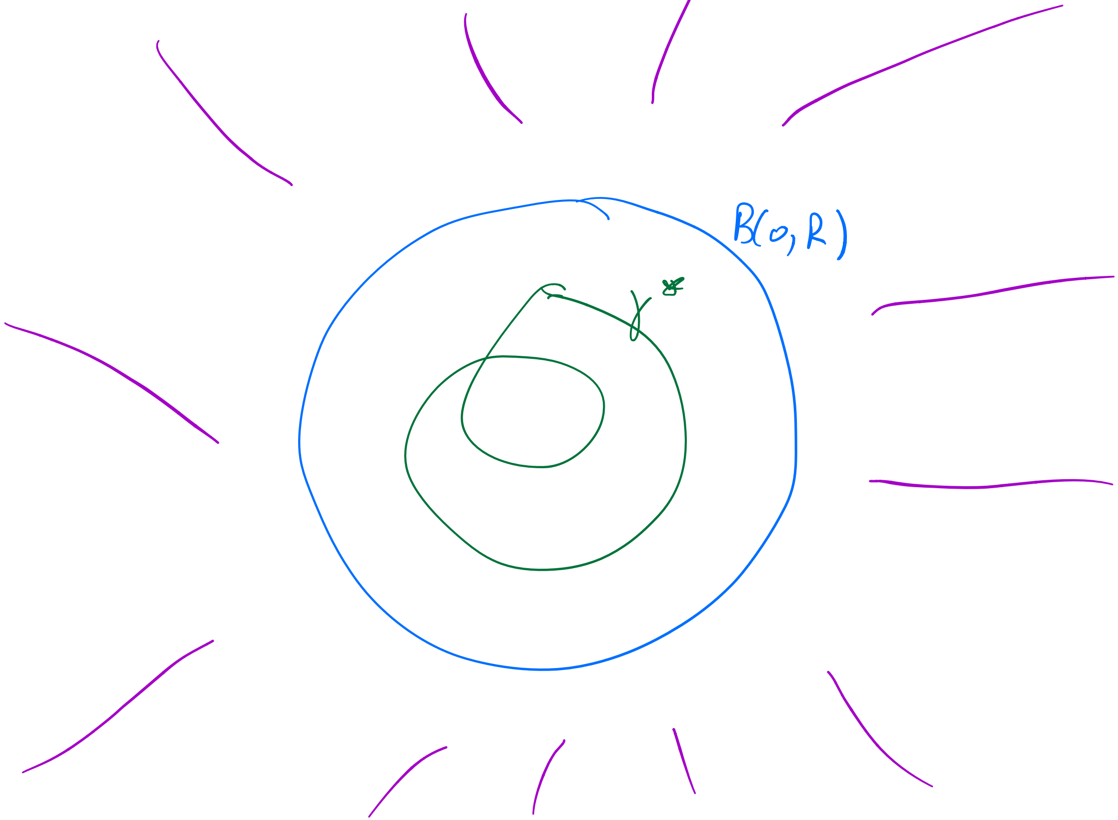
Remark By the power series expression we can calculate the derivatives of $g(w) = I_f(\gamma, w)$ at z_0:

$$g^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

If γ is a closed path then γ^* is compact and hence bounded. Thus there is an $R > 0$ such that the connected set $\mathbb{C} \backslash B(0, R) \cap \gamma^* = \emptyset$. It follows that $\mathbb{C} \backslash \gamma^*$ has exactly one unbounded connected component.



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Since

$$|\int_\gamma \frac{d\zeta}{\zeta - z}| \leq \ell(\gamma). \sup_{\zeta\in\gamma^*} |1/(\zeta - z)| \to 0$$

as $z \to \infty$ it follows that $I(\gamma, z) = 0$ on the unbounded component of $\mathbb{C}\backslash\gamma^*$.

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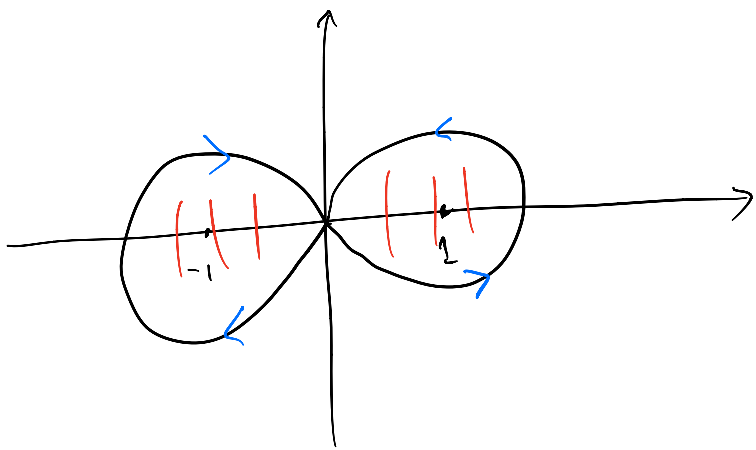
Definition

Let $\gamma \colon [0, 1] \to \mathbb{C}$ be a closed path. We say that a point z is in the inside of γ if $z \notin \gamma^*$ and $I(\gamma, z) \neq 0$. The previous remark shows that the inside of γ is a union of bounded connected components of $\mathbb{C} \backslash \gamma^*$. (We don't, however, know that the inside of γ is necessarily non-empty.)

Example

Suppose that $\gamma_1 : [-\pi, \pi] \to \mathbb{C}$ is given by $\gamma_1 = 1 + e^{it}$ and $\gamma_2 : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma_2(t) = -1 + e^{-it}$. Then if $\gamma = \gamma_1 \star \gamma_2$, γ traverses a figure-of-eight and it is easy to check that the inside of γ is $B(1,1) \cup B(-1,1)$ where $I(\gamma, z) = 1$ for $z \in B(1,1)$ while $I(\gamma, z) = -1$ for $z \in B(-1,1)$.



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Remark

It is a theorem, known as the Jordan Curve Theorem, that if $\gamma \colon [0,1] \to \mathbb{C}$ is a simple closed curve, so that $\gamma(t) = \gamma(s)$ if and only if $s = t$ or $s, t \in \{0, 1\}$, then $\mathbb{C} \backslash \gamma^*$ is the union of precisely one bounded and one unbounded component, and on the bounded component $I(\gamma, z)$ is either 1 or -1. If $I(\gamma, z) = 1$ for z on the inside of γ we say γ is positively oriented and we say it is negatively oriented if $I(\gamma, z) = -1$ for z on the inside.

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This is the single most important theorem of the course. Almost all important facts about holomorphic functions follow from it. Sample applications:

1. If f is holomorphic then it is C^1 and in fact infinitely differentiable.
2. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded then it is constant.
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For most of our applications we will need a simpler case of the theorem for starlike domains. We defer the discussion of the general case to later lectures.

Definition

A triangle or triangular path T is a path of the form $\gamma_1 \star \gamma_2 \star \gamma_3$ where $\gamma_1(t) = a + t(b-a)$, $\gamma_2(t) = b + t(c-b)$ and $\gamma_3(t) = c + t(a-c)$ where $t \in [0,1]$ and $a, b, c \in \mathbb{C}$. (Note that if $\{a, b, c\}$ are collinear, then T is a degenerate triangle.) That is, T traverses the boundary of the triangle with vertices $a, b, c \in \mathbb{C}$. The solid triangle $\mathcal{T}$ bounded by T is the region

$$\mathcal{T} = \{t_1 a + t_2 b + t_3 c : t_i \in [0,1], \sum_{i=1}^{3} t_i = 1\},$$

with the points in the interior of $\mathcal{T}$ corresponding to the points with $t_i > 0$ for each $i \in \{1,2,3\}$. We will denote by $[a, b]$ the line segment $\{a + t(b-a) : t \in [0,1]\}$, the side of T joining vertex a to vertex b. When we need to specify the vertices a, b, c of a triangle T, we will write $T_{a,b,c}$.

Theorem

(Cauchy's theorem for a triangle): Suppose that $U \subseteq \mathbb{C}$ is an open subset and let $T \subseteq U$ be a triangle whose interior is entirely contained in U. Then if $f\colon U \to \mathbb{C}$ is holomorphic we have

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Idea of proof. 1. $f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z)$. So if γ is 'small' close to z_0 $\int_\gamma f(z)dz = \int_\gamma (z - z_0)\psi(z)dz$ which by the estimation lemma and since $\psi(z) \to 0$, is much smaller than length(γ).

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has a primitive so $\int_\gamma g(z)dz = 0$!

$\int az + b = \frac{az^2}{2} + bz$

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2. Assuming that $I = |\int_T f(z)dz| \neq 0$ we will subdivide T into 4 smaller triangles and represent the integral as sum of the integrals on the smaller triangles. One of the integrals of the smaller triangles will be at least $I/4$. We will keep subdividing till we get a very small triangle where by part 1 the integral will be smaller than expected, contradiction.

Suppose $I = |\int_T f(z)dz| > 0$. We build a sequence of smaller and smaller triangles T^n, as follows: Let $T^0 = T$, and suppose that we have constructed T^i for $0 \le i < k$. Then take the triangle T^{k-1} and join the midpoints of the edges to form four smaller triangles, which we will denote S_i $(1 \le i \le 4)$.
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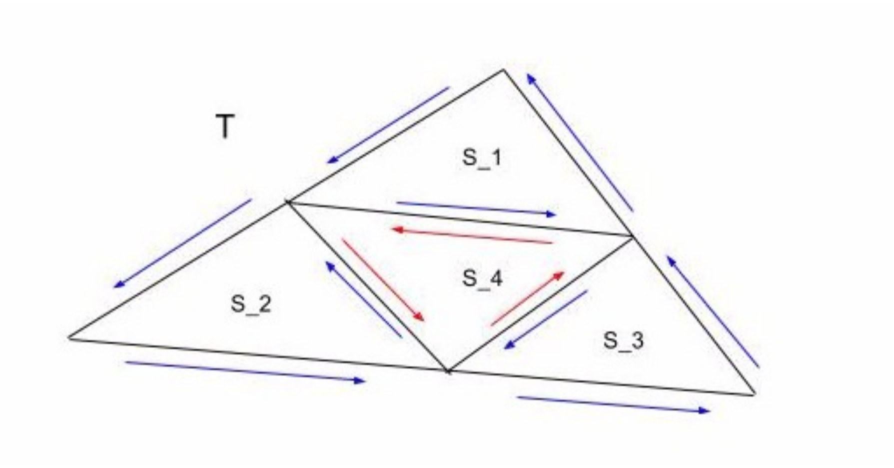

Figure: Subdivision of a triangle

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where $\psi(z) \to 0 = \psi(z_0)$ as $z \to z_0$.

$$\int_{T^k} f(z)dz = \int_{T^k} (z - z_0)\psi(z)dz$$

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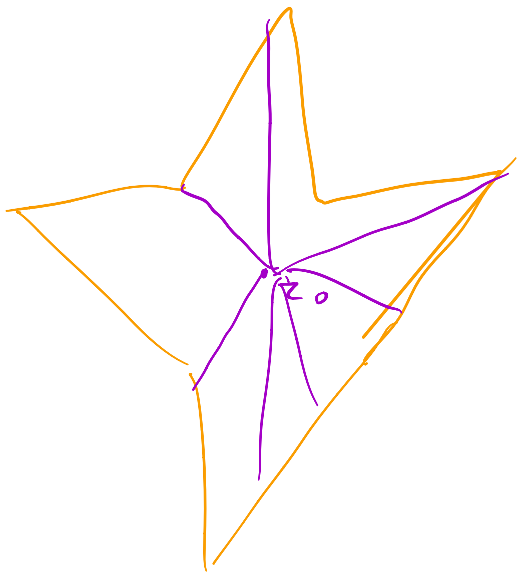
So $4^k I_k \to 0$ as $k \to \infty$. On the other hand, by construction $I_k \geq I/4^k \Rightarrow 4^k I_k \geq I > 0$, contradiction. $\square$

Definition

Let X be a subset in $\mathbb{C}$. We say that X is *convex* if for each $z, w \in U$ the line segment between z and w is contained in X. We say that X is star-like if there is a point $z_0 \in X$ such that for every $w \in X$ the line segment $[z_0, w]$ joining z_0 and w lies in X. We will say that X is star-like with respect to z_0 in this case. Thus a convex subset is thus starlike with respect to every point it contains.



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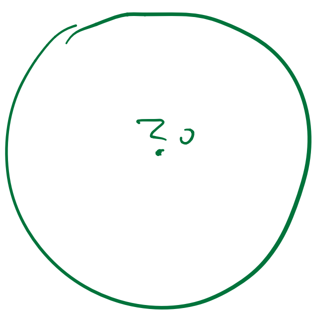
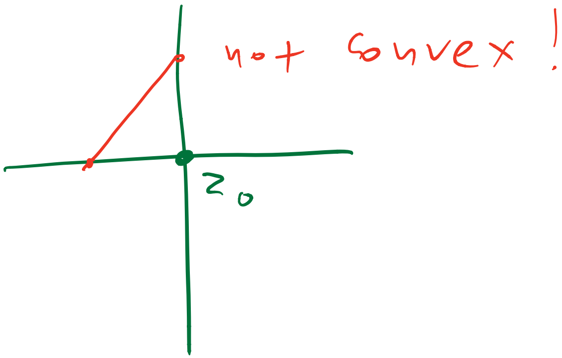
Example. A disk (open or closed) is convex, as is a solid triangle or rectangle. On the other hand the union of the xy-axes is starlike with respect to 0 but not convex.





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Theorem

(Cauchy's theorem for a star-like domain): Let U be a star-like domain. Then if $f: U \to \mathbb{C}$ is holomorphic and $\gamma: [a, b] \to U$ is a closed path in U we have

$$\int_\gamma f(z)dz = 0.$$

Proof. It suffices to show that f has a primitive in U.
Let $z_0 \in U$ such that for every $z \in U$, $\gamma_z = z_0 + t(z - z_0)$, $t \in [0, 1]$ is contained in U. We claim that

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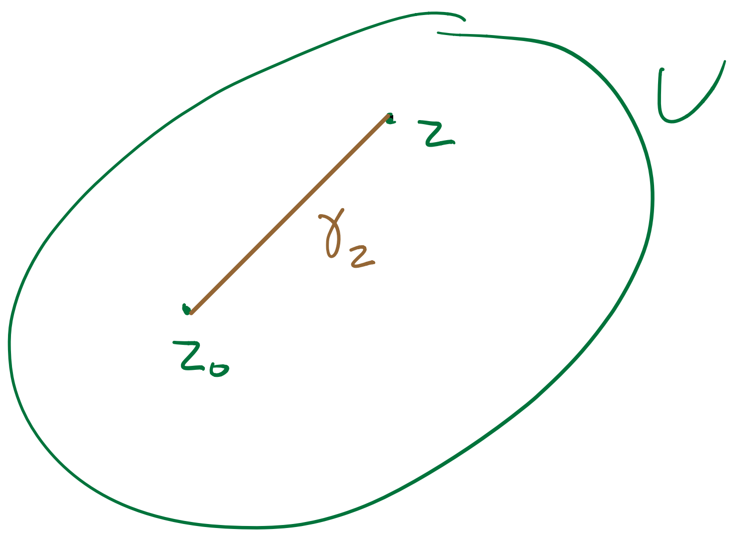

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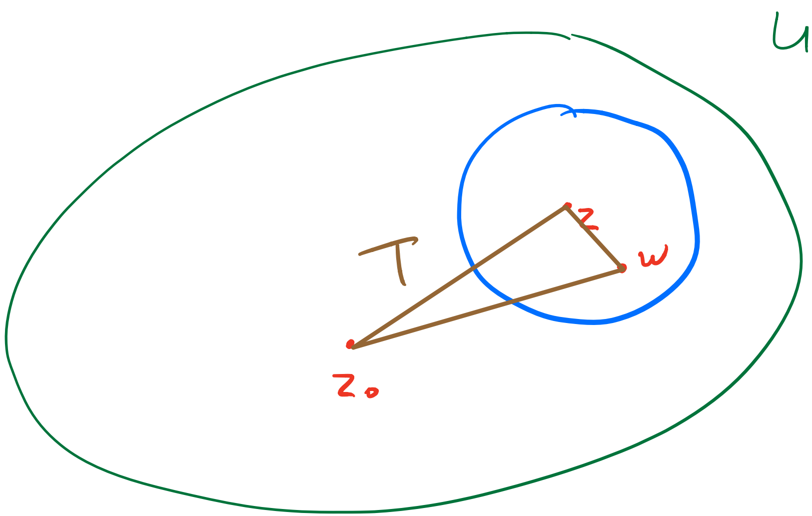

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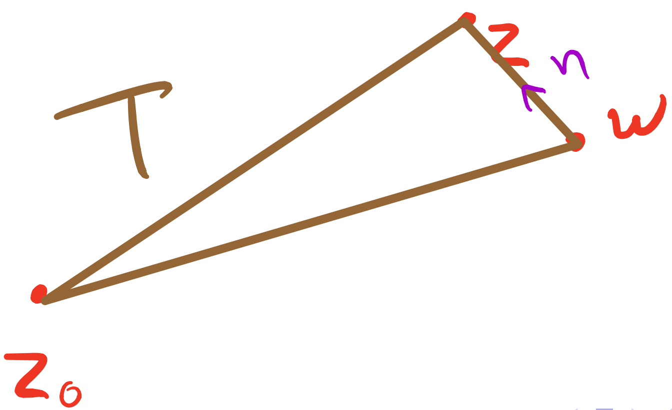

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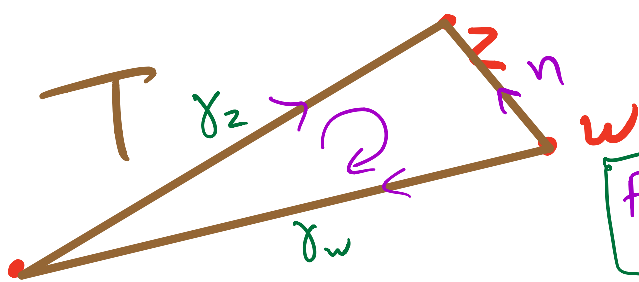

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Suppose that D_1 and D_2 are primitive domains and $D_1 \cap D_2$ is connected. Then $D_1 \cup D_2$ is primitive.

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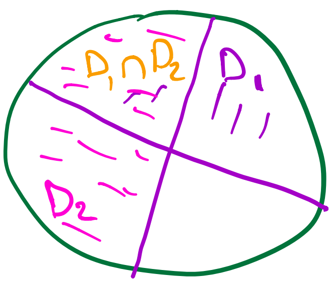

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Indeed each D_1, D_2 are convex, so they are primitive. $D_1 \cap D_2$ is connected so by the lemma $D_1 \cup D_2$ is primitive.

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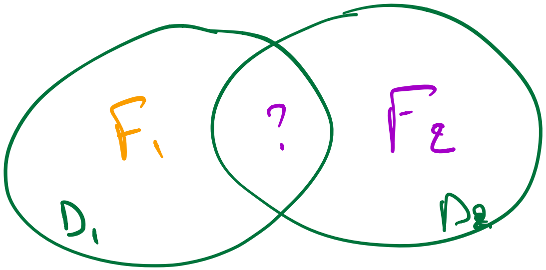

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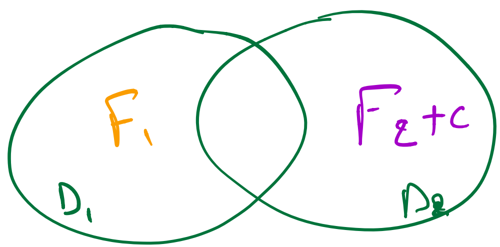
If $F\colon D_1 \cup D_2 \to \mathbb{C}$ is a defined to be F_1 on D_1 and $F_2 + c$ on D_2 then F is a primitive for f on $D_1 \cup D_2$. $\square$



Theorem

(Cauchy's Integral Formula.) Suppose that $f\colon U \to \mathbb{C}$ is a holomorphic function on an open set U which contains the disc $\bar{B}(a,r)$. Then for all $w \in B(a,r)$ we have

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Consider a circle $\gamma(w,\epsilon)$ centered at w and contained in $B(a,r)$. Pick two anti-diametric points on $\gamma(w,\epsilon)$ and join them by straight segments to points on γ.

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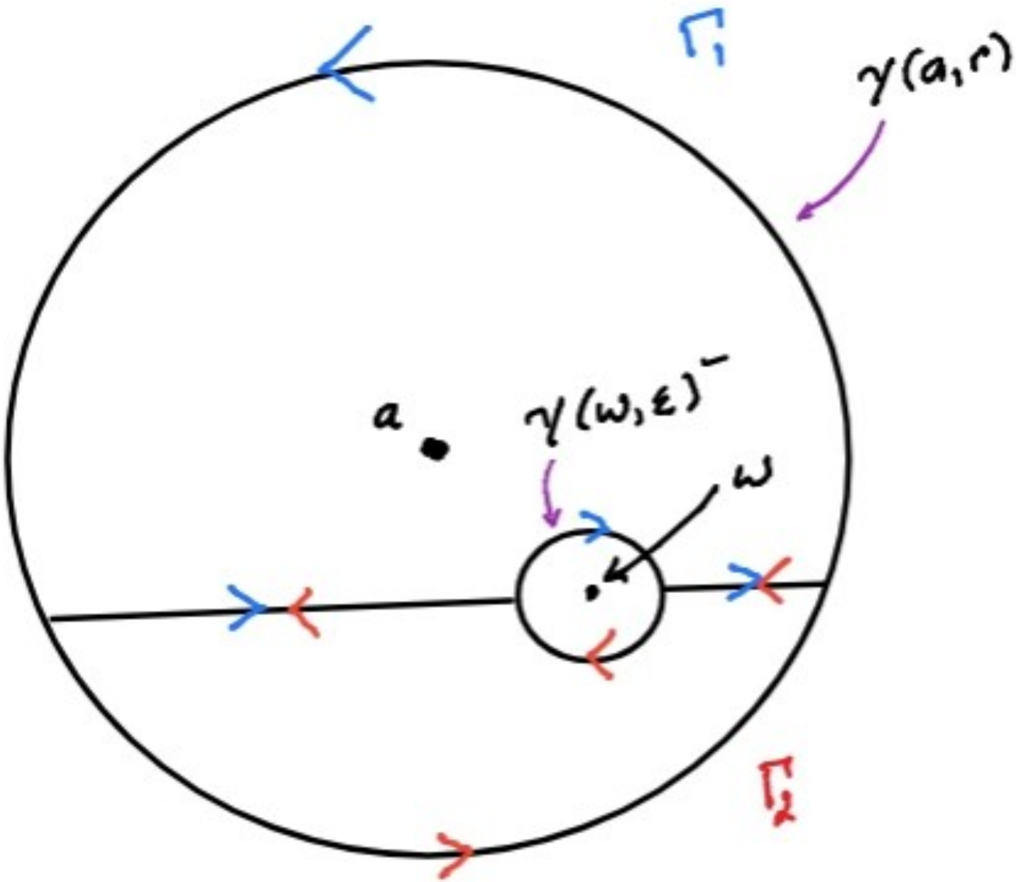

$$\int_{\Gamma_1} \frac{f(z)}{z-w} dz + \int_{\Gamma_2} \frac{f(z)}{z-w} dz = \int_{\gamma(a,r)} \frac{f(z)}{z-w} dz - \int_{\gamma(w,\epsilon)} \frac{f(z)}{z-w} dz.$$

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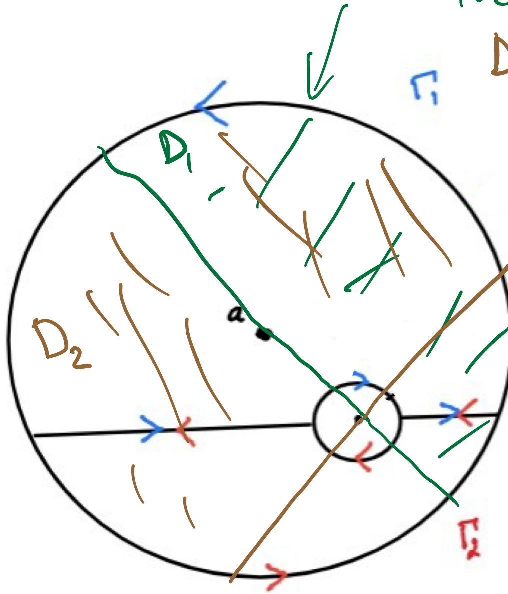

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$\Downarrow$

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It follows that

$$\frac{1}{2\pi i}\int_{\gamma(w,\epsilon)} \frac{f(z) - f(w)}{z - w} dz = 0$$

and

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Corollary

If $f\colon U \to \mathbb{C}$ is holomorphic on an open set U, then for any $z_0 \in U$, the $f(z)$ is equal to its Taylor series at z_0 and the Taylor series converges on any open disk centred at z_0 lying in U. Moreover the derivatives of f at z_0 are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz. \tag{1}$$

For any $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ with $z_0 \in B(a, r)$.

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Proof.

We showed when we studied winding numbers that

$$I_f(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz,$$

is analytic in z and its derivatives are given by the formula above. $\square$

Definition

Recall that a function which is locally given by a power series is said to be *analytic*. We have thus shown that any holomorphic function is actually analytic.